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# On a functional-analysis approach to orthogonal sequences problems

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**Abstract.** Let $T$ be a bounded linear injective operator from a Banach space $X$ into a Hilbert space $H$ having dense range and let a sequence $\{x_n\} \subset X$ be such that $\{Tx_n\}$ is orthogonal. We study properties of $\{Tx_n\}$ depending on properties of $\{x_n\}$. We also study the "opposite situation", i.e. the action of an operator $T : H \to X$ onto orthogonal sequences.

### Una aproximación a problemas de sucesiones ortogonales por análisis funcional

**Resumen.** Sea $T$ un operador lineal acotado e inyectivo de un espacio de Banach $X$ en un espacio de Hilbert $H$ con rango denso y sea $\{x_n\} \subset X$ una sucesión tal que $\{Tx_n\}$ es ortogonal. Se estudian propiedades de $\{Tx_n\}$ dependientes de propiedades de $\{x_n\}$. También se estudia la "situación opuesta", es decir la acción de un operador $T : H \to X$ sobre sucesiones ortogonales.

## 1. Introduction

There are numerous investigations in the theory of orthogonal sequences connected with the following problem of Banach ([11], Problem 86):
*Given an orthonormal and uniformly bounded sequence of measurable functions $\{\phi_n(t)\}$, can one always complete it, using functions with the same bound, to a sequence which is orthonormal and complete?. Consider the case when infinitely many functions are necessary for completion.*

This problem was first solved by S. Kaczmarz [7] in 1936. Later other solutions were found (see [8]). In [15] an "abstract" approach to this problem was proposed. Namely, in [15] a couple $X \hookrightarrow H$, where $H$ is a separable Hilbert space, $X$ is a Banach space densely, non-compactly embedded in $H$, was considered. It was proved that then there exists an orthonormal sequence $\{e_n\} \subset H$ such that it is bounded in $X$, admits no extension to a complete orthonormal sequence in $H$, using elements from $X$, and the closed linear span of $\{e_n\}$ in $H$ has an infinite codimension in $H$. This gives, in particular, a negative answer to Banach's question.

We can describe our approach to orthogonal sequences problems (like Banach's problem) in the following way. Let $X$ be a Banach space, $H$ be a separable Hilbert space. Consider a bounded, injective operator $T : X \to H$ having non-closed, dense range $T(X)$ in $H$. Following [4] we call $T$ with these properties "*a dense embedding operator.*" Suppose $\{Tx_n\}$ is an orthonormal sequence in $H$. We study properties of $\{Tx_n\}$ in $H$ depending on properties of $\{x_n\}$ in $X$.

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We also consider the "opposite situation" in the following sense: $T$ is a dense embedding operator acting from a separable Hilbert space $H$ into a Banach space $X$. In the last case we start from the following result in the theory of orthonormal sequences in the couple of spaces $L_2(0,1)\hookrightarrow L_1(0,1)$ (see [19]): There exists an orthonormal sequence which is a (Schauder) basis in $L_1(0,1)$, but not in $L_2(0,1)$.

## 2. Extension of orthonormal sequences

We start with some definitions and notations. Denote by $S_X$ the unite sphere of a Banach space $X$. The distance of an element $x \in X$ to a subset $A \subset X$ will be denoted by $d(x, A)$ ; $[x_n]_1^\infty$ denotes the closed linear span of a sequence $\{x_n\}$. Let $\{e_n\}$ be an orthonormal sequence in a Hilbert space $H$. We say that an orthonormal sequence $\{y_n\}$ is *an extension* of $\{e_n\}$ if there exists an orthonormal sequence $\{z_n\}$ such that

$$\{y_n\} = \{e_n\} \cup \{z_n\} ,$$

The following proposition shows, in particular, that we can find an orthonormal sequence $\{y_n\}$ in $T(X)$ such that it cannot be extended in $H$ to an orthonormal sequence even adding only one element from $T(X)$.

**Proposition 1** *Let $T$ be a dense embedding operator acting from a Banach space $X$ into a separable Hilbert space $H$. Then there exists an orthonormal sequence $\{y_n\} \subset T(X)$ such that $\operatorname{codim}[y_n]_1^\infty = \infty$ and*

$$([y_n]_1^\infty)^\perp \ \cap \ T(X) = \{0\} . \tag{1}$$

PROOF. Let $\{v_n\} \subset T(X)$ be a complete sequence in $H$ such that each element is repeated infinitely many times and let $\epsilon_n \downarrow 0$ . By [14] there exists a closed infinite dimensional subspace $Z$ in $H$ such that $Z \cap T(X) = \{0\}$ . We proceed by induction. For elements $\{z_i, y_i\}_{i=1}^n \subset H$ we will denote $H_n = \operatorname{span}(z_i, y_i)_{i=1}^n$ . On the first step we take $z_1 \in S_Z$ . By Lemma 6 from [15] there exists $y_1 \in T(X) \cap S_{z_1^\perp}$, such that $d(v_1, H_1) < \varepsilon_1$. On the $n$-th step we take $z_n \in S_Z \cap H_{n-1}^\perp$, and by Lemma 6 from [15] choose $y_n \in T(X) \cap H_{n-1}^\perp \cap S_{z_n^\perp}$ such that $d(v_n, H_n) < \varepsilon_n$. So we obtain a complete orthonormal sequence $\{z_n, y_n\}_{n=1}^\infty$ where $z_n \in Z$, $y_n \in T(X)$. By the construction $([y_n]_1^\infty)^\perp = [z_n]_1^\infty \subset Z$. The proof is complete. ■

If $T : X \to H$ is a non-compact dense embedding operator we can ask whether there exists a *bounded* sequence $\{x_n\} \subset X$ such that $y_n = Tx_n$ possesses the properties described in Proposition 1. The following proposition, the proof of which actually coincides with the proof of Proposition 3 from [15], holds.

**Proposition 2** *Let $T$ be a non-compact dense embedding operator acting from a Banach space $X$ into a separable Hilbert space $H$. Then there exists a bounded sequence $\{x_n\} \subset X$ such that $\{Tx_n\}$ is orthonormal, $\operatorname{codim}[Tx_n]_1^\infty = \infty$ and for $y_n = Tx_n$ the property $(1.1)$ holds.*

Now let $\{Tx_n\} \subset T(X)$ be a given orthonormal sequence. How "far" $\{Tx_n\}$ may be extended to an orthonormal sequence in $H$ by using elements from $T(X)$? In order to investigate this problem we put: $Y = [Tx_n]_1^\infty$, $X_0 = T^{-1}(Y^\perp)$ and

$$m = \operatorname{codim}_{Y^\perp} T(X_0) \ , \quad m \in \{0\} \cup \mathbb{N} \cup \{\infty\} . \tag{2}$$

**Theorem 1** *a) If the restriction $T|_{X_0}$ is non-compact then there exists an orthonormal extension of $\{Tx_n\}$ by images $\{z_j\}$ of some bounded sequence in $X$ such that*

$$(Y, \{z_j\})^\perp \ \cap \ T(X) = \{0\} \ \ and \ \ \operatorname{codim}_H [Y, z_j]_1^\infty = m . \tag{3}$$

*b) If $T|_{X_0}$ is compact then it does not exists a complete orthonormal extension of $\{Tx_n\}$ by images of some bounded sequence in $X$.*

PROOF. To prove a) we consider two cases:
1) $m < \infty$. Then the restriction $T|_{X_0}$ is an isomorphism. Let $\{z_j\}$ be any orthonormal basis in $T(X_0)$ (automatically infinite, becouse $T|_{X_0}$ is non-compact). Since $T|_{X_0}$ is an isomorphism, the sequence $\{T^{-1}z_j\}$ is bounded. Next we have $(Y, \{z_j\})^\perp \cap T(X) = (\{z_j\})^\perp \cap T(X_0) = \{0\}$ and $\operatorname{codim}_H [Y, z_j]_1^\infty = \operatorname{codim}_{Y^\perp} T(X_0) = m$, and (3) holds.
2) $m = \infty$. Here we also consider two cases:
(i) $T|_{X_0}$ is an isomorphism. Let $\{z_j\}$ be an orthonormal basis in $T(X_0)$. The condition (3) is verified in the same manner as in the case 1).
ii) $T|_{X_0}$ is not an isomorphism. In view of Proposition 2 there exists a bounded sequence $\{x_j\} \subset X_0$ such that $z_j = Tx_j \in Y^\perp \cap T(X_0)$ is orthonormal in $H$,

$$(\{Tx_j\})^\perp \cap T(X_0) = \{0\} \text{ and } \dim Y^\perp/[z_j]_1^\infty = \infty .$$

It is clear that $\{z_j\}$ is the desired extension.

To prove b) notice that the existence of a bounded sequence $\{x_j\} \subset X_0$ such that $\{Tx_j\}$ is orthonormal, contradicts the compactness of $T|_{X_0}$.
The proof of the theorem is complete. ■

Recall that an operator $T : X \to Y$ is called *strictly singular* if the restriction of $T$ to any infinite-dimensional subspace of $X$ is not an isomorphism.

**Corollary 1** *Let $T : X \to H$ be a strictly singular operator. Suppose that $\{x_n\}$ is a bounded sequence in $X$ such that $\{Tx_n\}$ is orthonormal and $T|_{T^{-1}(\{Tx_n\}^\perp)}$ is non-compact. Then there exists an orthonormal extension $\{Tx_n\} \cup \{Ty_n\}$ of $\{Tx_n\}$ with $\dim(\{Tx_n\} \cup \{Ty_n\})^\perp = \infty$ , $\sup \|y_n\| < \infty$ and such that it has no extension to an orthonormal basis in the whole space $H$ by elements of $T(X)$.*

PROOF. Put $Y = [Tx_n]_1^\infty$ , $X_0 = T^{-1}(Y^\perp)$. Since $T$ is strictly singular, $\operatorname{codim}_{Y^\perp} T(X_0) = \infty$ and the application of part (a) of Theorem 1 concludes the proof. ■

A classical example of strictly singular operator is the natural embedding

$$C[a, b] \hookrightarrow L_2(a, b) .$$

Thus by Corollary 1 we have

**Corollary 2** *Let $\{e_n(t)\}$ be the normalized trigonometric sequence in $L_2(0, 2\pi)$ and let $\{u_n\}$ be any subsequences whith infinite complementation. Then there exists an orthonormal extension $\{u_n\}_1^\infty \cup \{y_n\}_1^\infty$ such that $\dim(\{u_n\} \cup \{y_n\})^\perp = \infty$, every function $y_n(t)$ is continuous, $\sup_n \max_{t\in[0,2\pi]} |y_n(t)| < \infty$ and there is no any orthonormal extension of $\{u_n\} \cup \{y_n\}$ by using elements of $C[0, 2\pi]$.*

## 3. Disjointness of orthonormal sequences and operator ranges

Let $T : X \to H$ be a dense embedding operator and let $\{e_n\} \subset H$ be an orthonormal sequence in $H$. We say that $\{e_n\}$ and $T(X)$ are *disjoint* if

$$[e_n]_1^\infty \cap T(X) = \{0\} . \tag{4}$$

The disjointness of an orthonormal sequence and the operator range is not a rare property (see [1], where some examples are given). We ask about a possibility of extending $\{e_n\}$ with property (4) keeping this property for the extended sequence.

**Theorem 2** *Let $\{e_n\}$ be an orthonormal sequence with the property $(4)$. Then there is an orthonormal basis $\{x_n\}$ of $H$ such that*
*(i)* $\{x_{2n}\} \supset \{e_n\}$ , $\operatorname{card}(\{x_{2n}\}\backslash\{e_n\}) = \infty$ ,
*(ii)* $[x_{2n-1}]_1^\infty \cap T(X) = [x_{2n}]_1^\infty \cap T(X) = \{0\}$ .

This theorem follows from Corollary 3.4 of [3] (for a similar but weaker result see [2]). We give a proof for the sake of completeness. The proof is based on

**Lemma 1** *Let $U$ be a linear subspace of a Hilbert space $H$ such that: $U \setminus \{0\} = \cup_{n=1}^{\infty} U_n$, where $U_n$ are convex closed bounded sets and there exists a countable dimensional dense in $H$ subspace $Z$ for which $Z \cap U = \{0\}$. Then there are closed infinite dimensional subspaces $V, W$ in H so that $V \oplus W = H$, and $V \cap U = W \cap U = \{0\}$.*

PROOF. Let $\{z_n\}$ be an algebraic basis of $Z$. We construct by induction sequences $\{v_n\}$ and $\{w_n\}$ in $Z$ so that for every $n$:

$1^\circ$. $z_n \in Z_{2n-1}^{n-1}$ and $v_{2n-1} \perp Z_{2n-2}^{n-1}$ , where $Z_k^l = \operatorname{span}(v_1, \ldots, v_k, w_1, \ldots, w_l)$.

$2^\circ$. $w_n \perp Z_{2n-1}^{n-1}$ and $\langle w_n, U_n \rangle > 0$, where the last formula denote $\langle w_n, u \rangle > 0 \ \forall u \in U_n$.

$3^\circ$. $v_{2n} \perp Z_{2n-1}^{n}$ and $\langle v_{2n}, U_n \rangle > 0$,

On the first step we:

1. Put $v_1 = z_1$.
2. Choose, by Hahn-Banach theorem, an element $w_1 \in Z$ such that $w_1 \perp v_1$ and $\langle w_1, U_1 \rangle > 0$.
3. Since $Z_1^1 \cap U_1 = \varnothing$, there exists an element $v_2 \in Z$ such that $v_2 \perp Z_1^1$ and $\langle v_2, U_1 \rangle > 0$.

Let the collection $(v_1, \ldots, v_{2n}, w_1, \ldots, w_n)$ with the properties $1^\circ - 3^\circ$ is constructed. Then on the $n+1$ step we

1. Choose an element $v_{2n+1} \in Z$ such that $z_{n+1} \in Z_{2n+1}^{n}$ and $v_{2n+1} \perp Z_{2n}^{n}$.
2. Choose an element $w_{n+1} \in Z$ such that $w_{n+1} \perp Z_{2n+1}^{n}$ and $\langle w_{n+1}, U_{n+1} \rangle > 0$.
3. Choose an element $v_{2n+2} \in Z$ such that $z_{2n+2} \perp Z_{2n+1}^{n+1}$ and $\langle v_{2n+2}, U_{n+1} \rangle > 0$.

Put $V = [v_n]_1^\infty$ and $W = [w_n]_1^\infty$. By the construction, all $v_n$ and $w_n$ are orthogonal, hence $V \perp W$. By $1^\circ$ , $Z \subset U + V$, hence $V + W = H$. The item $2^\circ$ guarantees that $V \cap U_n = \varnothing$; the item $3^\circ$ guarantees that $W \cap U_n = \varnothing$ for any $n$.

PROOF OF THEOREM 2. Let $P$ be the orthogonal projection of $H$ onto $(\{e_n\})^\perp$. Then the range $U = PT(X)$ satisfies all conditions of Lemma 2.2. (see [14]). Hence there exist (infinite dimensional) subspaces $V$ and $W$ such that $V \oplus W = (\{e_n\})^\perp$ and $V \cap U = W \cap U = \{0\}$. It is easy to check that $V \cap T(X) = \{0\}$ and $(W \oplus [e_n]_1^\infty) \cap T(X) = \{0\}$. The rest of the proof is clear. ■

**Example 1** Let $0 < a_1 < a_2 < \ldots < 1$. Put

$$e_n(t) = \begin{cases} (a_{n+1} - a_n)^{-1/2} & t \in [a_n, a_{n+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $\{e_n\}$ is an orthonormal sequence in $L_2(0,1)$ and

$$[e_n]_1^\infty \ \cap \ C[0,1] = \{0\} \ .$$

So we can apply Theorem 2. It follows that there exists an orthonormal basis $\{x_n\}$ of $L_2(0,1)$ such that

$$\{x_{2n}\} \supset \{e_n\} \quad \text{and} \quad [x_{2n-1}]_1^\infty \cap C[0,1] = [x_{2n}]_1^\infty \cap C[0,1] = \{0\} \ .$$

**Example 2** Take $H = l_2$. Let $\{i_n\}$ be the sequence of prime numbers and $\sum (a_m)^2 = 1$ , $\sum a_m = \infty$. Put $e_n = (0 \ldots 0, a_1, 0 \ldots 0, a_2, 0 \ldots)$ where $a_1$ is located at the $i_n$-th place, $a_2$ is located at the $(i_n)^2$-th place,..., $a_m$ is located at the $(i_n)^m$-th place, and so on.

It is easy to verify that $[e_n]_1^\infty \cap l_1 = \{0\}$ . Therefore we can apply Theorem 2. It follows that there exists an orthonormal basis $\{x_n\}$ of $l_2$ such that

$$\{x_{2n}\} \supset \{e_n\} \quad \text{and} \quad [x_{2n-1}]_1^\infty \cap l_1 = [x_{2n}]_1^\infty \cap l_1 = \{0\} \ .$$

## 4. The image of orthonormal sequence under the action of dense embedding operator

In this section we prove a generalization of the main result from [16].

**Theorem 3** *Let $T$ be a dense embedding operator acting from a separable Hilbert space $H$ into a Banach space $X$. Let $\{x_n\} \subset X$ be an arbitrary sequence and $\{\varepsilon_n\}$ be arbitrary positive sequence numbers. Then there exists an orthogonal sequence $\{e_n\} \subset H$ such that*

$$\|x_n - Te_n\| < \varepsilon_n \quad and \quad \operatorname{codim}[e_n]_1^\infty = \infty \; .$$

PROOF. We denote by $\mathcal{J}$ the canonical isometry of $H$ onto $H^*$, i.e. $(\mathcal{J}x)(y) = \langle x, y\rangle$. It is easy to see that the adjoint operator $T^*$ is a dense embedding from $X^*$ into $H^*$. We construct sequences $\{e_n\} \subset H$ , $\{f_n\} \subset H^*$ such that for every $n$

(i) $f_i(e_j) = 0 \;\; i, j = 1, \ldots, n \;, \; f_n \notin \operatorname{span}(T^*(X^*); \{\mathcal{J}e_i \;, \; f_i\}_1^{n-1})$ ;

(ii) $\langle e_n, e_i\rangle = 0$ for $i < n$ ; $\;\mathcal{J}e_n \notin T^*(X^*) + F_n$ ,
where $F_n = \operatorname{span}(\{\mathcal{J}e_i\}_1^{n-1}, \{f_i\}_1^n)$ ;

(iii) $\|x_n - Te_n\| < \varepsilon_n$ .

To construct these sequences we use an inductive process. Choose $f_1 \in H^* \setminus T^*(X^*)$. Then the linear subspace $T\mathcal{J}^{-1}(f_1^\perp)$ is dense in $X$. It follows that there exists $e_1 \in \mathcal{J}^{-1}(f_1^\perp)$, $\mathcal{J}e_1 \notin T(X^*) + F_1$ for which

$$\|x_1 - Te_1\| < \varepsilon_1 \; .$$

It is easy now to check that properties (i) – (iii) hold.

Now suppose that we have constructed $\{e_i, f_i\}_1^{n-1}$ with properties (i) – (iii). Since $\dim H^*/T^*(X^*) = \infty$ we can choose an element $f_n \in (\{\mathcal{J}e_i\}_1^{n-1})^\perp \subset H^*$, $f_n \notin \operatorname{span}(T^*(X^*), \{\mathcal{J}e_i, f_i\}_1^{n-1})$. Because $F_n \cap T^*(X^*) = \{0\}$, the linear subspace $T\mathcal{J}(F_n^\perp)$ is dense in $X$. There exists $e_n \in \mathcal{J}(F_n^\perp)$, $\mathcal{J}e_n \notin T(X^*) + F_n$ such that

$$\|x_n - Te_n\| < \varepsilon_n \; .$$

We only need to check condition (ii). Since $e_n \in \mathcal{J}(F_n^\perp)$,

$$\langle e_n, e_i\rangle = \mathcal{J}e_i(e_n) = 0 \quad \text{for} \quad i < n \; .$$

This means that (ii) holds.

By the construction, $\{e_n\}$ is orthogonal. In view of (i) $\{e_n\} \subset (\{\mathcal{J}(f_n)\})^\perp$. It follows from (ii) that $\dim[f_n]_1^\infty = \infty$. So we have $\operatorname{codim}[e_n]_1^\infty = \infty$ and the proof is complete. ∎

Recall that a complete minimal sequence $\{x_n\} \subset X$ with the biorthogonal functionals $\{f_n\} \subset X^*$ is called *an $M$-basis* of a Banach space $X$ if $\{f_n\}$ is total over $X$ (i.e. for each $x \in X, \; x \neq 0$ there is an $n$ such that $f_n(x) \neq 0$).

It is well-known that each separable Banach space has an $M$-basis and that each $M$-basis has *a stability subsequence* i.e. positive sequence $\{\varepsilon_n\}$ so that $\|x_n - y_n\| < \varepsilon_n$ implies that $\{y_n\}$ is an $M$-basis in $X$ too. From Theorem 3 it follows

**Corollary 3** *Suppose that $T : H \to X$ satisfies the conditions of Theorem 3.1. There exists an orthonormal sequence $\{e_n\} \subset H$ such that $\{Te_n\}$ is an $M$-basis of $X$ and $\operatorname{codim}[e_n]_1^\infty = \infty$.*

PROOF. To use Theorem 3 we can take any $M$-basis $\{x_n\} \subset X$ and its stability sequence $\{\varepsilon_n\}$ . ∎

**Corollary 4 ([16])** *Suppose $T : H \to X$ satisfies the conditions of the Theorem 3.1. If $X$ has a basis then there exists an orthonormal sequence $\{e_n\} \subset H$ such that $\{Te_n\}$ is a basis of $X$ and $\operatorname{codim}[e_n]_1^\infty = \infty$.*

Recall that a sequence $\{x_n\}$ in a Banach space $X$ is said to be *a deficiently minimal* if it becomes minimal after deleting some finite set of elements.

**Corollary 5** *Let $T : H \to X$ satisfies the conditions of Theorem 3.1. Then there exists an orthonormal sequence $\{e_n\} \subset H$ such that $\{Te_n\}$ is not deficiently minimal in $X$ and $\operatorname{codim}[e_n]_1^\infty = \infty$ .*

PROOF. It is proved in [4] that there is a decomposition $H = H_1 \oplus H_2$ such that $T|_{H_1} : H_1 \to X$ and $T|_{H_2} : H_2 \to X$, are dense embedding operators.

By using Corollary 3 find orthonormal sequences $\{u_n\} \subset H_1$ and $\{v_n\} \subset H_2$ possessing the properties
(i) $\{Tu_n\}$ and $\{Tv_n\}$ are $M$-bases in $X$ ,
(ii) $\operatorname{codim}_{H_1}[u_n]_1^\infty = \infty$.

Put $\{e_n\} = \{u_n\} \cup \{v_n\}$. It is easy to see that $\{e_n\}$ satisfies the conditions of corollary. ■

**Remark 1** It is interesting to compare Corollary 5 with the following result [13]:
*Let $T$ be a bounded linear operator acting from a separable Banach space $Z$ into a separable Banach space $X$ with $\dim \operatorname{Ker} T = \infty$. There is an $M$-basis in $Z$ such that its image by $T$ is an overcomplete sequence in $X$ (i.e. each its subsequence is complete in $X$).*

Now we show how to apply Theorem 3 for the construction of special orthonormal sequences in the whole scale $L_p = L_p(0, 1)$ , $1 \le p \le 2$ (see [17]).

**Corollary 6** *Let $1 \le r < 2$ be a fixed number. There exists an orthonormal sequence $\{e_n(t)\} \subset L_2$ which is a basis in every space $L_p$ , $1 \le p \le r$ but it is not complete in $L_p$ if $r < p \le 2$.*

PROOF. From [2] it follows that there exists a function Banach space $Y$ (depending on $r$) such that:
1)$L_p$ is densely embedded (by the identity embedding) in $Y$ if $r < p \le 2$.
2)$Y$ is densely embedded (by the identity embedding) in $L_r$.

By using a method of [4] we construct a subspace $N \subset L_2$ which is dense in $L_r$ and whose closure in $Y$ has infinite codimension (in $Y$).

Now let $\{x_n\}$ be a sequence which is a basis simultaneously in all $L_p$ (take for example the Haar system) and let $\{\varepsilon_n\}$ be a stability sequence of $\{x_n\}$ in all spaces $L_p$, $1 \le p < r$. In view of Theorem 3 there exists an orthonormal sequence $\{e_n\} \subset N$ such that $\|x_n - e_n\|_{L_r} \le \varepsilon_n$ . Thus $\{e_n\}$ is a basis in every space $L_p$, $1 \le p \le r$. Clearly, $\{e_n\}$ is incomplete in $Y$, hence in any $L_p$ for $r < p \le 2$.

## 5. Properties of the sequence $\{Te_n\}$

Let $T$ be a dense embedding operator acting from a separable Hilbert space $H$ into a Banach space $X$. Then $T^* : X^* \to H^*$ is a dense embedding operator too. Take a complete sequence $\{\phi_n\}$ in $H^*$ such that $\{\phi_n\} \subset T^*(X^*)$. By using Gram-Schmidt's orthogonalisation process pass from $\{\phi_n\}$ to the orthonormal basis $\{g_n\} \subset T^*(X^*)$. Let $\mathcal{J}$ be the canonical isometry which maps $H$ onto $H^*$ as follows:

$$\mathcal{J}y(x) = \langle y, x\rangle \ , \ x, y \in H \ . \tag{5}$$

Put $e_n = \mathcal{J}^{-1}(g_n)$ , $n = 1, 2, \ldots$. We claim that $\{Te_n\}$ is a complete minimal sequence in $X$. The completeness of $\{Te_n\}$ is clear. Check the minimality: $T^{*-1}g_n(Te_m) = (T^*T^{*-1}g_n)(e_m) = g_n(e_m) = g_n(\mathcal{J}g_m) = \langle g_n, g_m\rangle = \delta_{nm}$ .

Thus we have found an orthonormal basis $\{e_n\}$ in $H$ such that $\{Te_n\}$ is a complete minimal sequence in $X$. Our aim is to construct $\{e_n\}$ such that $\{Te_n\}$ would have a stronger property. Namely $\{Te_n\}$ is an $M$-basis in $X$. Let us consider the following problems:
(i) Does there exist for every $\delta > 0$ an orthonormal basis $\{e_n\}$ in $H$ such that $\{Te_n\}$ is an $M$-basis in $X$ , $\|Te_n\| < \delta$ and

$$\lim_{n\to\infty} \|Te_n\| = 0? \tag{6}$$

(ii) Suppose that $T$ is non-compact. Does there exist an orthonormal basis $\{e_n\}$ in $H$ such that $\{Te_n\}$ is an $M$-basis in $X$ and

$$\inf_n \|Te_n\| > 0?$$

Let us notice that similar problems to (i) and (ii) were investigated in [12], [6], [5] and [18].

The problems similar to (i) for operators in a Hilbert space, were investigated by L. Gurvits in eighties. As far as we know those results of L. Gurvits were not published.

Actually we will investigate (i), (ii) in an even more general form.

Let us recall that an operator $T$ acting from a Banach space $X$ into a Banach space $Y$ is said to be *the* $\Phi_+$ *operator* if the image $T(X)$ is a closed subspace in $Y$ and $\dim(\operatorname{Ker} T) < \infty$. It is well known that $T$ is a $\Phi_+$ operator if and only if its restriction to some finite codimensional subspace is an isomorphism.

**Lemma 2** *Let $T$ be a non-$\Phi_+$ operator acting from a separable Hilbert space $H$ into a Banach space $X$. Suppose $\{y_n\}$ is a complete sequence in $H$. Then for every positive $\{\varepsilon_n\}$ there exists an orthonormal basis $\{z_n\}$ in $H$ such that $\operatorname{span}\{z_n\} = \operatorname{span}\{y_n\}$ and for every $n$*

$$||Tz_{2n-1}|| < \varepsilon_n \ .$$

PROOF. Put $L = \operatorname{span}\{y_n\}$. We construct $\{z_n\}$ using an inductive process. Since $T$ is a non-$\Phi_+$ operator it follows that there exists a $z_1 \in S_L$ such that $\|Tz_1\| < \varepsilon_1$. There are two possible cases:
a) $y_1 \notin \operatorname{span} z_1$. We orthogonalize $z_1$, $y_1$ and obtain $z_2$.
b) $y_1 \in \operatorname{span} z_1$. Take an arbitrary element $z_2 \in z_1^\perp \cap S_L$.

On the next step we choose $z_3 \in (z_1, z_2)^\perp \cap S_L$ such that $\|Tz_3\| < \varepsilon_2$. If $y_2 \notin \operatorname{span}(z_1, z_2, z_3)$ we orthogonalize $(z_1, z_2, z_3, y_2)$ and obtain $z_4$. If $y_2 \in \operatorname{span}(z_1, z_2, z_3)$ we take an arbitrary element $z_4$ in $(z_1, z_2, z_3)^\perp \cap S_L$. Continuing in the same way we construct $\{z_n\}$ with desired properties. ■

**Proposition 3** *Let $T$ be a non-$\Phi_+$ operator acting from a separable Hilbert space $H$ into a Banach space $X$. Then for every $\varepsilon > 0$ there exists an orthonormal basis $\{e_n\} \subset H$ such that for every $n$*

$$||Te_n|| < \varepsilon \ \ and \ \ \lim_{n\to\infty} \|Te_n\| = 0 \ . \tag{7}$$

PROOF. We use the sequence $\{z_n\}$ from Lemma 2. Let $\{u_n\}$ be an orthonormal basis in $H$ obtained as a permutation of $\{z_n\}$ by the following way: $u_{2^k} = z_{2k}$ , $\forall k$ ; next we write $\mathbb{N} \setminus \{2^k\} = \{n_k\}$ ($n_1 < n_2 < \ldots$) and put $u_{n_k} = z_{2k-1}$ ($k = 1, 2, \ldots$). Construct an orthonormal basis $\{e_n\}$ in $H$ as follows: put $e_1 = u_1$ , $e_2 = u_2, \ldots, e_n = \sum_{2^{k-1}+1}^{2^k} a_{ni} u_i$ , $2^{k-1} < n \le 2^k$, $k = 2, 3, \ldots$, where $(a_{ni})_{n,i=2^{k-1}+1}^{2^k}$ is the unitary Olevskii's matrix (see [10], p.45). In particular the following property holds

$$\max_{2^{k-1}<n\le 2^k} \|Te_n\| \le (1+\sqrt{2}) \max_{2^{k-1}<i\le 2^k} \|Tu_i\| + 2^{-\frac{k-1}{2}} \|Tu_{2^k}\|.$$

It is clear that for sufficiently small $\{\varepsilon_n\}$ the sequence $\{e_n\}$ has the desired properties. The proof is complete. ■

It turns out that for a dense embedding operator $T$ an orthonormal basis $\{e_n\}$ can be chosen in such manner that $\{Te_n\}$ possesses minimality properties and (7) holds.

Recall that an $M$-basis $\{x_n\} \subset X$ with the biorthogonal sequence $\{f_n\} \subset X^*$ is called *1-norming* if the subspace $\operatorname{span}\{f_n\} \subset X^*$ is 1-norming, i.e. for every $x \in X$

$$||x|| = \sup\{f(x) : \ f \in \operatorname{span}\{f_n\}, \ ||f|| = 1\} \ .$$

**Theorem 4** *Let $T$ be a dense embedding operator acting from a separable Hilbert space $H$ into a Banach space $X$ and $\varepsilon > 0$. Then there exists an orthonormal basis $\{e_n\} \subset H$ such that $\{Te_n\}$ is 1-norming $M$-basis in $X$ ,*

$$||Te_n|| < \varepsilon \ \ and \ \ \|Te_n\| \to 0 \ \ as \ \ n \to \infty \ .$$

PROOF. First we construct an orthonormal basis $\{z_n\}$ in $H$ such that $\{Tz_n\}$ is a 1-norming $M$-basis in $X$ and $\lim_{n\to\infty} \|Tz_{2n-1}\| = 0$. For this aim we choose $\{f_n\} \subset X^*$ such that $\mathrm{span}\,\{f_n\}$ be a 1-norming subspace in $X^*$. Let $\mathcal{J} : H \to H^*$ be a canonical map: for $x, z \in H$, $\langle \mathcal{J}x, z\rangle = \langle x, z\rangle$. Put $y_n = \mathcal{J}^{-1}T^* f_n$. Apply Lemma 2 with the complete sequence $\{y_n\}$ and construct an orthonormal basis $\{z_n\}$ in $H$ such that $\mathrm{span}\,\{z_n\} = \mathrm{span}\,\{y_n\}$ and $||Tz_{2n-1}|| < \varepsilon_n$. Check that $\{Tz_n\}$ is a 1-norming $M$-basis in $X$.
1) Clearly, $\{Tz_n\}$ is complete in $X$.
2) Put $g_n = (T^*)^{-1}\mathcal{J}z_n$. Then

$$\langle Tz_n, g_m\rangle = \langle Tz_n, (T^*)^{-1}\mathcal{J}z_m\rangle = \langle z_n, T^*(T^*)^{-1}\mathcal{J}z_m\rangle = \\ \langle z_n, \mathcal{J}z_m\rangle = \langle z_n, z_m\rangle = \delta_{nm}.$$

3) Since $\mathrm{span}\,\{z_n\} = \mathrm{span}\,\{y_n\}$ we have $\mathrm{span}\,\{\mathcal{J}z_n\} = \mathrm{span}\,\{\mathcal{J}y_n\}$. It follows that $\mathrm{span}\,\{(T^*)^{-1}\mathcal{J}z_n\} = \mathrm{span}\,\{(T^*)^{-1}\mathcal{J}y_n\}$. But $(T^*)^{-1}\mathcal{J}z_n = g_n$ and $(T^*)^{-1}\mathcal{J}y_n = f_n$. Therefore $\mathrm{span}\,\{y_n\} = \mathrm{span}\,\{f_n\}$ and $\{Tz_n\}$ is 1-norming $M$-basis.

To finish the proof we apply the method of construction of the orthonormal basis $\{e_n\}$ from Proposition 3. The orthonormal basis $\{e_n\}$ has all desired properties. The proof is complete. ■

In order to investigate problem (ii) we make use of the measure of non-compactness (see [9]). Let $T$ be an operator acting from a Banach space $X$ into a Banach space $Y$. Define $C(T) = \inf\{\|T|_M\| \ : \ M$ is a finite codimensional subspace in $X\}$. It is known (see [9]) that $T$ is compact if and only if $C(T) = 0$.

**Lemma 3** *Let $T$ be a non-compact operator acting from a separable Hilbert space $H$ into a Banach space $X$. Suppose that $\{x_n\}$ is a complete sequence in $H$. Then there exists an orthonormal basis $\{z_n\}$ in $H$ such that* $\mathrm{span}\,\{z_n\} = \mathrm{span}\,\{y_n\}$ *and*

$$\inf_n \|Tz_{2n-1}\| > 0\ .$$

PROOF. We use an inductive process. Put $L = \mathrm{span}\,\{y_n\}$. On the first step we take $z_1 \in S_L$ such that $\|Tz_1\| > \frac{C(T)}{2}$. As it was done in the proof of Lemma 2 we consider two cases:
a)If $y_2 \notin \mathrm{span}\, z_1$ we orthogonalize $z_1, y_2$ and obtain $z_2$.
b)If $y_2 \in \mathrm{span}\, z_1$ we choose an arbitrary element $z_2 \in z_1^\perp \cap S_L$.
On the second step we take $z_3 \in (z_1, z_2)^\perp \cap S_L$ such that $\|Tz_3\| > \frac{C(T)}{2}$.

We continue the construction in the same manner as it was done in the proof of Lemma 2. The proof is complete. ■

Now we are ready to prove the following

**Theorem 5** *Let $T$ be a non-compact dense embedding operator acting from a separable Hilbert space $H$ into a Banach space $X$. Then there exists an orthonormal basis $\{e_n\}$ in $H$ such that $\{Te_n\}$ is 1-norming $M$-basis in $X$ and*

$$\inf_n \|Te_n\| > 0\ .$$

PROOF. By using Lemma 3 and the method of the proof of Theorem 4 construct an orthonormal basis $\{z_n\}$ in $H$ such that $\{Tz_n\}$ is a 1-norming $M$-basis in $X$ , and

$$\inf_n \|Tz_{2n-1}\| = \delta > 0\ .$$

Enumerate the sequence $\{z_{2n}\}$ in such a way that $\{z_{2n}\} = \{u_n\} \cup \{v_n\}$ , $\|Tu_n\| < \delta/4$ and $\|Tv_n\| \geq \delta/4$. Put

$$\{e_n\} = \{v_n\} \cup \{(z_{2n-1} - u_n)/2\} \cup \{(z_{2n-1} + u_n)/2\}\ .$$

It is clear that $\|Te_n\| > \delta/4$ and that $\{e_n\}$ has all desired properties.

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