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## A SEPARABLE SPACE WITH NO SCHAUDER DECOMPOSITION

G. ALLEXANDROV, DENKA KUTZAROVA, AND A. PLICHKO

(Communicated by Dale Alspach)

**ABSTRACT.** We combine some known results to remark that there exists a separable Banach space which fails to have a Schauder decomposition. It can be chosen as a subspace of Gowers-Maurey space without any unconditional basic sequence.

The following problem was raised in [Si] (Problem 15.1, p. 494): Does every separable Banach space have a Schauder decomposition? This question goes back to J. R. Retherford [R].

Recall that a sequence  $\{X_n\}_{n=1}^{\infty}$  of closed subspaces of a Banach space  $X$  is said to be a Schauder decomposition of  $X$  if every  $x \in X$  has a unique representation of the form  $x = \sum_{n=1}^{\infty} x_n$ , with  $x_n \in X_n$  for every  $n$ .

Let  $GM$  be Gowers-Maurey space which does not contain any unconditional basic sequence [GM]. As was observed by W. B. Johnson,  $GM$  has in fact a stronger property, namely it is hereditarily indecomposable (*H.I.*); i.e., no infinite-dimensional closed subspace can be written as a direct sum  $Y \oplus Z$ , where  $Y$  and  $Z$  are infinite-dimensional closed subspaces. It is known that every block subspace of  $GM$  contains uniform copies of  $\ell_1^n$ . This follows from the lower  $f$ -estimate and Krivine's theorem as in [S]. Then, by Szankowski's refinement of Enflo's criterion (see [LT2, p. 111, Remark 1]), we immediately obtain the following.

**Proposition.** *There exists a subspace  $X$  of  $GM$  which does not have the compact approximation property (*C.A.P.*).*

*Remark 1.* For the same purpose we can as well use other *H.I.* spaces constructed after the breakthrough of W. T. Gowers and B. Maurey. For example, there are subspaces without the *C.A.P.* of the super-reflexive *H.I.* spaces in [F] in the case when they contain uniform copies of  $\ell_p^n$  for  $p \neq 2$ . One can also use the asymptotic  $\ell_1$  hereditarily indecomposable spaces constructed in [AD] and [ADKM]. The existence of uniform copies of  $\ell_1^n$  in these spaces follows directly from the definition and one does not need to apply Krivine's theorem. Therefore, they also have subspaces without the *C.A.P.*

**Corollary.** *The space  $X$  is an example of a separable Banach space with no Schauder decomposition.*

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*Proof.* Assume the contrary, i.e.  $X$  has a Schauder decomposition  $\{X_n\}_{n=1}^\infty$ .

Case 1.  $\{X_n\}_{n=1}^\infty$  is a finite-dimensional decomposition. This is impossible since the existence of an *F.D.D.* implies *B.A.P.* which in turn implies *C.A.P.* (see [LT1]) and this contradicts the above Proposition.

Case 2. There exists  $m$  such that  $X_m$  is infinite-dimensional. Denote  $Y = [X_n : n \neq m]$ . Then  $X = X_m \oplus Y$  which is also impossible because  $X_m$  and  $Y$  are closed infinite-dimensional subspaces of  $X$ ,  $X$  is a closed subspace of  $GM$ , and  $GM$  is *H.I.*  $\square$

*Remark 2.* Clearly, the result is true hereditarily in all the above mentioned *H.I.* spaces, e.g. we have that every subspace of  $GM$  has a further subspace which has no Schauder decomposition.

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