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LIMIT THEOREMS FOR RANDOM ELEMENTS IN IDEALS OF ORDER-BOUNDED ELEMENTS OF FUNCTIONAL BANACH LATTICES

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For a sequence of independent random elements belonging to an ideal of order-bounded elements of a Banach lattice, we investigate the asymptotic relative stability of extremal values, the law of large numbers for the pth powers, and the central limit theorem.

Assume that E is a Banach lattice with norm $\|\cdot\|$ and modulus $|\cdot|$ and X is a random element with values in E. For arbitrary elements $x_1, \ldots, x_n \in E$, the expressions $\sup_{1 \le k \le n} x_k$ and $\left(\sum_1^n |x_k|^p\right)^{1/p}$, $1 \le p < \infty$, are meaningful (see [1]). Therefore, for a sequence (X_n) of independent copies of X, it is natural to introduce and investigate the quantities

$$Z_n = \sup_{1 \le k \le n} X_k \quad \text{and} \quad Z_n^{(p)} = \left(\sum_1^n |X_k|^p\right)^{1/p}.$$

In a separable Banach lattice, these values are Borel random elements [2].

In the present work, we establish conditions under which the following asymptotic relations hold almost surely as $n \to \infty$:

$$\frac{Z_n}{b_n} \xrightarrow{\text{a.s.}} \mathfrak{S} X, \tag{1}$$

$$\frac{Z_n^{(p)}}{n^{1/p}} \xrightarrow{\text{a.s.}} \mathfrak{S}_p X \tag{2}$$

where $\xrightarrow{\text{a.s.}}$ means the almost-sure convergence in the norm of the space E, $\mathfrak{S} X$ and $\mathfrak{S}_p X$ are certain nonzero elements in E, and b_n is a numerical sequence.

By analogy with the one-dimensional case, relation (1) is called the almost-sure relative stability of the sequence Z_n, and relation (2) is called the law of large numbers for the pth powers. Parallel with relations (1) and (2), we also investigate conditions under which the central limit theorem is true, namely,

$$\frac{S_n}{n^{1/2}} \xrightarrow{\mathcal{D}} G \tag{3}$$

as $n \to \infty$; here, $\xrightarrow{\mathcal{D}}$ denotes the weak convergence of distributions of random elements, G is a Gaussian random element in E, and $S_n = \sum_1^n X_k$.

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The problem of generalization of one-dimensional results for the scheme of sums S_n to Banach spaces has been extensively studied in the last 25 years [3, 4]; however, we know a few works devoted to the investigation of conditions of validity of relations (1) and (2) in the infinite-dimensional case [2, 5, 6]. The asymptotic behavior of extremal values and the law of large numbers for random variables on the real straight line are well studied (see [7, 8] and the bibliography therein).

In what follows, we denote by E_+ the set of positive elements of the Banach lattice E and by $E_{(u)}$ the ideal generated by the element $u \in E_+$, i.e., $E_{(u)} = \{x \in E : \exists \lambda > 0,\ |x| \le \lambda u\}$. Then

$$\|x\|_u = \inf\{\lambda > 0 : |x| \le \lambda u\}$$

is a norm in $E_{(u)}$.

We study relations (1)–(3) under the following main condition: $X \in E_{(u)}$ a.s. for certain $u \in E_+$ or, which is the same,

$$\|x\|_u < \infty \quad \text{a.s.} \tag{4}$$

Below, we present necessary definitions and notation from the theory of Banach lattices. A set A of a Banach lattice E is called order-bounded if there exists $u > 0$ such that $|x| \le u$ for every $x \in A$.

We say that a Banach lattice E is σ-complete if, for an arbitrary order-bounded sequence (x_n), $x_n \in E$, there exist the least upper bound $\sup_{n \ge 1} x_n$ and the greatest lower bound $\inf_{n \ge 1} x_n$.

As an example of σ-complete Banach lattices, one can mention Banach lattices dual to other Banach lattices. In particular, a reflexive Banach lattice is σ-complete. Any Köthe functional space (see the definition below) is also a σ-complete lattice. The lattice $C[0, 1]$ is not σ-complete [1, p. 4].

A Banach lattice E is called σ-order-continuous if

$$\inf_{n \ge 1} x_n = 0 \Rightarrow \|x_n\| \to 0 \quad \text{as} \quad n \to \infty$$

for every decreasing sequence $x_1 \ge x_2 \ge \ldots$.

Köthe functional spaces are an important example of Banach lattices. Let us give its definition. Let (T, Λ, μ) be a complete σ-finite measurable space. A Köthe functional space E on T is a Banach space (of equivalence classes) of locally integrable functions on (T, Λ, μ) that satisfies the following conditions:

(i) if $|x(t)| \le |y(t)|$ a.e., $x(t)$ is measurable, and $y \in E$, then $x \in E$ and $\|x\| \le \|y\|$;

(ii) for a set $A \in \Lambda$ with $\mu(A) < \infty$, the characteristic function $I(A) = \{I(t, A), t \in T\}$ belongs to E.

The classical spaces $L_p(\mu)$, $1 \le p \le \infty$, and a Banach space with unconditional basis (in equivalent norm) are examples of Köthe functional spaces.

In what follows, we use the well-known statement presented below.

Proposition 1. *Every separable σ-complete Banach lattice is σ-order-continuous [1, p. 7] and order-isometric to a certain Köthe functional space [1, p. 29].*

A Banach lattice E is called q-concave, $1 \le q < \infty$, if there exists a constant $D_{(q)} < \infty$ such that

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \le D_{(q)} \left\| \left(\sum_{i=1}^{n} |x_i|^q\right)^{1/q} \right\|$$

for any finite collection of elements $(x_i)_1^n \subset E$. A Banach lattice E has a lower q-bound if there exists a constant $C_{(q)} < \infty$ such that

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \le C_{(q)} \left\| \sum_{i=1}^{n} x_i \right\|$$

for any finite collection of pairwise disjunctive elements $(x_i)_1^n \subset E$. A q-concave Banach lattice has a lower q-bound and, hence, its norm is q-order-continuous [1, p. 83]. If, in addition, the measure μ is separable, then the q-concave Köthe space E is separable [9, p. 93].

We say that a Banach space E uniformly contains l_∞^n if there exists a sequence of n-measurable subspaces $E_n \subset E$ such that the Banach–Mazur distance $d\left(E_n, l_\infty^n\right)$ tends to 1 as $n \to \infty$. Every Banach lattice that does not uniformly contain l_∞^n is q-concave for certain $q < \infty$ [1, pp. 85, 91].

1. First, we establish relation (1) in the Gaussian case for Köthe functional spaces with σ-order-continuous norm. If X is a centered (i.e., $\mathbb{M}X = 0$) Gaussian random element, then all moments of its norm exist, i.e., $\mathbb{M}\|X\|^k < \infty$, $0 < k < \infty$ [3, pp. 257, 258]. Therefore, the mean-square deviation of the random element X

$$\mathfrak{S}X = \left(\frac{\pi}{2}\right)^{1/2} \mathbb{M}\,|X| \tag{5}$$

exists. Since $\mathbb{M}\gamma = 0$, $\mathbb{M}\gamma^2 = 1$, and $\mathbb{M}\,|\gamma| = (\pi/2)^{-1/2}$ for a standard Gaussian numerical random variable γ, we can conclude that, for a Köthe space, this definition coincides with the standard definition, namely, $\mathfrak{S}X = \left\{\sigma(t) = \left(\mathbb{M}|X(t)|^2\right)^{1/2},\ t \in T\right\}$.

Theorem 1. *Suppose that E is a separable Köthe space with σ-order-continuous norm, X is a centered Gaussian random element in E, and condition (4) is satisfied. Then the sequence Z_n is relatively stable a.s., i.e., relation (1), where*

$$b_n = \begin{cases} \left(2\ln(n)\right)^{1/2}, & n \ge 3, \\ 1, & n < 3, \end{cases}$$

and $\mathfrak{S}X$ is defined by equality (5), is true.

Proof. Since there exists a measurable isomorphism from (T, Λ, μ) onto a certain measurable space (S, Σ, ν) with $\nu(S) = 1$ that preserves sets of measure zero, we can assume that $\mu(T) = 1$. Let us prove that

$$\mu\left\{t \in T\colon \lim_{n \to \infty} \frac{Z_n(t)}{b_n} = \sigma(t)\right\} = 1 \quad \text{a.s.} \tag{6}$$

and there exists a random element $Y \in E$ such that

$$\left|\frac{Z_n(t)}{b_n}\right| \leq Y(t) \quad \text{a.s.} \tag{7}$$

for all $n \geq 1$.

Since an abstract analog of the Lebesgue theorem on dominated convergence [10, p. 72] holds in Köthe spaces with σ-order-continuous norm, relations (6) and (7) yield (1).

For real Gaussian random variables, the property of almost-sure relative stability is known [7, p. 203]. Thus, for every $t \in T$, we have

$$\lim_{n \to \infty} \left|\frac{Z_n(t)}{b_n} - \sigma(t)\right| = 0 \quad \text{a.s.}$$

Hence, according to the Fubini theorem, we get

$$\mu\left\{t \in T\colon \lim_{n \to \infty} \left|\frac{Z_n(t)}{b_n} - \sigma(t)\right| = 0\right\} = 1 \quad \text{a.s.}$$

Thus, condition (6) is satisfied.

Since $X \in E_{(u)}$, we have

$$|X| \leq \tau u, \quad \tau = \|X\|_u, \quad \text{a.s.} \tag{8}$$

Considering the Gaussian random element X in the normed space $E_{(u)}$, we get [4, p. 59; 11, p. 120]

$$\mathbb{P}(\tau > s) \leq C_1 \exp\left(-C_2 s^2\right), \tag{9}$$

where C_1 and C_2 are certain constants dependent only on the correlation operator of the random element X. We set

$$\tau_n = \|X_n\|_u, \quad n \geq 1, \qquad \tau_\infty = \sup_{n \geq 1}\left(\frac{\tau_n}{b_n}\right).$$

Then, taking (8) into account, we obtain

$$\left|\frac{Z_n}{b_n}\right| \leq \sup_{1 \leq i \leq n}\left(\frac{\tau_i}{b_i}\right) u \leq \tau_\infty u \quad \text{a.s.} \tag{10}$$

It is easy to verify that the random variable τ_∞ is bounded a.s. Indeed, τ_n are independent copies of τ, and estimate (9) yields

$$\mathbb{P}(\tau_\infty > s) \leq \sum_{n=1}^{\infty} \mathbb{P}(\tau_n > s b_n) \leq C_3 s^2 \exp\left(-C_4 s^2\right)$$

for $s > 1$ (see Lemma 2 in [2]). This and (10) imply that the sequence (Z_n/b_n) is bounded, i.e., condition (7) is also satisfied.

Corollary 1. *Suppose that* $X = (\eta_n)$ *is a Gaussian random element in the space* $E = c_0$, $\mathbb{M}\eta_n = 0$, $\mathbb{M}\eta_n^2 = \sigma_n^2$, *and* $\mathfrak{S}X = (\sigma_n) \in c_0$. *Then relation (1), where* b_n *is defined in Theorem 1, is true.*

Proof. As is known, for a sequence of random variables $\eta_n \to 0$ a.s., $n \to \infty$, there exists a sequence of positive numbers $u_n \to 0$ such that $\eta_\infty = \sup_{n\ge 1} |\eta_n/u_n| < \infty$ a.s. (see [12, p. 59]). This implies that, for a Gaussian random element X in c_0, there exists a positive element $u = (u_n) \in c_0$ and a bounded random variable η_∞ such that $|X| \le \eta_\infty u$ a.s., i.e., $X \in c_{0(u)}$. In view of Theorem 1, this completes the proof of Corollary 1 (c_0 is a Köthe space with σ-order-continuous norm).

Below, we give an analog of Theorem 1 for one class of abstract Banach lattices.

Corollary 2. *Suppose that* E *is a separable* σ*-complete Banach lattice,* X *is a centered Gaussian random element in* E, *and condition (4) is satisfied. Then the sequence* Z_n *is relatively stable a.s., i.e., relation (1), where* b_n *is defined in Theorem 1 and* $\mathfrak{S}X$ *is given by equality (5), is true.*

Proof. Indeed, since an order isometry preserves the order, least upper bound, and greatest lower bound and is continuous, we conclude that Corollary 2 follows from Proposition 1 and Theorem 1.

Remark 1. It follows from the results of [2] and Proposition 1 that, in a separable Banach lattice that does not uniformly contain l_∞^n, equality (1) holds for any centered Gaussian random element.

Below, we present (without proof) another result concerning the almost-sure relative stability of the sequence (Z_n).

Proposition 2. *If* E *is a Banach space with unconditional basis* (e_n), $X = \sum_n \eta_n\sigma_n e_n$ *is a Gaussian random element in the space* E, $\mathbb{M}\eta_n = 0$, $\mathbb{M}\eta_n^2 = 1$, $\mathfrak{S}X = \sum_n \sigma_n e_n \in E$, *and the components* η_n *are independent, then relation (1) is true.*

The known results concerning the almost-sure relative stability of (1) allow us to make the following conjecture:

Hypothesis 1. *In a separable Banach lattice, relation (1) holds for every centered Gaussian random element* X.

2. Consider the law of large numbers for the pth powers (2). In what follows, X is an arbitrary (not necessarily Gaussian) random element in the Banach lattice E. We set

$$\mathfrak{S}_p X = \left\{\sigma_p(t) = \left(\mathbb{M}\,|X(t)|^p\right)^{1/p},\ t \in T\right\}, \quad 1 \le p < \infty.$$

Theorem 2. *Suppose that* E *is a separable Köthe space with* σ*-order-continuous norm and* X *is a centered random element in* E *for which condition (4) is satisfied and*

$$\mathbb{M}\|X\|_u^p < \infty. \tag{11}$$

Then the law of large numbers for the p*th powers (2) holds for* X.

Proof. Assume that $\mu(T) = 1$. As in Theorem 1, it is necessary to verify that

$$\mu\left\{t \in T\colon \lim_{n\to\infty} \frac{Z_n^{(p)}(t)}{n^{1/p}} = \sigma_p(t)\right\} = 1 \quad \text{a.s.}$$

and, for all $n \geq 1$,

$$\left|\frac{Z_n^{(p)}(t)}{n^{1/p}}\right| \leq Y(t), \quad t \in T, \quad \text{a.s.} \tag{12}$$

Since, by virtue of conditions (4) and (11),

$$|X(t)| \leq \|X\|_u u \tag{13}$$

and $\sigma_p(t)$ exists, we establish that, according to the Kolmogorov law of large numbers [8, p. 337], for every $t \in T$ we have

$$\lim_{n\to\infty}\left|\frac{Z_n^{(p)}(t)}{n^{1/p}} - \sigma_p(t)\right| = 0 \quad \text{a.s.}$$

By using the Fubini theorem, we get

$$\mu\left\{t \in T\colon \lim_{n\to\infty}\left|\frac{Z_n^{(p)}(t)}{n^{1/p}} - \sigma_p(t)\right| = 0\right\} = 1 \quad \text{a.s.}$$

It remains to verify the existence of a random element $Y \in E$ for which relation (12) is true. It follows from the definition of $Z_n^{(p)}(t)$ and relation (13) that

$$\frac{|Z_n^{(p)}(t)|}{n^{1/p}} \leq \left(\frac{1}{n}\sum_{i=1}^{n}\|X_i(t)\|_u^p\right)^{1/p} u(t) \leq \eta^{1/p} u(t),$$

where

$$\eta = \sup_{n\geq 1}\left(\frac{1}{n}\sum_{i=1}^{n}\|X_i(t)\|_u^p\right).$$

Estimate (11) and the Kolmogorov law of large numbers yield

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\|X_i(t)\|_u^p = \mathbb{M}\|X(t)\|_u^p < \infty \quad \text{a.s.}$$

Hence, $\eta < \infty$ a.s. and, setting $Y(t) = \eta^{1/p} u(t)$, we get (12).

Recall that a random element X in a Banach space E is called pre-Gaussian if there exists a Gaussian random element $G(X)$ in E such that X and $G(X)$ have the same correlation operator. For a centered pre-Gaussian random element X, we define the mean-square deviation by the equality

$$\mathfrak{S}X = \mathfrak{S}_2X = \left(\frac{\pi}{2}\right)^{1/2}\mathbb{M}|G(X)|. \tag{14}$$

Since a separable σ-complete Banach lattice is order-isometric to a certain Köthe space (Proposition 1) and $S\mathfrak{S}X = \mathfrak{S}SX$ for a linear isometry S, we obtain the following results:

Corollary 3. *If X is a centered pre-Gaussian random element in a separable σ-complete Banach lattice and conditions (4) and (11) are satisfied for $p = 2$, then the law of large numbers (2), where $p = 2$ and $\mathfrak{S}_2X$ is defined by equality (14), is true.*

Corollary 4. *Suppose that X is a centered pre-Gaussian random element in a separable σ-complete Banach lattice E and there exists $u \in E$ such that*

$$|X| \leq u \quad a.s. \tag{15}$$

Then the law of large numbers (2), where $p = 2$ and $\mathfrak{S}_2X$ is defined by equality (14), is true.

In [6], the law of large numbers for squares was established for q-concave Köthe spaces. The analysis of this result shows that it can be generalized as follows:

Proposition 3. *Suppose that E is a separable q-concave Köthe space, $1 \leq p \leq q < \infty$, and $\sigma_r(t) \in E$ for certain $r > q$. Then the law of large numbers for the pth powers (2) is true.*

3. Consider the central limit theorem (3). First, we formulate one result obtained in [6], which is necessary for what follows: For a separable q-concave $(q < \infty)$ Köthe space E and a centered random element X, the condition

$$\sigma_q(t) \in E \tag{16}$$

is sufficient for the validity of the central limit theorem (3).

Theorem 3. *Suppose that E is a separable q-concave $(q < \infty)$ Köthe space and X is a centered random element in E for which conditions (4) and (11) are satisfied for $p = 2$. Then the central limit theorem (3) holds and*

$$\mathbb{M}\left\|\frac{S_n}{n^{1/2}}\right\|^2 \to \mathbb{M}\|G\|^2 \tag{17}$$

as $n \to \infty$.

Proof. First, we prove the theorem under the additional condition

$$|X| \leq cu \quad \text{a.s.}, \tag{18}$$

where $c > 0$ is a nonrandom constant. It follows from (18) that $\mathfrak{S}_q X \le c u$, whence, taking into account that $u \in E$, we get $\mathfrak{S}_q X \in E$.

Thus, condition (16) is satisfied, and the central limit theorem (3) holds for X.

For fixed $\lambda > 0$, we set

$$X^{(\lambda)} = \begin{cases} X & \text{for } \|X\|_u \le \lambda, \\ 0 & \text{for } \|X\|_u > \lambda, \end{cases}$$

$$X_i^{(\lambda)} = \begin{cases} X & \text{for } \|X_i\|_u \le \lambda, \\ 0 & \text{for } \|X_i\|_u \le \lambda, \end{cases}$$

$$M^{(\lambda)} = \mathbb{M} X^{(\lambda)}, \qquad S_n^{(\lambda)} = \sum_{i=1}^{n} \left(X_i^{(\lambda)} - M^{(\lambda)} \right).$$

Denoting by $I(A)$ the characteristic function of the set A, by definition we get

$$\left| X_i^{(\lambda)} \right| \le \lambda u, \qquad \left| X^{(\lambda)} \right| \le \lambda u,$$

$$\left| M^{(\lambda)} \right| = \left| \mathbb{M} X^{(\lambda)} - \mathbb{M} X \right| = \left| \mathbb{M} I\left(\|X\|_u > \lambda \right) X \right| \le |u| \, \mathbb{M} I\left(\|X\|_u > \lambda \right) \|X\|_u . \tag{19}$$

If condition (11) is satisfied and $p = 2$, then estimate (18) and the central limit theorem (3) are true for the difference $X^{(\lambda)} - M^{(\lambda)}$. According to the known result of Pisier [4, p. 278], the central limit theorem (3) holds for a random element X if $\forall \varepsilon > 0 \ \exists \lambda > 0$ such that

$$\sup_{n \ge 1} \frac{\mathbb{M} \left\| S_n - S_n^{(\lambda)} \right\|}{n^{1/2}} < \varepsilon. \tag{20}$$

To verify (20), we use the known estimate [13]

$$C^{-1} \mathbb{M} \left\| \sum_{i=1}^{n} \left(|Y_i|^2 \right)^{1/2} \right\| \le \mathbb{M} \left\| \sum_{i=1}^{n} Y_i \right\| \le C \mathbb{M} \left\| \sum_{i=1}^{n} \left(|Y_i|^2 \right)^{1/2} \right\|,$$

which is true for a q-concave Banach lattice $(q < \infty)$ and independent random elements Y_i, $\mathbb{M} Y_i = 0$. The last inequality and (19) yield

$$\frac{\mathbb{M} \left\| S_n - S_n^{(\lambda)} \right\|}{n^{1/2}} \le C \mathbb{M} \left\| \left(\frac{1}{n} \sum_{i=1}^{n} \left| X_i - X_i^{(\lambda)} + M^{(\lambda)} \right|^2 \right)^{1/2} \right\|$$

$$\le C_1 \mathbb{M} \left\| \left(\frac{1}{n} \sum_{i=1}^{n} I\left(\|X_i\|_u > \lambda \right) \left| X_i^{(\lambda)} \right|^2 + \left| M^{(\lambda)} \right|^2 \right)^{1/2} \right\|$$

$$\leq C_1 \mathbb{M} \left\| \left(\frac{1}{n} \sum_{i=1}^{n} I\big(\| X_i \|_u > \lambda \big) \| X_i \|_u^2 |u|^2 + \big(\mathbb{M} I\big(\| X \|_u > \lambda \big) \| X \|_u \big)^2 |u|^2 \right)^{1/2} \right\|$$

$$\leq C_1 \| u \| \Big(\mathbb{M} I\big(\| X \|_u > \lambda \big) \| X \|_u^2 + \big(\mathbb{M} I\big(\| X \|_u > \lambda \big) \| X \|_u \big)^2 \Big)^{1/2}.$$

For $p = 2$, it follows from estimate (11) that

$$\mathbb{M} I\big(\| X \|_u > \lambda \big) \| X \|_u^2 \to 0 \quad \text{as} \quad \lambda \to \infty,$$

which yields inequality (20).

The convergence of moments (17) follows from the well-known general results.

Corollary 5. *Suppose that E is a separable Banach lattice that does not uniformly contain l_∞^n, and X is a centered random element in E for which conditions (4) and (11) are satisfied for $p = 2$. Then the central limit theorem (3) and relation (17) are true.*

Proof. According to Proposition 1, the lattice E is isometric to a certain separable Köthe space, which does not uniformly contain l_∞^n. Since a Banach lattice that does not uniformly contain l_∞^n is q-concave for certain $q < \infty$, we can apply Theorem 3.

In [14], for every Banach spaces that uniformly contains l_∞^n, an example of a pre-Gaussian random element bounded in norm for which the central limit theorem is not satisfied was constructed.

It turns out that, even in certain Banach spaces that simultaneously have the type $2 - \delta$ and cotype $2 + \delta$, $\delta > 0$, there exist bounded pre-Gaussian random elements for which the central limit theorem is not satisfied [15]. Recall that a Banach space B is of the type p, $1 \leq p \leq 2$, and cotype q, $q > 2$, if there exists a constant $C < \infty$ such that, for any finite collection of elements (x_i) from B, we have, respectively,

$$\mathbb{M} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \leq C \left(\sum_{1}^{n} \| x_i \|^p \right)^{1/p}$$

and

$$\mathbb{M} \left\| \sum_{1}^{n} \varepsilon_i x_i \right\| \geq C \left(\sum_{1}^{n} \| x_i \|^q \right)^{1/q},$$

where (ε_i) is a sequence of independent symmetric Bernoulli random variables.

The result presented below shows that if, in a Banach lattice, we replace the boundedness in norm by the order boundedness, then the conditions for the validity of the central limit theorem can be significantly weakened.

Proposition 4. *Suppose that E is a separable Banach lattice and X is a centered random element in E. Then the following assertions are true:*

(i) if E does not uniformly contain l_∞^n and there exists $u \in E_+$ such that condition (15) is satisfied, then X satisfies the central limit theorem (3) and

$$\mathbb{M}\left\|\frac{S_n}{n^{1/2}}\right\|^p \;\to\; \mathbb{M}\|G\|^p$$

as $n \to \infty$ *for every* $1 \le p < \infty$;

(ii) if E *uniformly contains* l_∞^n, *then there exists a centered pre-Gaussian random element* X *with values in* E *and a nonrandom element* $u \in E_+$ *such that condition (15) is satisfied but* X *does not satisfy the central limit theorem (3).*

Proof. Assertion (i) is contained in Corollary 5.

Let us prove assertion (ii). If E uniformly contains l_∞^n, then it obviously does not have any finite cotype. Therefore, it does not have any lower q-bound [1, p. 88]. Hence, for any ε and n, there exists a collection $(x_i)_1^n$ of pairwise disjunctive elements of E such that

$$\max_{1\le i\le n}|a_i| \;\le\; \left\|\sum_{i=1}^n a_i x_i\right\| \;\le\; (1+\varepsilon)\max_{1\le i\le n}|a_i|$$

for any collection of numbers $(a_i)_1^n$ [1, p. 91]. Since

$$\left|\sum_{i=1}^n a_i x_i\right| \;=\; \left|\sum_{i=1}^n a_i |x_i|\right|$$

for any disjunctive collection, we can assume that all x_i are nonnegative.

Hence, by analogy with [16], we can conclude that, for any sequence of positive numbers α_k convergent to zero, there exists a sequence of positive elements $x_k \in E$ such that $\|x_k\| = \alpha_k$ and the series $\sum x_k$ is unconditionally convergent.

Let $\alpha_k = 1/\ln\ln\ln(k+15)$ be a sequence of independent random variables (ξ_k) such that $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = 1/\ln(k+7)$ and $\mathbb{P}(\xi_k = 0) = 1 - 2/\ln(k+7)$. We set

$$X \;=\; \sum_{i=1}^\infty \xi_i x_i.$$

According to [14], the random element X is pre-Gaussian and does not satisfy the central limit theorem. We set

$$u \;=\; \sum_{i=1}^\infty x_i.$$

It is clear that

$$|X| \;\le\; \sum_{i=1}^\infty |x_i| \;=\; \sum_{i=1}^\infty x_i \;=\; u,$$

i.e., condition (15) is satisfied.

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