SUPERSTRICTLY SINGULAR AND SUPERSTRICTLY COSINGULAR OPERATORS *

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Abstract

We prove some known and new results on superstrictly singular operators and establish full duality between superstrictly singular and superstrictly cosingular operators.

1 Introduction

Unless otherwise stated, in this paper the *space* denotes a Banach space and the *operator* denotes a bounded linear operator.

Definition 1 [6]. An operator T from a space X into a space Y is called *strictly singular* if there does not exist a number $c > 0$ and an infinite dimensional subspace $E \subset X$ such that $\|Tx\| \geq c\|x\|$ for all x in E.

The following local version of this definition is natural.

Definition 2. An operator T is called *superstrictly singular* (*SSS* for short) if there does not exist a number $c > 0$ and a sequence of subspaces $E_n \subset X$, dim $E_n = n$, such that

$$\|Tx\| \geq c\|x\| \quad \text{for all } x \text{ in } \cup_n E_n \ . \tag{1}$$

Put for an operator T

$$b_n(T) = \sup \inf_{x \in S(E_n)} \|Tx\| \ , \tag{2}$$

where supremum is taken over all n-dimensional subspaces $E_n \subset X$ and $S(E_n)$ is the unit sphere of E_n.

Obviously

$$\|T\| = b_1(T) \geq b_2(T) \geq \cdots \geq 0 \ ,$$

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T is SSS if and only if

$$b_n(T) \to 0 \quad \text{as} \quad n \to \infty \tag{3}$$

and the greatest constant c for which (1) is satisfied, is equal to $\lim_{n\to\infty} b_n(T)$.

The numbers $b_n(T)$, which are called the *Bernštein numbers*, were considered in Approximation Theory (see [21] and the references given there). They are among the classical widths. However those widths are considered mostly for compact sets. Evidently, every compact operator is SSS. We almost do not see noncompact SSS operators in [21]. For operators, defined on a Hilbert space, the Bernštein numbers are connected with so-called AMD numbers (see [2]).

SSS operators were introduced implicitly in [14] and explicitly, under the name "$\mathfrak{S}_0^*$-operators" or "operators of class C_0^*", in [10]-[12]. Probably, the difference can be explained by absence of good letter $\mathfrak{S}$ in Kharkiv. Implicitly these operators was considered also in [23]. The term "superstrictly singular operator" was introduced in [4]; where this class was investigated by technique of superideals. Our note is inspired by the paper [4]. We do not try specially to calculate the exact values of $b_n(T)$, but we shall determine whether they converge to zero or not.

Obviously, every SSS operator is strictly singular. It is easy to see that T has finite rank if and only if $b_n(T) = 0$ begining with some number N. Observe also, that if T is infinite dimensional then the supremum in (2) can be taken over subspaces E_n such that $T|_{E_n}$ are injective, only. Then the formula (2) turns into following

$$b_n(T) = \sup \frac{1}{\|(T|_{E_n})^{-1}\|} \tag{$2'$}$$

where the supremum is taken over all n-dimensional subspaces E_n such that the restrictions $T|_{E_n}$ are injective.

2 Some properties of SSS operators

V.Milman [11] has shown, using the spectrum of continuous function and Dvoretzky's theorem, that SSS operators form a closed ideal, i.e. that a sum of SSS operators is SSS, a composition (right or left) of SSS operator with bounded operator is SSS and $L(X,Y)$-norm limit of SSS operators is SSS. Using superideal technique and the Dvoretzky theorem, this result was proved in [4] also. We shall deduce Milman's result from following Theorem 1. This theorem was proved in [11] too (using the spectrum and Dvoretzky theorem). Our proof of Theorem 1 is elementary and does not use Dvoretzky's theorem.

Theorem 1. *An operator $T : X \to Y$ is SSS if and only if for every sequence of subspaces $E_n \subset X$, $\dim E_n = n$, there exist subspaces $F_n \subset E_n$ such that $\dim F_n \to \infty$ and*

$$\|T|_{F_n}\| \to 0 \ \ as \ \ n \to \infty \ .$$

The “if" part of this statement is evident. To prove the “only if" part we need

Lemma 1. *For every $\varepsilon > 0$ there exists a sequence $k(n) \to \infty$ as $n \to \infty$ such that every normed space E_n , $\dim E_n = n$, contains subspaces $G_1, \dots, G_{k(n)}$, each are of dimension $k(n)$, whose finite dimensional decomposition constant is less than $1 + \varepsilon$.*

The proof of this lemma is deduced easily from Mazur's lemma on basic sequences ([8], p.4). Lemma 1 also follows immediately from the Dvoretzky theorem, but we prefer not to use the Dvoretzky theorem here. □

Proof of Theorem 1. Let $\varepsilon > 0$. Construct, using Lemma 1, subspaces $(G_i^n)_{i=1}^{k(n)}$ of E_n so that $\dim G_i^n = k(n) \to \infty$ and $(G_i^n)_{i=1}^{k(n)}$ have, for fixed n, the finite dimensional decomposition constant less than $1 + \varepsilon$.

Choose, using (2), elements x_i^n in $S(G_i^n)$, $i = 1, \dots, k(n)$ such that $\|Tx_i^n\| \leq b_{k(n)}(T)$ (because of finite dimensional nature of E_n, infimums in (2) are attained).

Now choose a sequence $1 \leq l(n) \leq k(n)$ so that $l(n) \to \infty$ and $l(n) \cdot b_{k(n)}(T) \to 0$ as $n \to \infty$. Put $F_n = \mathrm{lin}(x_i^n)_{i=1}^{l(n)}$. Then for $x \in S(F_n)$, $x = \sum_{i=1}^{l(n)} a_i x_i^n$, we have by finite dimensional decomposition nature

$$|a_1| < 1 + \varepsilon \ , \ |a_2| < 2(1 + \varepsilon) \ , \dots , \ |a_{l(n)}| < 2(1 + \varepsilon).$$

Hence

$$\sum_{i=1}^{l(n)} |a_i| \|Tx_i^n\| \leq (\sum |a_i|) b_{k(n)}(T) < 2(1 + \varepsilon) l(n) b_{k(n)}(T) \to 0 \text{ as } n \to \infty \ . \ \square$$

Remark 1. Theorem 1 is, to a certain extent, a development of the well known Kato's result [6]: *Let X and Y be infinite dimensional spaces. Assume that $T : X \to Y$ is an operator such that the restriction of T to any subspace of finite codimension is not an isomorphism. Then for every $\varepsilon > 0$ there is an infinite dimensional subspace E of X so that $T|_E$ is compact and $\|T|_E\| \leq \varepsilon$.*

Corollary 1. *An operator T is SSS if and only if for every sequence of finite dimensional subspaces $E_n \subset X$, $\dim E_n \to \infty$, there are subspaces $F_n \subset E_n$ such that $\dim F_n \to \infty$ and*

$$\|T|_{F_n}\| \to 0 \quad as \quad n \to \infty \ .$$

Recall that for given $\varepsilon > 0$, a space E is called $(1+\varepsilon)$-*Euclidean* if there exists an operator U from E onto some Euclidean space with

$$(1-\varepsilon)\|x\| \leq \|Ux\| \leq (1+\varepsilon)\|x\| \quad \forall x \in E \ .$$

Application of Dvoretzky theorem gives immediately
Corollary 2. *The following conditions are equivalent:*

(i) T is SSS;

(ii) for every $\varepsilon > 0$ and every sequence of $(1+\varepsilon)$-Euclidean subspaces $E_n \subset X$ with $\dim E_n \to \infty$, we have

$$\inf_{x \in S(E_n)} \|Tx\| \to 0 \quad as \quad n \to \infty \ ;$$

(iii) for every $\varepsilon > 0$ and every sequence of $(1+\varepsilon)$-Euclidean subspaces $E_n \subset X$ with $\dim E_n \to \infty$, there are subspaces $F_n \subset E_n$, $\dim F_n \to \infty$ such that

$$\|T|_{F_n}\| \to 0 \quad as \quad n \to \infty \ .$$

Corollary 3. *A sum $T+U$ of two SSS operators is SSS.*

Proof. Let $E_n \subset X$ be an arbitrary sequence of subspaces, $\dim E_n \to \infty$. There are, by Theorem 1, subspaces $F_n \subset E_n$ so that $\dim F_n \to \infty$ and $\|T|_{F_n}\| \to 0$. By definition, there exists a sequence $x_n \in S(F_n)$, such that $\|Ux_n\| \to 0$. Then

$$\|(T+U)x_n\| \to 0 \ \text{ as } \ n \to \infty \ . \ \Box$$

The fact that a composition of SSS and bounded operator is SSS, is proved very simply.
Corollary 4. *Let $X = X_1 \oplus X_2$, $T_1 : X_1 \to Y$ and $T_2 : X_2 \to Y$ be SSS. Then the operator $T : X \to Y$ defined by the formula $T(x_1 + x_2) = T_1x_1 + T_2x_2$, $x_1 \in X_1$, $x_2 \in X_2$ is SSS.*

Proposition 1. *An $L(X,Y)$-norm limit of SSS operators (T_k) is SSS.*

Proof. Suppose otherwise, i.e. that there exists $c > 0$ and a sequence of subspaces $E_n \subset X$, $\dim E_n \to \infty$, with the property (1) for T. Choose k such that $\|T - T_k\| < c/2$. Since T_k is SSS, $\inf\{\|T_kx\| : x \in S(E_n)\} < c/2$ for some n. Hence $\inf\{\|Tx\| : x \in S(E)\} < c$. This contradicts (1). $\Box$

The following statement is contained in [11] and [4] but we give a more transparent proof.

Theorem 2. *If T is not SSS then for every sequence $\varepsilon_n > 0$ there are $(1+\varepsilon_n)$-Euclidean subspaces $E_n \subset X$, $\dim E_n \to \infty$, and a constant $d > 0$ so that for every n*

$$(d - \varepsilon_n)\|x\| \leq \|Tx\| \leq (d + \varepsilon_n)\|x\| \quad \forall x \in E_n \ .$$

Proof. Of course, we can not do without the Dvoretzky theorem here. Applying it twice, we obtain the existence of $(1+\varepsilon_n)$-Euclidean subspaces $F_n \subset X$, $\dim F_n \to \infty$, and a constant $c > 0$ such that

$$c\|x\| \le \|Tx\| \quad \forall x \in \cup_n F_n \ ,$$

and the subspaces TF_n are $(1+\varepsilon_n)$-Euclidean too.

To continue the proof we use the easily-proved

Lemma 2. *Let a segment $[a,b]$, $\varepsilon > 0$ and a natural number l are given. Then there exists a natural number $k = k(l,a,b,\varepsilon)$ such that whatever numbers $d_1,\dots,d_l$ in $[a,b]$ we take, there exists $d \in [a,b]$ and a subset $d_{i_1},\dots,d_{i_k}$ in $d_1,\dots,d_l$ so that*

$$|d_{i_1} - d| < \varepsilon, \dots, |d_{i_k} - d| < \varepsilon \ .$$

Moreover $k(l,a,b,\varepsilon) \to \infty$ as $l \to \infty$.

And also

Lemma 3. *Let $0 < a \le b$, $0 < \varepsilon < a$ and $l \in \mathbb{N}$. Then there exists a natural number $k = k(l,a,b,\varepsilon)$ such that $k \to \infty$ as $l \to \infty$ and for any operator T from l-dimensional Euclidean space F onto l-dimensional Euclidean space G satisfying the condition*

$$a\|x\| \le \|Tx\| \le b\|x\| \quad \forall\, x \in F$$

there is a subspace $E \subset F$, $\dim E = k$, and a number $d \in [a,b]$ satisfying the condition

$$(d-\varepsilon)\|x\| \le \|Tx\| \le (d+\varepsilon)\|x\| \quad \forall\, x \in E \ . \tag{4}$$

Proof of Lemma 3. Let us choose in F an orthonormal basis $(e_i)_1^l$ such that $(Te_i)_1^l$ is orthogonal basis in G (see f.e. [3], p.175). There exists, by Lemma 2, $d \in [a,b]$ and natural numbers $i_1,\dots,i_k$ for which

$$|\|Te_{i_1}\| - d| < \varepsilon, \dots, |\|Te_{i_k}\| - d| < \varepsilon \ .$$

Then for any numbers $(a_j)_1^k$, we have

$$\|T(\sum_{j=1}^k a_j e_{i_j})\|^2 = \sum_{j=1}^k |a_j|^2 \|Te_{i_j}\|^2 \le \sum |a_j|^2 (d+\varepsilon)^2 = (d+\varepsilon)^2 \|\sum a_j e_{i_j}\|^2 \ . \tag{5}$$

Put $E = \mathrm{lin}(e_{i_j})_{j=1}^k$. The right inequality in (4) follows from (5). The left one is verified in the same way. □

Now it is clear how to finish the proof of Theorem 2. □

Remark 2. An infinite dimensional version of Theorem 2 is true also : *If T is an isomorphism from one Hilbert space into another then there exists an infinite dimensional subspace $E \subset H$ such that the restriction $T|_E$ is proportional to an isometry up to ε.* The same fact is true for l_p spaces, $1 \leq p < \infty$ and c_0 [11].

Note more two properties of SSS operators.

Proposition 2. *Let $T : X \to Y$ be an operator and let $\overline{T}$ be the operator T from X into $\overline{TX} \subset Y$. Then $\overline{T}$ is SSS if and only if T is SSS, moreover for every n*

$$b_n(T) = b_n(\overline{T}).$$

Proposition 3. *Let $T : X \to Y$ be an operator and $\widehat{T} : X/\ker T \to Y$ be the corresponding quotient operator. If $\widehat{T}$ is SSS then T is SSS too, moreover for every n*

$$b_n(T) \leq b_n(\widehat{T}) .$$

Remark 3. An example of quotient map $T : l_1 \to l_1/Z \simeq l_2$ shows that the converse statement is not true (we shall show further that T is SSS; of course $\widehat{T}$ is an isometry).

3 Examples

a) **Strictly singular operators which are non SSS**

As is well known, every strictly singular operator on l_p , $1 \leq p < \infty$ or on c_0 is compact ([8], p.76). Every operator from l_p into l_q , $p \neq q$, $1 \leq p, q < \infty$, and from l_p into c_0 and from c_0 into l_p, is strictly singular ([8], p.75). Moreover, every operator from l_p into l_q for $p > q$ and from c_0 into l_p, is compact (Pitt's theorem ([8], p.76). Let us give

Example 1. A (strictly singular) operator from l_p into l_q , $1 < p < q < \infty$, which is not SSS.

Denote by E_n a subspace of l_p which consists of sequences with supports contained between 2^n and 2^{n+1}. Let $F_n \subset E_n$ be a subspace spanned by 2^n-dimensional version of Rademacher functions and G_n be its complement spanned by the remaining 2^n-dimensional version of Walsh functions. From Khinchine's inequality ([8], p.66) it follows that there are constants $a, b > 0$ and $d_n > 0$ such that

$$ad_n\|x\|_p \leq \|x\|_q \leq bd_n\|x\|_p \quad \text{for every } x \text{ in } F_n ,$$

and also that the projections in E_n onto F_n along G_n both in norm l_p and in norm l_q are uniformly bounded.

Define an operator T in the following way: if $x \in F_n$, put $Tx = d_n x$; if $x \in G_n$, put $Tx = x$. Then we extend T to a linear and bounded operator from l_p into l_q . $\square$

Remark 4. For $p = 2\ ,\ q > 2$ a similar example was given in [10] and [11].

Remark 5. It will follow from next Proposition 6 that each operator from l_1 into $l_p\ ,\ 1 < p < \infty$, is SSS. We shall consider soon c_0-valued operators .

Corollary. *Let* $1 < p < q < \infty$. *For every infinite dimensional* $\mathcal{L}_p$*-space* X *and* $\mathcal{L}_q$*-space* Y *(in the sense of* [7]*) there exists a strictly singular operator from* X *into* Y *which is non SSS.*

Indeed, any infinite dimensional $\mathcal{L}_p$-space contains a complemented subspace isomorphic to l_p [7].

Let us consider c_0-valued operators .

Proposition 4. *Let* X *be a Banach space with a finite dimensional decomposition* (E_n). *Then there exists an injective operator from* X *into* c_0 *which is non SSS. If* X *has no subspaces isomorphic to* c_0 *then this operator is strictly singular. There exists an operator from an arbitrary separable Banach space into* c_0 *which is non SSS.*

Proof. Of course, we can assume $\dim E_n \to \infty$. Take $\varepsilon > 0$. Choose, using the local universality of c_0, subspaces $F_n \subset c_0$ which are $(1+\varepsilon)$-isometric to E_n, have disjoint supports and let $T : E_n \to F_n$ be these $(1+\varepsilon)$-isometries. It remains to extend T to an injective operator from X into c_0. To prove the last statement of the proposition it is sufficient to use the well known fact that any infinite dimensional separable space has a quotient with basis ([8], p.10). $\square$

b) **SSS operators which are non compact**

A natural example of noncompact SSS operator is the identity inclusion $l_p \hookrightarrow l_q$ for $1 \le p < q \le \infty$ [12]. Probably, the widths $b_n(\hookrightarrow)$ for these inclusions were calculated or estimated in the Approximation Theory. In [21] the inclusion $l_1 \hookrightarrow l_2$ (a width of octahedron) is considered. We shall deduce the result of [12] from a somewhat general statement on an inclusion into c_0. However first recall

Lemma 4. [9],[12]. *Let* X *be some linear space of functions of natural argument vanishing at infinity. Then every subspace* $E_n \subset X$, $\dim E_n = n$, *contains a function* $x(i)$ *such that* $\max\{|x(i)| : i \in \mathbb{N}\}$ *is attained in at least* n *points.*

Proof. Obviously, every function $|x(i)|\ ,\ x \in X$ attains maximum in at least one point. Let for n the lemma be proved. Consider an arbitrary subspace $E_{n+1} \subset X$, $\dim E_{n+1} = n+1$. By supposition, there exists a function $y \in E_{n+1}$ so that $\max |y(i)|$ is attained at least in n points $(i_k)_1^n$. Without loss of generality we can assume that the maximum is attained exactly in n points. A subspace

$E^n = \{x \in X : x(i_k) = 0 \text{ for } k = 1, \ldots, n\}$ has codimension n, hence has a non-zero intersection with E_{n+1}. Let $0 \neq z \in E^n \cap E_{n+1}$ and $\max |z(i)| = |z(j)|$. Since $z(i_k) = 0$, then $j \neq i_k$, $k = 1, \ldots, n$. Then the family $\{\lambda y + (1-\lambda)z : 0 \leq \lambda \leq 1\}$ contains a function, which attains the maximum of modulus in $n+1$ points $(i_k)_{k=1}^{n+1}$ (i_{n+1} may be not equal to j). □

Proposition 5. *Let X be a space with a symmetric basis (e_n) which is not equivalent to the standard basis of c_0. Then the natural inclusion $X \hookrightarrow c_0$ is SSS.*

Proof. Let $E_n \subset X$, $\dim E_n = n$. Choose, using Lemma 4, an element $x_n(i) \in E_n$, whose maximum modulus is attained at least in n points and is equal to one. Obviously, $\|x_n\|_X \to \infty$, which proves that our operator is SSS. □

Remark 6. From the proof of Proposition 5 it follows that for the inclusion $\hookrightarrow_p$ of l_p to c_0, we have

$$b_n(\hookrightarrow_p) = n^{-1/p} \ .$$

Corollary 1. *The natural inclusion $l_p \hookrightarrow l_q$ for $1 \leq p < q < \infty$ is SSS.*

Proof. Indeed,

$$\|x\|_q^q = \sum |x(i)|^q = \sum |x(i)|^p |x(i)|^{q-p} \leq (\sum |x(i)|^p) \max |x(i)|^{q-p} = \|x\|_p^p \|x\|_0^{q-p} \ .$$

If $\|x\|_p = 1$ then $\|x\|_q \leq \|x\|_0^{(q-p)/q}$. This and Proposition 5 implies SSS of our inclusion. □

Remark 7. From the proof of Corollary 1 it follows that for the inclusion $\hookrightarrow_{pq}$ of l_p to l_q , $p < p$

$$b_n(\hookrightarrow_{pq}) = n^{(p-q)/pq} \ .$$

Corollary 2. *Let $1 < p < q < \infty$, X be an infinite dimensional $\mathcal{L}_p$-space and Y be an infinite dimensional $\mathcal{L}_q$-space. Then there exists an SSS and non compact operator from X into Y.*

To prove this it is sufficient to recall the arguments of corollary of Example 1.

Corollary 3. *There exists an SSS and non compact operator in the space $L_p(0,1)$, $1 \leq p < \infty$, $p \neq 2$.*

Proof. Let X be a subspace of space L_p spanned by Rademacher functions, Y be its complement (for $p \neq 1$), U be a complemented subspace of L_p isometric to l_p , V be its complement.

If $p < 2$ then there exists an SSS and non compact operator from U into X (Corollary 1). If $p > 2$ then there exists an SSS and non compact operator from X into U (Corollary 1, too).

In both case let us consider an arbitrary compact operator from V (resp. Y) into L_p and extend our operator onto whole space (see Corollary 4 to Theorem 1). □

Remark 8. For $2 < p < \infty$ Corollary 3 was proved in [12]. Of course, Corollary 3 remains true for every space containing complemented $L_p(0,1)$, for example, for $L_p(\mathbb{R})$, $1 \leq p < \infty$, $p \neq 2$.

c) **Spaces, where** $SSS(X,Y) = L(X,Y)$

Definition 7. We say that a space X *contains no uniformly complemented almost Euclidean subspaces* if there exists $\varepsilon > 0$ such that for every sequence of $(1+\varepsilon)$-Euclidean subspaces $E_n \subset X$, $\dim E_n \to \infty$, the relative projection constants $\lambda(E_n, X) \to \infty$.

Definition 9 [16]. A space X is called *locally π-Euclidean* if there is a $d \geq 1$ so that for all n and $\varepsilon > 0$ there is an N such that every N-dimensional subspace of X contains a $(1+\varepsilon)$-isomorph of l_2^n which is d-complemented in X.

Every $\mathcal{L}_1$-space or $\mathcal{L}_\infty$-space does not contain uniformly complemented almost Euclidean subspaces ([7], Corollaries 6,2). The well known Pisier space in which the norms of all n-dimensional projections grow with n [18] does not contain such subspaces. Locally π-Euclidean spaces coincide with B*-convex* spaces i.e. spaces which contain no l_1^n uniformly [17]. We shall use both terms B-convex and locally π-Euclidean.

Proposition 6. *Let X contain no uniformly complemented almost Euclidean subspaces and Y be B-convex. Then every operator T from X into Y is SSS.*

Proof. Suppose contrary to our claim, that there is a sequence of $(1+\varepsilon)$-Euclidean subspaces $E_n \subset X$, $\dim E_n \to \infty$, such that

$$\inf_{x \in S(E_n)} \|Tx\| \geq c > 0 \quad \text{for every} \quad n \, .$$

Passing, if necessary, to subspaces, we may suppose that $F_n := TE_n$ are $(1+\varepsilon)$-Euclidean and d-complemented. Let P_n be a projection of norm d of the space Y onto F_n. Put $Q_n = (T^{-1}|_{F_n})P_nT$. Evidently, Q_n is a projection of X onto E_n and

$$\|Q_n\| \leq \|T^{-1}|_{F_n}\| \cdot \|P_n\| \cdot \|T\| \leq \frac{d}{c}\|T\| \, .$$

Contradiction. □

Remark 9. Proposition 6 is inspired by the following

Proposition MP [14]. *Let T be an operator from the space $C(S)$ of all continuous functions on a compact S into l_2. Then*

$$b_n(T) \leq 1/\sqrt{p_n} \, ,$$

where p_n is the projection constant of n-dimensional Euclidean space.

Proof. Probably, we have assume in Proposition MP $\|T\| \leq 1$. Let E_n be an n-dimensional subspace of $C(S)$ such that $T|_{E_n}$ is one to one and $F_n = TE_n$. Then there is a projection $Q_n : C(S) \to E_n$ with $\|Q_n\| \leq \|T^{-1}|_{F_n}\|$. As is well known, the projection constant q_n of subspace E_n satisfies the inequality $p_n \leq q_n \cdot \|T\|\|T^{-1}|_{F_n}\|$. Hence,

$$p_n \leq q_n \cdot \|T^{-1}|_{F_n}\| \leq \|Q_n\| \cdot \|T^{-1}|_{F_n}\| \leq \|T^{-1}|_{F_n}\|^2 \leq b_n(T)^{-2}.$$

It implies Proposition MP. □

d) **Absolutely summing and narrow operators**

An operator $T : X \to Y$ is called *p-absolutely summing* if there is a constant K so that, for every choice of an integer n and vectors $\{x_i\}_{i=1}^n$ in X we have

$$(\sum_{i=1}^n \|Tx_i\|^p)^{1/p} \leq K \sup_{\|f\|=1} (\sum_{i=1}^n |f(x_i)|^p)^{1/p} .$$

The following proposition was known to the authors of [14].

Proposition 7. *Every p-absolutely summing operator T, $1 \leq p < \infty$, is SSS.*

Proof. Since any p-absolutely summing operator is q-absolutely summing if $p < q$ ([8], p.65), we can assume $p > 2$. Suppose T is not SSS. Take subspaces E_n from Theorem 2. Obviously, for "almost orthogonal" bases $\{x_i\}_{i=1}^n$ in these spaces we can not find a common K in the definition of p-absolutely summing operator. □

Note, that not every p-absolutely summing operator is compact.

An operator T from $L_p(0,1)$ to a space Y is *narrow* if for each measurable subset $A \subset (0,1)$ and each $\varepsilon > 1$ there is $x \in L_p$ such that $x^2 = \chi_A$ and $\|Tx\| < \varepsilon$. For properties of narrow operators see [19]. In particular, every compact operator is narrow. There is a narrow operator which is not strictly singular.

Question. Let $1 < p < \infty$. Is every SSS operator $T : L_p \to Y$ narrow, for any space Y?

4 Connection between SSS of an operator and its duals

Let us consider first the connection between STS of an operator and its dual. A simple example such as embedding $L_\infty \hookrightarrow L_2$ shows that an operator can be SSS[1] even if its dual is not strictly singular. So, as in the case of strictly singular

[1] A simple proof of SSS of this embedding is contained, in fact, in ([20], Theorem 5.2).

operators, for guarantee of connection between SSS of operator and its dual , we have to impose some additional conditions. It is found, that for SSS operators such conditions are more natural than those for strictly singular one.

Probably, the following corollary of the principle of local reflexivity is well known.

Lemma 5. *Let X be a locally π-Euclidean space. Then X^* is "locally π^*-Euclidean"; more exactly, there is a $d^* \geq 1$ so that for all n and ε there is an N such that every N-dimensional subspace of X^* contains an n-dimensional $(1+\varepsilon)$-Euclidean and d-complemented in X^* subspace , moreover this complement is weakly* closed.*

Proof. Since X it locally π-Euclidean it is B-convex [17]; hence X^* is B-convex (This is well known and is easy to prove). So X^* is locally π-Euclidean.

Let given N-dimensional subspace of X^*, E_n be a subspace of X^* , $\dim E_n = n$, the existence of which guarantee Definition 4. Let F^n be a d-complement of E_n. Then the annihilator in the dual space $(F^n)^\perp \subset X^{**}$ has dimension n. Let $\varepsilon > 0$. By the principle of local reflexivity there exists a $(1+\varepsilon)$-isometry T from $(F^n)^\perp$ onto some subspace $X_n \subset X$ such that $\langle T\varphi, e\rangle = \langle e, \varphi\rangle$ for every $\varphi \in (F^n)^\perp$ and $e \in E_n$. Hence, we can take $X_n^\perp \subset X^*$ as the desired weakly* closed complement for E_n . $\square$

Theorem 3. *Let T be an SSS operator from a B-convex space X into arbitrary space Y. Then T^* is SSS.*

Proof. Suppose T^* is not SSS. By Lemma 5, X^* is locally π^*-Euclidean. Hence there are subspaces $E_n \subset Y^*$ such that $\dim E_n \to \infty$, for some $c > 0$

$$\|T^* f\| \geq c\|f\| \quad \forall f \in \cup_n E_n$$

and subspaces $F_n = T^* E_n$ are uniformly complemented in X^*, moreover these complements G^n are weakly* closed. Put $X_n = (G^n)^\top \subset X$, where $\top$ denotes an annihilator in the predual space. Then $\dim X_n = n$ and for $x \in S(X_n)$

$$\|Tx\| = \sup_{f\in S(Y^*)} \langle Tx, f\rangle = \sup_{f\in S(Y^*)} \langle x, T^* f\rangle \geq \sup_{f\in S(E_n)} \langle x, T^* f\rangle \geq c \sup_{g\in S(F_n)} \langle x, g\rangle \geq \frac{c}{d}\|x\| . \square$$

Remark 10. Let us recall that a space is called *c-convex* if it does not contain uniformly l_∞^n. There exists an SSS operator T from a c-convex space X into a Hilbert space H such that T^* is not even strictly singular. As an example, we can take a quotient map $l_1 \to l_1/Z \simeq H$. The SSS of this quotient follows from Proposition 6. Theorem 3 is false for strictly singular operators ([5], p.47).

The following definition was introduced in [12]: A closed subspace E of a space X is said to we *partially complemented* if there exists an infinite dimensional closed subspace $F \subset X$, $E \cap F = 0$, for which $E + F = \overline{E + F}$. In [12] is stated

Theorem M. *If every closed subspace $E \subset X$, $\operatorname{codim} E = \infty$, is partially complemented, then the strict singularity of T implies the strict singularity of T^*.*

Prof. Milman informs us that the proof of Theorem M contains a gap and that his proof works only if X^* is norm separable. So, the following question is natural; Let X be separable and suppose that every infinite codimensional subspace of X is partially complemented. Must X^* be separable? Consider the James tree JT.

Let us now consider the connection between the SSS of operator and its second dual. The following (probably, well known) fact plays there a decisive role.

Corollary of the principle of local reflexivity. *Let Φ be a finite dimensional subspace of X^{**} and suppose that for some $c > 0$*

$$\|T^{**}\varphi\| > c\|\varphi\| \quad \forall\varphi \in \Phi \ .$$

Then there exists a subspace $E \subset X$, $\dim E = \dim \Phi$, such that

$$\|Tx\| > c\|x\| \quad \forall x \in E. \tag{6}$$

Proof. Let F be a finite dimensional subspace of Y^* which $(1 - \varepsilon)$-norms $T^{**}\Phi$, i.e. $\forall\varphi \in T^{**}\Phi$ there is $f \in S(F)$, for which $(1 - \varepsilon)\|\varphi\| \le \langle f, \varphi\rangle$. By the principle of local reflexivity, there exists a subspace $E \subset X$ and $(1 + \varepsilon)$-isometry $U : E \to \Phi$ such that

$$\langle x, g\rangle = \langle g, Ux\rangle \quad \forall x \in E \ , \ g \in T^*F.$$

Then for arbitrary $x \in E$ and $f \in F$

$$\langle Tx, f\rangle = \langle x, T^*f\rangle = \langle Ux, T^*f\rangle = \langle T^{**}Ux, f\rangle \ .$$

Therefore, for any $x \in E$ there is an element $f \in S(F)$ such that

$$\|Tx\| \ge |\langle Tx, f\rangle| = |\langle T^{**}Ux, f\rangle| \ge (1-\varepsilon)\|T^{**}Ux\| > (1-\varepsilon)c\|Ux\| > \frac{(1-\varepsilon)}{(1+\varepsilon)}c\|x\| \ .$$

So, (6) is satisfied for sufficiently small ε . □

Corollary. *If T is SSS then T^{**} is SSS too.*

Remark 11. The strict singularity of T does not imply the strict singularity of T^{**}. We shall give examples in the next section.

5 Superstrictly cosingular operators

Definition 5 [15]. An operator $T : X \to Y$ is called *strictly cosingular* if for every closed subspace $E \subset Y$ of infinite codimension, the map QT (where $Q : Y \to Y/E$ is a quotient map) has non-closed range.

It is easy to see ([15]) that if T^* is strictly singular, then T is strictly cosingular and if T^* is strictly cosingular then T is strictly singular.

The following quantitative characteristic of an operator T was introduced in [14] :

$$a_n(T) = \sup_{E^n} \inf\{\|\widehat{Tx}\|_{Y/E^n} : x \in X \ , \ \|\hat{x}\|_{X/T^{-1}E^n} = 1\} \ ,$$

where supremum is taken over all closed subspaces $E^n \subset Y$ of codimension n and caps denote the corresponding quotient classes.
Remark 12. More exactly, in [14]

$$a_n(T) = \sup_{E^n} \inf_{\|x\|=1} \|Tx\|_{Y/E^n} \ .$$

Note that with this definition $a_n(T) = 0$ for every operator with infinite dimensional kernel.
Definition 6. We call an operator T *superstrictly cosingular* (*SSCS* for short), if $a_n(T) \to 0$ as $n \to \infty$.

Evidently, every SSCS operator is strictly cosingular.

We do not know any papers expect [14] where the numbers $a_n(T)$ are considered. We do not know also, how the numbers $a_n(T)$ are connected with other widths. The principle of local reflexivity gives the complete duality between SSS and SSCS operators. Namely, the following is true

Theorem 4. *An operator T is SSS (SSCS) if and only if T^* is SSCS (respectively SSS).*

Proof. 1. *T^* is SSS $\Rightarrow$ T is SSCS.* Let E^n be a closed subspace of Y of codimension n and $F_n = (E^n)^{\perp}$. Then $\dim F_n = n$. Let $f_0 \in S(F_n)$ be an element such that

$$\|T^* f_0\| = \inf_{f \in S(F_n)} \|T^* f\| \ .$$

Put $\widehat{E}_n = X/T^{-1}E^n$. Then $(\widehat{E}_n)^* = T^* F_n$. Let $\hat{x} \in S(\widehat{E}_n)$ be an element such that

$$\langle \hat{x}, T^* f_0 \rangle = \sup_{f \in S(F_n)} \langle \hat{x}, T^* f \rangle \ .$$

Then for every representative $x \in \hat{x}$

$$\|\widehat{Tx}\| = \sup_{f \in S(F_n)} \langle \widehat{Tx}, f \rangle = \sup_{f \in S(F_n)} \langle Tx, f \rangle =$$
$$\sup_{f \in S(F_n)} \langle x, T^* f \rangle = \langle x, T^* f_0 \rangle \leq \|x\| \|T^* f_0\| \ .$$

Hence, $\|\widehat{Tx}\| \leqslant \|T^* f_0\|$ and implication 1 is proved.

2. *T is SSS$\Rightarrow$ T^* is SSCS*. By the Corollary of the principle of local reflexivity, T^{**} is SSS. The T^* is SSCS by item 1.

3. *T is SSCS $\Rightarrow$ T^* is SSS*. Let F_n be an n-dimensional subspace of Y^*. We can suppose, by ($2'$), that the restriction $T^*|_{F_n}$ is injective. Put $\widehat{E}_n = X/(T^* F_n)^\top$. Then $\dim \widehat{E}_n = n$. Take an element $\hat{x}_0 \in S(\widehat{E}_n)$ such that (the norm $\|\widehat{Tx}\|$ is taken in the quotient space $Y/F_n^\top$)

$$\|\widehat{Tx_0}\| = \inf_{\|\hat{x}\|=1} \|\widehat{Tx}\|.$$

Take an element $f \in S(F_n)$ such that

$$\langle Tx_0, f \rangle = \sup_{\|\hat{x}\|=1} \langle \widehat{Tx}, f \rangle \ .$$

Then

$$\|T^* f\| = \sup_{\|\hat{x}\|=1} \langle \hat{x}, T^* f \rangle = \sup_{x \in \hat{x}\ ,\ \|\hat{x}\|=1} \langle x, T^* f \rangle = \sup_{x \in \hat{x}\ ,\ \|\hat{x}\|=1} \langle Tx, f \rangle =$$
$$\sup_{\|\hat{x}\|=1} \langle Tx, f \rangle = \langle \widehat{Tx_0}, f \rangle \leq \|\widehat{Tx_0}\| \ .$$

4. *T^* is SSCS $\Rightarrow$ T is SSS*. By item 3, T^{**} is SSS and hence T is also. □

Theorem 4 allows to carry results, known for SSS operators, onto SSCS operators. Let us present some of them.

Corollary 1. *SSCS operators form a closed ideal.*

Proof. See Corollary 3 of Theorem 1 and Proposition 1.

Corollary 2. *If an operator T is SSCS then there exists a sequence of closed subspaces $E^n \subset Y$ of codimension n such that $X/T^{-1}E^n$ are $(1+\frac{1}{n})$-Euclidean and*

$$\inf\{\|\widehat{Tx}\|_{Y/E^n} : \|\hat{x}\|_{X/T^{-1}E^n} = 1\} \to \ 0 \ \ as \ \ s \to \infty \ .$$

Proof. See Corollary 2 of Theorem 1.

Corollary 3. *The natural embedding $l_p \hookrightarrow l_q$, where $1 < p < q < \infty$, is SSCS.*

Proof. See Corollary 1 of Proposition 5.

Corollary 4. *Let X be B-convex and suppose that Y does not contain uniformly complemented almost Euclidean subspaces. Then every operator T from X into Y is SSCS.*

Proof. See Proposition 6.

Corollary 5. *Let T be an SSCS operator from an arbitrary space X into a B-convex space Y. Then T^* is SSCS too.*

Proof. See Theorem 6.

Theorem 4 allows us also to prove non SSS or non SSCS of some particular operators. Let us present examples.

Corollary 6. *An operator $T : L_1(0, 2\pi) \to c_0$ which put into correspondence to a function from L_1 sequence of its Fourier coefficients, is neither SSS nor SSCS, although it is strictly singular.*

Proof. As is known ([22]) this operator is strictly singular but its adjoint is neither strictly singular nor strictly cosingular. It remains to use Theorem 2. □

Corollary 7. *The quotient map from l_1 onto $l_1/Z \simeq l_p$, $1 < p < \infty$, is SSS, by Proposition* 6. *The quotient map Q from l_1 onto $l_1/Y \simeq c_0$ is strictly singular but the adjoint map Q^* is not strictly cosingular* ([15]; *Example* 1). *So, Q is not SSS.*

6 SSS and uniformly strictly singular operators

In [13] the notion of uniformly strictly singular operator was introduced and it was shown that in well known Gowers-Maurey space every operator is a sum of scalar and uniformly strictly singular operators. After some small correction, the definition of [13] reads as follows.

Definition 3. Let X be a space with a basis (e_k) and Y be an arbitrary space. An element in X is called a *block* if it has finite support, that is , if it is a finite linear combination of elements of the basis. The blocks v and w are said to be *consecutive* if the support of v (the set of elements of the basis that form v as a linear combination) ends before the support of w begins. One says that an operator $T : X \to Y$ is *uniformly strictly singular* with respect to the basis (e_k) if for every $\varepsilon > 0$ there exists a number N such that for arbitrary consecutive blocks $(x_n)_1^N$ there exists an element $x \in S(\text{lin}(x_n)_1^N)$ such that $\|Tx\| < \varepsilon$.

Let us recall that a sequence (x_n, f_n) , $x_n \in X$, $f_n \in X^*$, is called an M-*basis* if $f_n(x_m) = \delta_{nm}$, $[x_n]_1^\infty = X$ and for any $x \in X \setminus \{0\}$ there is n such that $f_n(x) \neq 0$. The M-basis is called 1-*norming* if

$$\|x\| = \sup\{|f(x)| : f \in \text{lin}(f_n)_1^\infty \ , \ \|f\| = 1\}$$

for every $x \in X$. Of course we can carry Definition 6 to M-basis. Evidently, every SSS operator is uniformly strictly singular with respect to any basis (and with respect to any M-basis).

Proposition 8. *Suppose that an operator $T : X \to Y$, where X is separable, is not SSS. Then T is not uniformly strictly singular with respect to some 1-norming M-basis.*

Proof. Let (x_n, f_n) be a 1-norming M-basis of X. Then for some $c > 0$ there is a sequence $n_k \uparrow \infty$ such that the subspaces $E_k = \mathrm{lin}(x_n : n_k \leq n < n_{k+1})$ contain subspaces F_k so that $\inf\{\|Tx\| : x \in S(F_k)\} > c$ and $\dim F_k \to \infty$. Hence, we can construct in every E_k a basis $e_{n_k+1}, \dots, e_{n_{k+1}}$ so that F_k is a span of some of the first elements of this basis and the biorthogonal functionals span $\mathrm{lin}(f_n : n_k < n \leq n_{k+1})$. Obviously, T is not uniformly strictly singular with respect to (e_n) . □

Question. Whether there exists a non SSS operator which is uniformly strictly singular with respect to each basis ?

Proposition 9. *Every operator T from l_p into a space Y of type q , $1 \leq p < q \leq 2$ is uniformly strictly singular with respect to the standard basis of l_p.*

Proof. Since Y has type q, there exists a constant M such that for an arbitrary collection $(y_n)_1^N$ in Y there are signs (θ_n) for which

$$\|\sum_1^N \theta_n y_n\| \leq M(\sum_1^N \|y_n\|^q)^{1/q} .$$

Let $y_n = Tx_n$ where $(x_n)_1^N \subset S(l_p)$ are consecutive blocks. Then

$$\|T(\sum \theta_n x_n)\| \leq M(\sum \|Tx_n\|^q)^{1/q} \leq M\|T\|N^{1/q} .$$

Since $\|\sum_{n=1}^N \theta_n x_n\| = N^{1/p}$, this proves Proposition 9. □

Let us give an

Example of an operator which is uniformly strictly singular with respect to one basis but is not uniformly strictly singular with respect to another basis, and hence is non SSS.

Let $T : l_p \to l_q$ be the operator from Example 1, where $1 < p < q \leq 2$. By Proposition 9, this operator is uniformly strictly singular with respect to the standard basis. Obviously, it is not uniformly strictly singular with respect to basis consisting of blocks $w_i^n : 1 \leq i \leq 2^n$, $n = 1, 2, \dots$ of Walsh systems.

Remark 13. Prof. R.Wagner informs us, that from results of [1] it follows the existence of strictly singular operator which is not uniformly strictly singular with respect to any basis.

His arguments are following: In [1] was constructed a strictly singular operator T from a space X with a basis $e_k \to 0$ weakly to a space Y with the property:

Whenever (e_{k_i}) are elements of the basis with $n < k_1 < \cdots < k_n$ then $T|_{\mathrm{lin}(e_{k_i})_1^n}$ is isometry.

Now take any other basis (x_k) in X. Fix an n and let $k_1 = n+1$. Approximate e_{k_1} by a finite combination of x_k. Let x_{m_1} be the last vector used to approximate e_{k_1}.

There exists a number k_2 such that none of $x_1, \ldots, x_{m_1}$ has a significant role in the expression of e_{k_2} as a combination of vectors x_k (because $e_k \to 0$ weakly). Approximate e_{k_2} by a finite combination of $x_k,\ k > m_1$. Let x_{m_2} be the last vector used to approximate e_{n_2}.

Continue until n is reached. The sequence $e_{k_1}, \ldots, e_{k_n}$ is almost a block sequence of (x_k). It is also satisfies $n < k_1 < \cdots < k_n$ so T is an isometry on $\mathrm{lin}(e_{k_i})_1^n$. Hence T is not uniformly strictly singular with respect to (x_k).

Obviously, we can take in this argumentation an M-basis instead of basis x_k.

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