Isotropic mappings and automatic continuity of polynomials, analytic and convex operators

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In this paper we study a large class of nonlinear mappings satisfying the Baire mapping and Borel graph theorems, which are well known to hold for linear operators in Banach spaces. In particular, this class includes polynomial operators on abelian groups and convex operators on Banach lattices. We also consider the automatic continuity of analytic operators.

1. Introduction

In the paper [16] we introduced the notion of isotropic mapping and studied some its properties. In particular, we showed that for isotropic mappings the theorems on Baire mapping and Borel graph are valid. The main examples of isotropic mappings were polynomial operators on Banach spaces. In this paper we consider a somewhat weaker notion of a pointwise isotropic mapping and investigate its properties. Main examples of pointwise isotropic mapping will be convex mappings of Banach lattices and polynomial operators on metric abelian groups, in particular, on linear metric spaces. Since we do not know the isotropy of analytic mappings in complex Banach spaces, the automatic continuity (more precisely, the fulfillment of Baire mapping theorem and Borel graph theorem) of such mappings we will investigated separately.

Let us indicate some papers on this topic. Polynomial operators on abelian groups were studied by Van der Lijn [18], [19]. A generalization of the Banach theorem on Baire mapping to analytic mappings was given by Zorn [21]. We will use essentially Zorn's results [21], [22] for study of analytic operators. The Closed Graph theorem for polynomial and even for analytic functionals in Fréchet spaces was proved by Drużkowski [4]. We will show that for Banach spaces this theorem can easily be deduced from the above mentioned Zorn's result on Baire mapping. Automatic continuity of homomorphisms of abelian groups and convex and polynomial operators in linear metric spaces in term of Christensen measurability was investigated by Fischer and Słodkowski [8] and Gajda [9]. We refer to [3] for basic results on analytic operators on Banach spaces.

2. Pointwise isotropic mappings

Definition 1. *Let X be a metric group (not necessarily abelian) with a shift invariant metric and a group operation "+" and let Y be a metric space. We call a mapping F from an open subset $D \subset X$ into Y* isotropic on D *if either it is continuous on D or there exists a sequence $x_i \in X$, $x_i \to 0$ such that for some number $c > 0$*

$$\overline{\lim_i}\, \operatorname{dist}(F(x + x_i), F(x)) \geq c \tag{2.1}$$

for each $x \in D$, where $\overline{\lim}$ is taken over all i such that $x + x_i \in D$ and dist *denotes the distance in Y.*

We call the maximal number c for which (2.1) is true, where the maximum is taken over all sequences $x_i \to 0$, *the isotropy constant of* F *on* D; it can be equal to ∞ also.

If $D = X$, we call the mapping F simply *isotropic* [16].

We assume the isotropy constant of a continuous on D mapping to equal zero. Conversely, when we speak about isotropic mappings with zero isotropy constant we mean everywhere continuous on D mappings.

Definition 2. *One say that a group of homeomorphisms G of metric space X* acts transitively *if for any elements $x, y \in X$ there exists a homeomorphism $g \in G$ such that $g(x) = y$.*

Definition 3. *Let X, Y be metric spaces and a group of homeomorphisms G acts transitively on X. We call a mapping F from an open subset $D \subset X$ into Y* pointwise isotropic on D with respect to the group G *if either it is continuous on D or there is a sequence $x_i \in X$, $x_i \to x_0$, such that*

$$\overline{\lim_i} \operatorname{dist}(F(g(x_i)), F(g(x_0))) > 0$$

for every $g \in G$ such that $g(x_0) \in D$.

We gave the general definition, but in this paper we will only consider metric groups X. Then we will consider G to be the group of left shifts (we could also take the right shifts). Namely we give

Definition 3′. *Let X be a metric group and Y be a metric space. We call a mapping F from an open subset $D \subset X$ to Y* pointwise isotropic on D *if either it is continuous on D or there is a sequence $x_i \in X$, $x_i \to 0$, such that*

$$\overline{\lim_i} \operatorname{dist}(F(x + x_i), F(x)) > 0$$

for every $x \in D$.

If $D = X$, we call the mapping F simply pointwise isotropic .

A linear space is considered as a group with respect to addition. We use, as in [1], the symbol "+" for the group operation in X, since we will use this definition to linear metric spaces. We consider a subset D, since nonlinear mappings are often not defined on the whole space.

We showed in [16] that every polynomial operator in a Banach space is isotropic with isotropy constant equal to 0 or ∞. We also showed that for every $c > 0$ there exists an isotropic mapping with isotropy constant equal to c. Let us give an example of pointwise isotropic but not isotropic mapping.

Example. Let X be a normed space considered as a group over addition and $S_n = \{x \in X : n - 1 \le ||x|| < n\}$, $n = 1, 2, \ldots$. Let us introduce a new metric on X by

$$d(x, y) = \begin{cases} \max(\frac{1}{n} : x \text{ or } y \text{ belongs to } S_n) & \text{if } x \ne y, \\ 0 & \text{if } x = y. \end{cases}$$

It is easy to see that d is a discrete metric on X (but not shift invariant). Denote by F the identical mapping from $(X, ||\ ||)$ onto (X, d). Then for any sequence $x_i \in X$, $||x_i|| \to 0$, $x_i \ne 0$ and $x \in S_n$

$$\frac{1}{n} \le d(x + x_i, x) \le \frac{1}{n-1}$$

for sufficiently large i.

Proposition 1. *Let $F : D \to Y$ be an isotropic (pointwise isotropic) mapping.*

a) *If X is a metric group and $x_0 \in X$, then $F(x+x_0)$ is isotropic (pointwise isotropic) on $D-x_0$.*
b) *If, moreover, Y is a metric group with a shift invariant metric and $y_0 \in Y$ then $F(x)+y_0$ is isotropic (pointwise isotropic) on D.*
c) *If Y is a linear metric space with a shift invariant metric and $a \in \mathbb{R}$ (or $\mathbb{C}$), then $aF(x)$ is isotropic (pointwise isotropic) on D.*

The proof is simple.

Now we give main properties of pointwise isotropic mappings connected with continuity (some similar properties for isotropic mappings are given in [16]).

A subset M of linear space X is called M-O-set if the space is not the union $\cup_1^\infty M_n$ of a sequence of sets $M_n = a_n M + x_n$, where $x_n \in X$ and (a_n) are scalars. This notion was used by Mazur and Orlicz [13] and Zorn [21] for study of polynomial and analytic operators in Banach spaces. By analogy with the notion of M-O-set in a linear space we introduce the following

DEFINITION 4. *Let D be a subset of a group X. We define a subset $M \subset X$ to be a Z-*set with respect to D *if for every sequence $x_i \in X$*

$$\bigcup_{i=1}^{\infty}(M+x_i) \not\supset D.$$

If $D = X$, we will say M to be a Z-set.

A subset of Z-set is a Z-set, but nothing is maintained about finite or countable unions of Z-sets. Of course, every M-O-set of a linear space is a Z-set. The opposite statement is not true. For example, the unit ball of a nonseparable Banach space is a Z-set but is not an M-O-set. One can give a nonconstructive example of a Z-set M which is not an M-O-set in an infinite dimensional separable Banach space too. It is sufficient to consider a normed Hamel basis and take as M its convex hull.

Question 1. Is there a separable Banach space and a Borel Z-set which is not a M-O-set? In particular, is such a set in $\mathbb{R}$?

Theorem 1. *Let F be a pointwise isotropic mapping from an open subset D of a metric group X into a metric space Y and M be a Z-set with respect to D. If F is continuous on $D\backslash M$, then it is continuous on D.*

Proof. Suppose that F is discontinuous on D. Let (x_i) be a sequence from Definition $3'$. Then for $\tilde{M} = X\backslash M$ the set

$$N = \tilde{M} \cap \Big(\bigcap_{i=1}^{\infty} (\tilde{M} - x_i) \Big)$$

intersects D. Let $x_0 \in N \cap D$. Then $x_0 \in \tilde{M} \cap D$ and $x_0 \in \tilde{M} - x_i$ i.e. $x_0 + x_i \in \tilde{M}$. Since $x_i \to 0$ and D is open, $x_0 + x_i \in \tilde{M} \cap D$ beginning with some index i_0. Since F is pointwise isotropic on D,

$$F(x_0 + x_i) \not\to F(x_0).$$

Thus F is discontinuous on $\tilde{M} \cap D = D\backslash M$. □

Theorem $1'$. *Let F be an isotropic mapping from an open subset D of metric group X into a metric space Y. If the isotropy constant of F equals to infinity and M is a Z-set with respect to D then the restriction of $F|_{D\backslash M}$ is unbounded at some point from $D\backslash M$.*

Proof. The proof is similar to the proof of Theorem 1. Let x_i be a sequence from Definition 1. Then for $\tilde{M} = X\backslash M$ the set $N = \tilde{M} \bigcap(\cap_{i=1}^{\infty}(\tilde{M} - x_i))$ intersects D. Let $x_0 \in N \cap D$. Then $x_0 \in \tilde{M} \cap D$ and $x_0 \in \tilde{M} - x_i$, i.e. $x_0 + x_i \in \tilde{M}$. Beginning with some index i_0, $x + x_i \in D$. By the assumption of the theorem

$$\overline{\lim_i} \operatorname{dist}(F(x_0 + x_i), F(x_0)) = \infty,$$

which proves the theorem. □

A simple example of a Z-set is a meager subset of complete metric group, i.e., a countable union of nowhere dense sets. Since the union of countable many meager sets in a complete metric space is a meager set, any meager set in a complete metric group is a Z-set with respect to every open set D.

Definition 5 [21]. *A mapping F from a metric space X into a metric space Y is called B-*continuous *if there exists a meager subset $M \subset X$ such that $F|_{X\backslash M}$ is continuous.*

The following theorem is classical.

Theorem B [1, I.§3]. *An additive B-continuous operator from a complete metric group X into a metric group Y is continuous.*

Evidently, an additive operator is isotropic. So Theorem 1 is a generalization of Theorem B.

Let us note a connection between B-continuity and other similar notions. A mapping F has the *Baire property* (in the broad sense) if for every open subset $U \subset Y$

$$F^{-1}(U) = (A\backslash B) \cup C,$$

where A is open and B, C are meager sets ([11], §32.I). Evidently, every B-continuous mapping has the Baire property. For separable spaces Y, the B-continuous mappings coincide with mappings which have the Baire property in the broad sense ([11], §32.II). A subset M of metric space X is called *perfect* if it is closed and contains no isolated points. A mapping F from a metric space X into a metric space Y satisfies the *condition of Baire in the restricted sense* (satisfies the condition of Baire in terms of [1]) if for each nonempty perfect set $M \subset X$ there is a meager in M subset $N \subset M$ such that the restriction $F|_{M\backslash N}$ is continuous ([11, §32.IV], [1], Introduction, §9). Finally, the mapping F is called *Baire mapping* (measurable in the terminology of [1]) if it belongs to the smallest class of mappings that includes continuous mappings and is closed under pointwise limits. Every Baire mapping satisfies the Baire condition in the restricted sense and thus is B-continuous ([1], Introduction, §9). Simple examples show that the inverse statement is not true.

Let us mention some simple properties of B-continuous mappings [21]. The pointwise limit of a sequence of continuous mappings is B-continuous. If the space Y is separable, then any Borel mapping $F : X \to Y$ is B-continuous. If X, Y are metric groups and $F, G : X \to Y$ are B-continuous and $x_0 \in X$, then $F(x) + G(x)$ and $F(x + x_0)$ are B-continuous. If X, Y are linear metric spaces and a is scalar, then $F(ax)$ and $aF(x)$ are B-continuous too.

Corollary 1. *Let M be a meager subset of a complete metric group X and suppose that a pointwise isotropic mapping F from an open subset D of X into a metric space Y is continuous on $D\backslash M$. Then it is continuous on D.*

Corollary 1′. *If a pointwise isotropic mapping F from an open subset D of complete metric group X into a metric space Y is B-continuous on D, then it is continuous on D.*

Proof. Let $N \subset D$ be a meager subset in D. Then N is meager in X, hence a Z-set. By Theorem 1, F is continuous on D. □

Corollary 2. (A generalization of Theorem B).*A Baire pointwise isotropic mapping from an open subset D of a complete metric group X into a metric space Y is continuous.*

Proof. Since the limit of B-continuous mapping is B-continuous [21], any Baire mapping is B-continuous. □

Corollary 3 (A nonlinear inverse operator theorem). *Let F be a continuous bijective mapping from a Polish (i.e. complete separable metric) space X onto a complete metric group Y such that F^{-1} is pointwise isotropic. Then F^{-1} is continuous.*

Proof. Since in our condition F^{-1} is Baire (this follows from [11], Chap. 3 §39.4 and Chap. 2, §31.9), it is sufficient to apply Corollary 2. □

Theorem 2. *A pointwise isotropic mapping F from an open subset D of an arbitrary metric group X into a Polish space Y that has Borel graph is continuous.*

Proof. First we suppose that the group X is separable. Let $\Gamma = \{(x, F(x)) : x \in D\}$ be the graph of F. We let $F_*(x) = (x, F(x))$, $x \in D$. It is clear that F_* is a bijective mapping from X onto Γ and F_*^{-1} is a Borel mapping ([11, §39.5]) so F_* is a Baire mapping [11, §31.9]. If F is discontinuous at some point of D then, according to Definition 3, there exists a sequence $x_i \to 0$ as $i \to \infty$ such that

$$\overline{\lim_i} \operatorname{dist}(F(x + x_i), F(x)) > 0$$

for each $x \in D$. So

$$\overline{\lim_i} \operatorname{dist}((x + x_i, F(x + x_i)), (x, F(x))) > 0.$$

Therefore F_* is pointwise isotropic. But according to Corollary 2 this mapping is necessarily continuous on D. So, the mapping F is continuous too.

Now let X be an arbitrary complete metric group. If F is discontinuous at some point $x_0 \in D$, then there is a closed separable subgroup $X_0 \subset X$ such that F, which is defined on the open subset $D_0 = D \cap X_0$ of the group X_0, is discontinuous at x_0. The graph of mapping $F : D_0 \to Y$ is the intersection of Borel set Γ and a closed subset $X_0 \times Y \subset X \times Y$. So from the first part of this proof it follows that $F|_{D_0}$ is a continuous mapping. This contradiction proves the theorem. □

As is well known, a weakly continuous linear operator in a Banach space is norm continuous. The following corollary generalizes this result to pointwise isotropic mappings.

Corollary 4. *Let F be a pointwise isotropic mapping from an open subset D of a Banach space X into a Banach space Y and let F be norm to weak continuous. Then F is norm to norm continuous.*

Proof. Suppose that F is not norm to norm continuous. Then F is not norm to norm continuous on some open subset D_0 of a separable subspace $X_0 \subset X$. So we can suppose that X is separable. Then $F(D_0)$ is weakly separable, hence norm separable.

As is well known (see, for example, [14]), on a separable Banach space the Borel σ-algebras generated by norm and by weak topology coincide. So F is norm to norm Borel mapping. So it is norm to norm Baire mapping ([11], Chap. 2, Sect. 31.9) and hence F is continuous (Corollary 2). $\square$

Remark. Since in Corollary 4 we do not use any separability assumption, this corollary do not follows immediately from Theorem 2. A suggestion to prove this corollary was given by J. Orihuela.

Let us give another example of a Z-set. Let X be a metric abelian group with a nonzero Borel quasi-invariant measure μ (i.e. $\mu(M) = 0 \Rightarrow \forall x \in X,\ \mu(M+x) = 0$). For example, we can take as X a locally compact metric abelian group with the Haar measure. Of course, in this group any Borel set M with $\mu(M) = 0$ is the Z-set.

Corollary 5. *If a pointwise isotropic mapping F from an open subset D of abelian group X with a nonzero quasi-invariant measure μ into a metric space Y is continuous on a set $D\backslash M$, where $\mu(M) = 0$, then it is continuous on D.*

Similarly to Definition 5 we introduce

Definition 5′. *A mapping F from a metric space X with a Borel measure μ to a metric space Y is called to be* H-continuous*, if there exists a set $M \subset X$, $\mu(M) = 0$, such that $F|_{X\backslash M}$ is continuous.*

Corollary 6. *If a pointwise isotropic mapping F from an open subset D of a metric abelian group X with a nonzero quasi-invariant measure μ into a metric space Y is H-continuous then it is continuous.*

Definition 6. *A mapping F from a metric space X with a Borel measure μ into a metric space Y is called* μ-measurable *if the preimage $F^{-1}(U)$ of any open set $U \subset Y$ is μ-measurable in X.*

Remark. It seems that the characterization function of a Cantor set of nonzero Lebesgue measure on $[0, 1]$ gives an example of μ-measurable but not H-continuous mapping.

Let us consider one more example of a Z-set.

Definition 7. *Let X be a Polish abelian group. A set $M \subset X$ is called* universally measurable *if it belongs to the completion of Borel σ-algebra with respect to any probability Borel measure. A universally measurable set is called a* Haar zero set *if there exists a probability Borel measure μ such that $\mu(M + x) = 0$ for every $x \in X$. M is called a* Christensen zero set *if there exists a Haar zero set N such that $M \subset N$. Finally, sets of the form $M \cup N$, where M is universally measurable and B is a Christensen zero set, are called* Christensen measurable.

These sets have the following properties which we formulate in the form of

Lemma 1 ([8], Proposition 1 and Corollary 1). *Christensen measurable sets form a shift invariant σ-algebra. Every Christensen zero set is a Z-set.*

Corollary 7. *If a pointwise isotropic mapping F from an open subset D of Polish abelian group X into a metric space Y is continuous on some set $D\backslash M$, where M is a Christensen zero set, then it is continuous on D.*

Similarly to Definition 5 we introduce

Definition ″. *A mapping F from an open subset D of a Polish abelian group X into a metric space Y is called* Ch-continuous *if there is a Christensen zero subset $M \subset D$ such that $F|_{D\backslash M}$ is continuous.*

Corollary 8. *If a pointwise isotropic mapping F from an open subset D of a Polish abelian group X into a metric space Y is Ch-continuous on D, then it is continuous on D.*

Definition 8 [8]. *A mapping F from an open subset of a Polish abelian group X into a separable metric space Y is called* Christensen measurable *if for every open set $U \subset Y$ the set $F^{-1}(U)$ is Christensen measurable in X.*

Proposition 2.

a) *If F_n, $n = 1, 2, ...$, are Christensen measurable and $F_n(x) \to F(x)$ for every $x \in D$, then F is Christensen measurable too.*
b) *If Y is a Polish group and $F, G : D \to Y$ are Christensen measurable, then $F(x) + G(x)$ is Christensen measurable too.*
c) *If Y is a separable linear metric space, $a \in \mathbb{R}$ (or $\mathbb{C}$) and F is Christensen measurable, then aF is Christensen measurable too.*

Proof. The proof follows from Lemma 1. To prove a) it is sufficient to use the classical formula $F^{-1}(U) = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} F_n^{-1}(U)$. To prove b) it is sufficient to use the existence for separable Y the sequence $V_n \times W_n \subset Y \times Y$ such that $\cup_k V_{n_k} \times W_{n_k} = U$ for every open $U \subset Y \times Y$. □

Let us give an example of isotropic mapping. Here we suppose X to be a linear metric space with shift invariant metric and Y to be a linear normed space.

Definition 9. *A mapping F from a convex open subset $D \subset X$ to Y is called* norm convex *if $\phi(x) := ||F(x)||$ is a convex functional, i.e. for any $x, y \in D$*

$$||F\left(\frac{x+y}{2}\right)|| \leq \frac{||F(x)|| + ||F(y)||}{2}.$$

Theorem 3. *A norm convex mapping F from a convex open subset $D \subset X$ into Y is isotropic with the isotropy constant equal to 0 or ∞.*

Proof. Let F be discontinuous on D. We can suppose, by Proposition 1, that $0 \in D$, $F(0) = 0$ and $||F(x_i)|| \geq c$ for some sequence $x_i \to 0$ as $i \to \infty$ and for some $c > 0$.

Evidently, for every $x \in D$ such that $2x \in D$

$$||F(x)|| = ||F\left(\frac{2x+0}{2}\right)|| \leq \frac{||F(2x)|| + ||F(0)||}{2} = \frac{||F(2x)||}{2}.$$

So for every k such that $2^k x \in D$

$$2^k ||F(x)|| \leq ||F(2^k x)||. \tag{2.2}$$

Choose a subsequence $(x_{i_k}) \subset (x_i)$ such that $u_k := 2^k x_{i_k} \to 0$ as $k \to \infty$. Such a subsequence exists by [17, p.4]. Of course we can suppose $u_k \in D$. Then by (2.2) for any n

$$||F\left(\frac{u_k}{2^n}\right)|| = ||F\left(\frac{2^k x_{i_k}}{2^n}\right)|| \geq 2^{k-n} ||F(x_{i_k})|| \geq 2^{k-n} c \to \infty \text{ as } k \to \infty. \tag{2.3}$$

Take any open ball $B \subset D$ with center at the origin. For any $x \in B$ we have that $-x \in D$ and starting with some index n, $u_k/2^n + x \in D$. Hence

$$||F\left(\frac{u_k}{2^{n+1}}\right)|| = ||F\left(\frac{\frac{u_k}{2^n} + x - x}{2}\right)|| \leq \frac{||F(\frac{u_k}{2^n} + x)|| + ||F(-x)||}{2}. \tag{2.4}$$

So

$$||F\left(\frac{u_k}{2^n}+x\right)|| \to \infty \text{ as } k \to \infty \text{ for any } n. \tag{2.5}$$

Now if (2.5) is satisfied for $y \in D$, $B \subset D$ is some ball with center at y such that $B - y \subset D$, and $x \in B$, then as in (2.4)

$$||F\left(\frac{u_k}{2^{n+1}}+y\right)|| = ||F\left(\frac{\frac{u_k}{2^n}+x+y-x}{2}\right)|| \le \frac{||F\left(\frac{u_k}{2^n}+x\right)||+||F(y-x)||}{2}$$

i.e. for this x (2.5) is satisfied.

Thus, the convexity and openness of D imply that (2.5) is satisfied for every $x \in D$. In particular, (2.5) is satisfied for $n = 1$. This means that F is isotropic with isotropy constant equal to ∞. □

Definition 9′. *A mapping F from a convex open subset D of linear metric space X into a Banach lattice Y is called* Jensen convex *if for any $x, y \in D$*

$$F\left(\frac{x+y}{2}\right) \le \frac{F(x)+F(y)}{2}.$$

It is called Jensen concave *if the converse inequality is satisfied above.*

Corollary 9. *Convex and concave mappings, which are defined on a convex open subset of a linear metric space X are isotropic with the isotropy constant equal to 0 or ∞.*

Proof. Indeed, if F is convex then it is norm convex and we can apply Theorem 3. If F is concave then $-F$ is convex and we can apply Theorem 3 and Proposition 1. □

Corollary 10 (compare with Theorem 2 of [8]). *Let F be a convex (or concave) Baire mapping from a complete linear metric space X to a Banach lattice Y. Then it is continuous.*

Proof is a combination of Corollary 8 and Corollary 2.

Question 2. Is an n-convex mapping (see the definition in [9]) pointwise isotropic?

3. Polynomial and analytic mappings

As we already noted, polynomial operators on abelian groups were studied in [18, 19].

Let X, Y be abelian groups. An operator $T : X \to Y$ is called additive if $T(x+y) = T(x) + T(y)$ and $T(-x) = -T(x)$. Let $X^n = X \times \cdots \times X$ be the n-th Cartesian product of a group X. A mapping $B_n(x_1, \dots, x_n)$ from X^n into Y is called *symmetric* if for any permutation σ of indices $\{1, \dots, n\}$

$$B_n(x_{\sigma(1)}, \dots x_{\sigma(n)}) = B_n(x_1, \dots, x_n)$$

and *n-additive* if it is additive in each of its arguments.

Throughout this section we will consider a group Y which contains no elements of finite order (i.e. $ky = 0$, $k \in \mathbb{N}$, $y \in Y$ implies $y = 0$).

A mapping $P_n : X \to Y$ is called a *homogeneous polynomial operator of degree n* if P_n is not identically zero and there is a number $k_n \in \mathbb{N}$ and an n-additive mapping B_n such that $k_n P_n(x) := B_n(x, \dots, x)$. A constant operator will be denoted by $P_0(x)$. An operator $P : X \to Y$ of the form $P(x) = P_0(x) + P_1(x) + \cdots + P_n(x)$ will be called a polynomial operator of degree n, provided that P_n is not identically zero. It is easy to see that for a linear space this definition coincides with the usual definition of polynomial operators (see e.g. [3]).

In [19] it is shown that for each homogeneous polynomial operator P_n of degree n there is a unique (up to a positive integer factor) symmetric n-additive operator B_n such that $k_nP(x) = B_n(x,\dots,x)$ for some integer k_n. Let us put

$$B_{n-k,k}(z,x) = B_n(\overbrace{z,\dots,z}^{n-k},\overbrace{x,\dots,x}^{k});$$

in particular

$$B_{n,0}(z,x) = k_nP_n(z). \tag{3.1}$$

It is easy to see [19] that $B_{n-k,k}(z,x)$ is homogeneous polynomial operator of degree k in x and $n-k$ in z and for every integer m, $P_n(mx) = m^nP_n(x)$. So the proof of the following proposition is evident.

Proposition 3 [19]. *If P is a polynomial operator of degree n and $x_0 \in X$, then $P_{x_0}(x) := P(x+x_0)$ is a polynomial operator of degree n.*

Let us recall that a topological group is called *uniformly dissipative* if there is neighborhood U of zero such that for every neighborhood V of zero there is an positive integer number m such that $nx \in U$ for some $n > m$ implies $x \in V$. For more detail information on uniformly dissipative groups see [5]. Note that if a metric group Y is uniformly dissipative then $by_j \to 0$ as $j \to \infty$ for some positive integer number b and $y_j \in Y$ implies $y_j \to 0$ as $j \to \infty$.

The Open Mapping Principle for Polish groups (see Exercise T to Chapter 6 of [10]) implies that any Polish divisible groups G without elements of finite order (as well as any subgroup of G) is uniformly dissipative. We recall that a group G is *divisible* if for every $a \in G$ and $n \in \mathbb{N}$ there is $x \in G$ with $nx = a$. It should be mentioned that each abelian group (without elements of finite order) is a subgroup of a divisible abelian group (without elements of finite order) [7, §24] and every divisible abelian group without elements of finite order has the structure of a linear space over the field $\mathbb{Q}$ of rational numbers, see [7, 23.1] or [20].

Theorem 4. *Let X, Y be metric abelian groups such that Y is uniformly dissipative and has no elements of finite order. Then every polynomial operator from X into Y is pointwise isotropic.*

For the proof of this theorem we need the following

Lemma 2. *There exists a matrix $(a_{kj})_{k,j=0}^n$, $a_{kj} \in \mathbb{Z}$, and an integer number $b \neq 0$ such that for any collection $y_0,\dots,y_n$ of elements of an abelian group Y which contains no elements of finite order we have*

$$by_k = \sum_{j=0}^n a_{kj}q(j), j = 0,\dots,n$$

where $q(j) := \sum_{k=0}^n j^ky_k$, $j = 0,\dots,n$.

Proof. The determinant of the matrix $(j^k)_{j,k=0}^n$ is the Vandermonde determinant, which is different from zero since $j = 0,\dots,n$ are distinct numbers. Hence, this matrix has a (unique) inverse (c_{kj}). It is clear that c_{kj} are rational. Let b denote the common denominator of c_{kj}. Put $a_{kj} = bc_{kj}$. Then

$$b\sum_{l,k=0}^n c_{kj}j^ly_l = \sum_{j=0}^n a_{kj}q(j),$$

thus

$$by_k = \sum_{j=0}^{n} a_{kj} q(j), \quad k = 0, \dots, n.$$

□

Corollary 11. *Let $P : X \to Y$ be a polynomial operator of degree n. Then its homogeneous components P_k, $k = 0, \dots, n$ are determined by*

$$bP_k(x) = \sum_{j=0}^{n} a_{kj} P(jx),$$

where a_{kj} and $b \neq 0$ are integers dependent on the degree n only.

Proof. Applying Lemma 2 to $y_k = P_k(x)$ and using that $P_k(jx) = j^k P_k(x)$ for every $x \in X$ and $j, k = 0, \dots, n$, we obtain the existence of numbers (a_{kj}) and b which satisfies the condition of Corollary 11. □

Remark. For polynomial operators in linear spaces these Lemma and Corollary are well known results [12].

Proof of Theorem 4. Thus, let $P(x) = \sum_{k=0}^{n} P_k(x)$, where P_k is a homogeneous polynomial operator of degree k. Suppose that P is discontinuous. By Proposition 1, we can suppose that P is discontinuous at zero. Then we can suppose that P_n is discontinuous at zero too. Indeed, in opposite case we can consider the polynomial operator $\bar{P} = P - P_n$. So, there exists a sequence $x_i \in X$, $x_i \to 0$ such that

$$\overline{\lim_i} \operatorname{dist}(P_n(x_i), P_n(0)) > 0.$$

Fixing $x \in X$ put $P_x(z) = P(z+x)$. It is clear that the homogeneous component of highest degree of $P_x(z)$ is equal to $P_n(z)$.

Applying Corollary 11 to $P_x(z)$ we get integers a_{nj}, $j = 0, 1, \dots, n$ and $b \neq 0$ such that

$$bP_n(z) = \sum_{j=0}^{n} a_{nj} P_x(jz) = \sum_{j=0}^{n} a_{nj} P(jz + x),$$

in particular

$$bP_n(x_i) = \sum_{j=0}^{n} a_{nj} P(jx_i + x), \quad i = 1, 2 \dots \tag{3.2}$$

Putting in (3.2) $z = 0$ we have

$$\sum_{j=0}^{n} a_{nj} P(x) = 0 \tag{3.3}$$

for every $x \in X$.

We show that the sequence $(x'_i)_{i=1}^{\infty}$ of points $(jx_i, j = 0, \dots, n,\ i = 1, \dots, \infty)$, which is enumerated somehow, satisfies Definition $3'$. Suppose to the contrary that for some $x \in X$ and each $0 \le j \le n$

$$P(jx_i + x) \to P(x)$$

as $i \to \infty$. Then

$$\sum_{j=0}^{n} a_{nj}[P(jx_i + x) - P(x)] \to 0$$

as $i \to \infty$.

Taking into account (3.3), from here we obtain $\sum_{j=0}^{n} a_{nj} P(jx_i + x) \to 0$ as $i \to \infty$. From (3.2) it follows that $bP_n(x_i) \to 0$ as $i \to \infty$. Since Y is a uniformly dissipative group, $P_n(x_i) \to 0$ as $i \to \infty$ and this contradicts the choice of (x_i).

Thus, for any $x \in X$

$$\overline{\lim_{i}}\, \mathrm{dist}(P(x_i' + x), P(x)) > 0$$

i.e. P is pointwise isotropic.

Corollary 12. *A Baire polynomial mapping from a complete metric abelian group X into a uniformly dissipative abelian metric group Y which contains no elements of finite order, is continuous.*

Corollary 13. *Let X be a metric abelian group and Y be a uniformly dissipative Polish abelian group containing no elements of finite order. If a polynomial operator from X into Y has a Borel graph then it is continuous.*

Corollary 13$'$. *Let X be a complete linear metric space and Y be a Polish linear metric space. If a polynomial operator from X into Y has the Borel graph then it is continuous.*

Now we consider analytic mappings.

Definition 11. *Let a mapping F is defined on an open subset D of a complex Banach space X and takes value in a complex Banach space Y. It is called* Gâteaux differentiable on D *if*

$$\lim_{t\to 0,\, t\in\mathbb{C}} \frac{F(x+th) - F(x)}{t} =: F_h'(x)$$

exists for every $x \in D$ and $h \in X$.

We recall some known results from the theory of Gâteaux differentiable mappings (see [21, 22]): For fixed h the mapping $F_h'(x)$ is Gâteaux differentiable on D. That is why the Gâteaux differentiable mappings in complex Banach spaces are called *Gâteaux analytic* (*G-analytic* in short); late on we shall use this term. The main result of [22] is

Theorem Z. *If a mapping $F : X \to Y$, defined on an open subset $D \subset X$ is G-analytic and B-continuous on D, then it is continuous on D.*

Corollary 3.$'$ (A generalization of Theorem B.) *A Baire G-analytic mapping from an open subset D of Banach space X into a Banach space Y is continuous.*

Corollary 4$'$ (see e.g. [15, p.65]). *A norm-to-weak continuous G-analytic mapping from an open subset D of Banach space X into a Banach space Y is norm to norm continuous.*

Proof. The proof repeat verbatim the proof of Corollary 4. □

Theorem 2$'$. *A G-analytic mapping F from an open subset D of arbitrary Banach space X into a separable Banach space Y, which has a Borel graph, is continuous.*

Proof. The proof of this theorem is a simple modification of the proof of Theorem 2. We shall describe this modification. Let F_* be the mapping from the proof of Theorem 2. Let us show that it is G-analytic on D. Indeed,

$$\lim_{t\to 0} \frac{F_*(x+th) - F_*(x)}{t} = \lim_{t\to 0} \frac{(x+ht, F(x+th)) - (x, F(x))}{t} =$$

$$\lim_{t\to 0}\frac{(th, F(x+th)-F(x))}{t} = (h, F'_h(x)).$$

So F_* is G-analytic. Since it is Baire (see the proof of Theorem 2), F_* is continuous on D by Corollary $3'$. Then the proof of Theorem 2 is repeated literally. □

Corollary $3''$. *A Christensen measurable G-analytic mapping F from an open subset D of separable Banach space X into a Banach space Y, is continuous.*

Proof. By [22, 2.4] for every $x_0 \in D$ there exists a ball B with center in x_0 such that the mapping F is developed in a series:

$$F(x_0 + x) = \sum_{n=0}^{\infty} P_n(x_0, x) \tag{3.4}$$

where $x_0 + x \in B$ and

$$P_n(x_0, x) = \frac{1}{n!}\frac{d^n F(x_0 + tx)}{dt^n}$$

is a polynomial operator of degree n.

Since the difference, the quotient of division by a scalar and limit of Christensen measurable functions are Christensen measurable (by Proposition 2), then $P_n(x_0, x)$ is Christensen measurable, hence continuous [9] in B. This and (3.4) imply that F is Baire in B, hence is continuous in B, by Corollary $3'$, in particular F is continuous at x_0. The arbitrariness of $x_0 \in X$ proves the corollary. □

Now we show how Gajda's result can be proved using the Fischer and Słodkowski theorem.

Theorem G [9]. *Let P be a polynomial operator from a complete separable metric linear space X into a separable metric space Y. If $P = P_0 + P_1 + \cdots + P_n$ is Christensen measurable then P is continuous.*

Proof. If we fix $h \in X$ then $P'_h(x)$ is also Christensen measurable. It is easy to see that $P^{(n-1)}_{h_1\ldots h_{n-1}} = (n-1)!B_n(h_1 \ldots h_{n-1}, x)$, where B_n is an n-linear operator corresponding to P_n. So $B_n(h_1 \ldots h_{n-1}, x)$, by [8] is continuous. So $B_n(x_1, \ldots x_n)$ and P_n are continuous [3]. Taking $P - P_n$ instead of P we obtain that P_{n-1} is also continuous. The induction in n finishes the proof. □

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