

The Radon-Nikodým Property Does Not Imply the Separable Complementation Property

Anatolij M. Plichko

Department of Mathematics, Pedagogical University, str. Shevchenko 1,
316050 Kirovograd, Ukraine

E-mail: aplichko@inetlab.kspu.kr.ua

and

David Yost¹

Department of Mathematics, King Saud University, P.O. Box 2455,
Riyadh 11451, Saudi Arabia

E-mail: dthoyost@ksu.edu.sa

Communicated by Gilles Pisier

Received May 8, 2000; revised September 18, 2000; accepted September 28, 2000

There is a Banach space X enjoying the Radon-Nikodým Property and a separable subspace Y which is not contained in any complemented separable subspace of X .

© 2001 Academic Press

Key Words: Banach space; complemented subspace; Radon-Nikodým Property.

In [9] a Banach space is said to have the *separable complementation property* (which we abbreviate SCP) if every separable subspace is contained in a complemented separable subspace. W. B. Johnson [4, p. 38] announced that every dual space with the Radon-Nikodým Property (RNP) has SCP; one proof of this can be found in [17, Proposition 2] and another in [7]. Diestel and Uhl [4, Problem 22] asked whether every Banach space with the Radon-Nikodým Property has SCP; here we give a counterexample. Of course our example is not isomorphic to any dual space. It is easy to check that every Banach space with both the weak RNP (as defined by Musial [13]) and the SCP already has the RNP; our example also shows that this result has no converse.

It may be useful to recall that a Banach space X has the *Radon-Nikodým Property* if the Radon-Nikodým Theorem is valid for measures taking values in X . It is clear that this property is invariant under renorming, and

¹ To whom correspondence should be addressed.

straightforward to prove that the ℓ_1 sum of a (possibly uncountable) family of Banach spaces, each with RNP, will also have RNP. These facts about RNP are sufficient for our purposes. An equivalent geometric property is that every non-empty bounded subset of X admits a slice (i.e. a non-empty intersection with a half-space) of arbitrarily small diameter. There are many other equivalent formulations; we refer to [4] or [5] for further enlightenment. Amongst other facts which we don't actually need here, Asplund spaces coincide with spaces whose duals have RNP. In particular, separable dual spaces have the Radon-Nikodým Property.

Of course the SCP is only interesting for non-separable spaces. It is fairly clear that for any $1 \leq p < \infty$ and any measure μ , the familiar spaces $L_p(\mu)$ have SCP. However $L_\infty(\mu)$ does not have any non-trivial separable complemented subspaces. Some other sufficient conditions for the separable complementation property are: being a Banach lattice not containing c_0 [9, p. 83], being the predual of a von Neumann algebra [8, pp. 111–112], having a so-called *projective generator* [6, Chapter 6, and references therein]. The latter class includes all weakly compactly generated spaces (for which the SCP was first established in [2, Lemma 4]) and all dual spaces with the RNP.

An earlier sufficient condition involves Markuševič bases. Recall that a Markuševič basis $(x_\gamma, f_\gamma)_{\gamma \in \Gamma}$ for a Banach space X is a biorthogonal system for which $(x_\gamma)_{\gamma \in \Gamma}$ generates a dense subspace of X and $(f_\gamma)_{\gamma \in \Gamma}$ generates a weak* dense subspace of X^* . For the moment, write $\Phi(f) = \{x_\gamma : f(x_\gamma) \neq 0\}$ for each $f \in X^*$. According to [14], a Markuševič basis $(x_\gamma, f_\gamma)_{\gamma \in \Gamma}$ is said to be *countably norming* if the collection of functionals $M = \{f \in X^* : \Phi(f) \text{ is countable}\}$ forms a norming subspace of X^* , i.e. if $\|x\| = \sup\{f(x) : f \in M, \|f\| = 1\}$ defines an equivalent norm on X . It was shown in [14, Theorem 1] that any Banach space with a countably norming Markuševič basis has the SCP. However it is not hard to see that for such a space, renormed as above, Φ will be a projective generator.

We note in passing that not every Banach space is isomorphic to a subspace of a Banach space with SCP. That ℓ_∞ is not was proved without statement in [15, Theorem 3]. More complicated examples had been discussed earlier by Musial [13, p. 162]; they depend on his result [13, Theorem 1] that if X has the weak RNP and is isomorphic to a subspace of a Banach space with SCP, then X already has the RNP.

Our idea is to show that certain renormings of $\ell_1(\omega_1)$ fail what might be called the λ -SCP, for arbitrarily large real numbers λ . (As usual, we denote by ω_1 the first uncountable ordinal; by definition it is also a cardinal number.) We build on the technique developed in [16, Section 8], where a weaker and more technical result about large families of projections was proved. We begin with the following simple combinatorial result [16, Lemma 8.2].

LEMMA 1. *There is an uncountable collection $\{N_\alpha: 0 \leq \alpha < \omega_1\}$ of infinite subsets of the integers such that*

(i) *the intersection of any two is finite,*

(ii) *if A is an uncountable subset of $[0, \omega_1)$, and β_1, \dots, β_k are ordinals outside A , then there is an infinite set $B \subset A$ and an integer $j \notin \bigcup_{i=1}^k N_{\beta_i}$ with $j \in N_\alpha$ for all $\alpha \in B$, and*

(iii) *given finitely many distinct ordinals $\alpha_1, \dots, \alpha_n$ in $[\omega_0, \omega_1)$, there exist distinct ordinals $\gamma_1, \dots, \gamma_n$, in $[0, \omega_0)$, such that the $2n$ sets $N_{\alpha_k} \setminus N_{\gamma_k}$ and $N_{\gamma_k} \setminus N_{\alpha_k}$ are pairwise disjoint.*

Proof. We simply recall the construction from [16], and leave the reader to check the details. Rather than the integers, we work on the dyadic tree, D . The collection of all its branches is undoubtedly uncountable and clearly the intersection of any two is finite. The collection of branches which, from some point onwards, turn only to the left is clearly countable; we will label it as $\{N_\alpha: 0 \leq \alpha < \omega_0\}$. For $\{N_\alpha: \omega_0 \leq \alpha < \omega_1\}$, we choose a subset of the remaining branches with cardinality ω_1 . ■

Now we fix one such family N_α . For each $\alpha < \omega_1$, we denote by y_α the characteristic function of N_α and by e_α the usual basis vector in $\ell_1(\omega_1)$. Then we can define a bounded linear operator $T: \ell_1(\omega_1) \rightarrow \ell_\infty$ by $T(e_\alpha) = y_\alpha$. To avoid confusion, we will write $\|\cdot\|_1$ and $\|\cdot\|_\infty$ for the natural norms on $\ell_1(\omega_1)$ and ℓ_∞ . For each integer n , we define an equivalent norm on $\ell_1(\omega_1)$ by

$$\|x\|_n = \max\{n^{-1} \|x\|_1, \|Tx\|_\infty\}.$$

Call this space X_n , and write L_n for the subspace $\ell_1(\omega_0)$. (For convenience, we will denote by $\ell_1(\beta)$ the subspace $\{x \in \ell_1(\omega_1) : x_\alpha = 0 \text{ for all } \alpha \geq \beta\}$.) For any Banach spaces Y and Z , we denote by $Y \oplus_1 Z$ their direct sum equipped with the norm $\|y + z\| = \|y\| + \|z\|$.

LEMMA 2. *Fix a natural number n . Then the space $X = X_n$ just defined has the following property: if Z is any Banach space whatsoever and U is any complemented separable subspace of $X \oplus_1 Z$ which contains L_n , then any projection from $X \oplus_1 Z$ onto U will have norm at least $\frac{1}{4}(n - 3)$.*

Proof. Throughout this proof, we omit the subscript n . We need to show that if V is any complement for U , then the natural projection $U \oplus V \rightarrow U$ has norm greater than $\frac{1}{4}(n - 3)$: to do this we will find an element of large norm, which is close to both U and V .

Write $P: X \oplus Z \rightarrow X$ for the natural projection. By separability, and the regularity of ω_1 , there is an index $\alpha_0 < \omega_1$ for which $P(U)$ is contained in $\ell_1(\alpha_0)$. In other words, $U \subset \ell_1(\alpha_0) + Z$. Note that $\alpha_0 \geq \omega_0$ because $P(U) \supseteq L$.

It is clear that each $x \in U$ is the limit of a sequence in $B(x, 1/n) \cap (\text{linsp}\{e_\alpha : \alpha < \alpha_0\} + Z)$. Since U is separable, we can then find a sequence $(w_m)_{m=1}^\infty$ in $(\text{linsp}\{e_\alpha : \alpha < \alpha_0\} + Z) \cap (U + B(0, 1/n))$ whose closure contains U .

For any $\alpha \geq \alpha_0$, the decomposition $U \oplus V$ applied to e_α gives us an index m_α such that $d(e_\alpha + w_{m_\alpha}, V) < n^{-1}$. Since $[\alpha_0, \omega_1)$ is uncountable, it must contain an uncountable subset A , every member α of which satisfies $d(e_\alpha + w, V) < n^{-1}$ for some fixed element $w = w_m$. This element w must have the form $w = \sum_{i=1}^k \lambda_i e_{\beta_i} + z$ for some scalars λ_i , ordinals $\beta_i \in [0, \alpha_0)$ and $z \in Z$.

Let $B \subset A$ and $j \in \bigcap_{\alpha \in B} N_\alpha \setminus \bigcup_{i=1}^k N_{\beta_i}$ be given by Lemma 1(ii). Being infinite, B contains at least n elements $\alpha_1, \dots, \alpha_n$. Lemma 1(iii) gives us corresponding finite ordinals $\gamma_1, \dots, \gamma_n$. We define now elements $u \in \ell_1(\omega_1)$ and $v \in \ell_\infty$ by $u = \sum_{i=1}^n e_{\alpha_i}$ and $v = \sum_{i=1}^n y_{\alpha_i}$. Clearly $Tu = v$ and $\|u\|_1 = n$. A moment's reflection shows that $\|\sum_{i=1}^n y_{\alpha_i} - v\|_\infty = 1$. If we put $x = \sum_{i=1}^n e_{\alpha_i} + nw$, then $x - nw - u$ will belong to X , although x itself need not. Now

$$\begin{aligned} \|x\| &= \left\| \sum_{i=1}^n e_{\alpha_i} + n \left(\sum_{i=1}^k \lambda_i e_{\beta_i} + z \right) \right\| \\ &= \left\| \sum_{i=1}^n e_{\alpha_i} + n \sum_{i=1}^k \lambda_i e_{\beta_i} \right\| + \|nz\| \\ &\geq \left\| T \left(\sum_{i=1}^n e_{\alpha_i} + n \sum_{i=1}^k \lambda_i e_{\beta_i} \right) \right\|_\infty + 0 \\ &= \left\| \sum_{i=1}^n y_{\alpha_i} + n \sum_{i=1}^k \lambda_i y_{\beta_i} \right\|_\infty \\ &\geq \left(\sum_{i=1}^n y_{\alpha_i} + n \sum_{i=1}^k \lambda_i y_{\beta_i} \right) (j) \\ &= \sum_{i=1}^n y_{\alpha_i} (j) + 0 = n \end{aligned}$$

and

$$\begin{aligned} d(x, U) &\leq \|x - nw - u\| + d(nw - u, U) \\ &= \max \left\{ \frac{1}{n} \|x - nw - u\|, \|T(x - nw - u)\| \right\} + d(nw, U) \\ &\leq \max \left\{ \frac{1}{n} \left\| \sum_{i=1}^n e_{\alpha_i} - u \right\|, \left\| \sum_{i=1}^n y_{\alpha_i} - v \right\| \right\} + 1 \\ &= 3. \end{aligned}$$

With less difficulty, $d(x, V) \leq \sum_{i=1}^n d(e_{\alpha_i} + w, V) < 1$.

To finish the proof, note that the natural projection $U \oplus V \rightarrow U$ has norm at least $(\|x\| - d(x, U))/(d(x, U) + d(x, V))$. ■

THEOREM 3. *There exists a Banach space X with the Radon-Nikodým Property but without the separable complementation property. Moreover, X has a strong Markušević basis, but no norming or countably norming Markušević basis, and no projective generator.*

Proof. If X is the ℓ_1 sum of the Banach spaces X_n , and L is the sum of the subspaces L_n , the previous Lemma shows that L is not contained in any complemented separable subspace of X . It is clear from our preliminary remarks that X has the Radon-Nikodým Property, and no countably norming Markušević basis. As for strong Markušević bases, this means [15, p. 638] that for every $\Delta \subset \Gamma$, the closed linear span of $\{x_\gamma; \gamma \in \Delta\}$ coincides with $\{f_\gamma; \gamma \in \Delta\}^0$. Such a basis can easily be constructed in X , since each component X_n has one. ■

Note that although each X_n is isomorphic to $\ell_1(\omega_1)$, their ℓ_1 direct sum is not even isomorphic to a dual space. We do not know whether it is isomorphic to a subspace of a dual space with RNP.

The Banach space just constructed has density character ω_1 . A modification of our argument will obviously yield examples with larger density characters. Rather than slogging through the transfinite combinatorics, it is easier to note that such examples can be constructed simply taking the direct sum of X and a suitably large Hilbert space.

The choice of the ℓ_1 norm for our direct sums was arbitrary; the same argument works equally well with the ℓ_p norm if p is finite. For the c_0 direct sum, we get a Banach space without the SCP, but also without the RNP.

We suspect that there are Banach spaces with RNP, but without any complemented infinite-dimensional separable subspaces. Our space is obviously not a counterexample to this harder problem. Kernels of particular quotient maps from $\ell_1(\alpha)$ onto ℓ_2 (which exist if α has cardinality at least the continuum) were shown in [15, Section 4] to be counterexamples to another complementation problem. They may turn out to be counterexamples to this problem as well.

Some related problems are discussed in [3, Chap. VI] and [16, Section 8]. (We refer to either of these, or to [6], for the definition of PRI.) In particular, [3, Problem VI.1] asks whether there is a subspace of $\ell_1(\omega_1)$ which does not have a PRI (Projectional Resolution of the Identity) in any equivalent norm. There it was suggested considering the kernels of quotient maps from $\ell_1(\omega_1)$ onto ℓ_∞ (which clearly exist under the Continuum Hypothesis). Such kernels are also candidates for a counterexample to the preceding problem, and to the more general problem of finding a Banach

space which fails the SCP but which embeds isomorphically into a dual space with RNP.

As the referee has pointed out, the existence of a non-separable hereditarily indecomposable space with RNP is also an open problem. Recall that a Banach space is *decomposable* if it is the direct sum of two closed infinite-dimensional subspaces; *hereditarily indecomposable* if it has no decomposable subspaces. The first infinite-dimensional examples of hereditarily indecomposable spaces [10] were all separable and reflexive; in particular they had RNP. We observed in [16, Proposition 3.2] that a hereditarily indecomposable space cannot be too large, in the sense that its cardinality must equal the continuum. Recently, S. Argyros has constructed the first example of a non-separable hereditarily indecomposable Banach space. He has informed us that his example is the dual of a separable space, and thus it cannot enjoy RNP.

We note that the following statement [5, p.208] is still true. "It is unknown to this day whether each Banach space with the Radon-Nikodým property admits an equivalent strictly convex norm." We are equally ignorant about locally uniformly convex norms. Fabian and Godefroy [7] proved that every dual space with RNP has an equivalent locally uniformly convex norm, but for non-dual spaces, the question remains open. That our example does admit such a norm follows from its possession of a strong Markušević basis [1], or by direct calculation.

Some recent progress on this question has been made by Moltó *et al.* [12]. Recall that a Banach space has the Kadets-Klee Property if its weak topology agrees with the norm topology on the unit sphere. It is an easy exercise to show that every locally uniformly convex space has this property. The following partial converse appears in [12]: if a Banach space has the Krein-Milman Property (which is implied by RNP) and the Kadets-Klee Property, then it has an equivalent locally uniformly convex norm.

ACKNOWLEDGMENT

We are indebted to Ondřej Kalenda for numerous helpful comments.

REFERENCES

1. G. Alexandrov and A. M. Plichko, On connection between strong M -bases and equivalent locally uniformly convex norms in Banach spaces, *C. R. Acad. Bulgare Sci.* **40** (2) (1987), 15–16. [In Russian]
2. D. Amir and J. Lindenstrauss, The structure of weakly compact sets in Banach spaces, *Ann. of Math.* **88** (1968), 35–46.

3. R. Deville, G. Godefroy, and V. Zizler, "Smoothness and Renormings in Banach Spaces," Pitman Monographs and Surveys in Pure and Applied Math., Vol. 64, Longman, Essex, 1993.
4. J. Diestel and J. J. Uhl, Jr., The Radon-Nikodým theorem for Banach space valued measures, *Rocky Mountain Math. J.* **6** (1976), 1–46.
5. J. Diestel and J. J. Uhl, Jr., "Vector Measures," Amer. Math. Soc., Providence, RI, 1977.
6. M. J. Fabian, "Gâteaux Differentiability of Convex Functions and Topology. Weak Asplund Spaces," Wiley-Interscience, New York, 1997.
7. M. Fabian and G. Godefroy, The dual of every Asplund space admits a projectional resolution of the identity, *Studia Math.* **91** (1988), 141–151.
8. N. Ghoussoub, G. Godefroy, B. Maurey, and W. Schachermayer, Some topological and geometrical structures in Banach spaces, *Mem. Amer. Math. Soc.* **378** (1987).
9. N. Ghoussoub and E. Saab, On the weak Radon-Nikodým property, *Proc. Amer. Math. Soc.* **81** (1981), 81–84.
10. W. T. Gowers and B. Maurey, The unconditional basic sequence problem, *J. Amer. Math. Soc.* **6** (1993), 851–874.
11. J. Lindenstrauss and L. Tzafriri, "Classical Banach Spaces II. Function Spaces," Springer-Verlag, Berlin, 1979.
12. A. Moltó, J. Orihuela, S. Troyanski, and M. Valdivia, Kadets and Krein-Milman properties, *C. R. Acad. Sci. Paris Sér. I Math.* **331** (2000), 459–464.
13. K. Musiał, The weak Radon-Nikodým property in Banach spaces, *Studia Math.* **64** (1979), 151–173.
14. A. M. Plichko, On projective resolutions of the identity operator and Markuševič bases, *Dokl. Akad. Nauk SSSR* **263**(3) (1982), 543–546. [In Russian; English transl.: *Soviet Math. Dokl.* **25** (1982), 386–389.].
15. A. M. Plichko, On bases and complements in nonseparable Banach spaces, *Sibirsk. Mat. J.* **25**(4) (1984), 155–162. [In Russian; English transl.: *Siber. Math. J.* **25** (1984), 636–641.].
16. A. M. Plichko and D. Yost, Complemented and uncomplemented subspaces of Banach spaces, *Extracta Math.* **15** (2000), 335–371.
17. B. Sims and D. Yost, Banach spaces with many projections, *Proc. Centre Math. Anal. Austr. Nat. Univ.* **14** (1986), 335–342.