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ON A PROBLEM OF “SCOTTISH BOOK” CONCERNING CONDENSATIONS OF METRIC SPACES ONTO COMPACTA

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It is proved that every Banach space of density ω_1 admits a condensation onto the Hilbert cube.

1. Introduction

In this note we consider a question posed in 1935 in the known book of Lviv mathematicians [1]. In the modern terminology this question sounds as follows.

Problem (S. Banach). *When does a metric (possibly Banach) space X admit a condensation (i.e. a bijective continuous map) onto a compactum (= compact metric space)?*

We do not know the origins of this question. An obvious impulse for its appearance could be a well known result stating that every continuous image of a compactum is compact. Another impulse could come from the note [2], where a condensation of the space of all irrationales onto $[0,1]$ was constructed.

Independently to S. Banach the problems concerning condensations were posed by P.S. Aleksandrov and investigated intensively by the Moscow topological school [3]–[6]. In particular, it was proven in [6] (see also [7]) that every separable absolute Borel space X condenses onto the Hilbert cube $Q=[-1,1]^\omega$, whenever X is not σ-compact. This result implies that every infinite-dimensional separable Banach space condenses onto the Hilbert cube. It is an easy exercise to show that each separable metrizable locally compact space (and thus each finite-dimensional Banach space) condenses onto a compactum. Thus every separable Banach space admits a condensation onto a compactum. In the meantime, the “Banach” part of the problem still remains open.

Question 1. *Does every Banach space of density $\le \mathfrak{c}$ condenses onto a compactum?*

The main result of this paper is the positive answer to this question for Banach spaces of density ω_1 or $\mathfrak{c}$.

In fact, for the density $=\mathfrak{c}$ the proof is quite easy: According to [8], every Banach space X of density $\mathfrak{c}$ is homeomorphic to the Banach space l^∞ of all bounded sequences equipped with the sup-norm. By [9, p.190], the closed unit ball B of

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l^∞ is homeomorphic to l^∞. Thus X is homeomorphic to B. But B as a set coincides with the Hilbert cube $Q = [-1,1]^\omega$. Moreover the identity map from B equipped with the norm-topology onto $B = Q$ endowed with the product topology is a condensation. Consequently, X condenses onto Q.

Since every Banach space of density ω_1 is homeomorphic to $l^2(\omega_1)$, the Hilbert space of density ω_1 [8], the proclaimed result follows from

Main Theorem. *The Hilbert space $l^2(\omega_1)$ condenses onto the Hilbert cube Q.*

Unlike to the case of density $\mathfrak{c}$, the proof of this theorem requires more sophisticated (and maybe even artificial) arguments involving some nontrivial results on the topology of probability measure spaces.

2. Preliminaries

All spaces considered are assumed to be metrizable. Denote by $P(X)$ the space of all probability Radon measures on a space X, equipped with the $*$-weak topology (for details see [10]). Let us recall that a measure μ on X is called *Radon* if for every $\varepsilon > 0$ there is a compact subset $K \subset X$ such that $\mu(X \setminus K) < \varepsilon$. If $f: X \to Y$ is a map then the formula $P(f)(\mu)(A) = \mu\big(f^{-1}(A)\big)$, A a Borel subset of Y, determines a continuous map $P(f): P(X) \to P(Y)$. A measure $\mu \in P(X)$ is called *discrete*, if $\mu(C) = 1$ for some countable subset $C \subset X$. Denote by $P_d(X)$ the subspace in $P(X)$ consisting of all discrete measures. Evidently for a map $f: X \to Y$ we have $P(f)\big(P_d(X)\big) \subset P_d(Y)$, i.e. the map $P_d(f) = P(f)|P_d(X): P_d(X) \to P_d(Y)$ is well-defined. We will need the following trivial fact.

Lemma 1. *If $f: X \to Y$ is a condensation, then the map $P_d(f): P_d(X) \to P_d(Y)$ is a condensation.*

A similar result holds also for the map $P(f)$, see [10].

Lemma 2. *If $f: X \to Y$ is a condensation of a separable Borel space X then $P(f): P(X) \to P(Y)$ is a condensation.*

We need also some facts on the topology of spaces $P(X)$ and $P_d(X)$.

Lemma 3 ([11] or [12]). *For every complete-metrizable space X of density ω_1 the probability measure space $P(X)$ is homeomorphic to $l^2(\omega_1)$.*

In the following lemmas $s = (-1,1)^\omega$ is the pseudo-interior of the Hilbert cube, and Ω_2, Λ_2 are certain special subsets in Q, the so-called absorbing sets for the classes of all absolute $F_{\sigma\delta}$-sets and $G_{\delta\sigma}$-sets, respectively. By $\mathcal{M}_0$ we denote the class of all compacta, $\mathcal{M}_1$ is the class of all Polish (= separable complete-metrizable) spaces, and $\mathcal{M}_2$ denotes the class of all separable absolute $F_{\sigma\delta}$-sets, see [13] or [14]. The following lemmas are proved in [12], see also [14, §5.6].

Lemma 4. *If X is a Polish infinite space then $P_d(X)$ is homeomorphic to one of the spaces Q, s, Ω_2, $Q \setminus \Lambda_2$.*

Recall that a space X is called a *Baire space* if the intersection of any countable collection of dense open subsets in X is dense in X.

Lemma 5. *For an infinite space X the space $P(X)$ is homeomorphic to*

(1) *Q iff $X \in \mathcal{M}_0$;*
(2) *s iff $X \in \mathcal{M}_1 \setminus \mathcal{M}_0$;*
(3) *Ω_2 iff $X \in \mathcal{M}_2 \setminus \mathcal{M}_1$ and X is not a Baire space;*
(4) *$Q \setminus \Lambda_2$ iff $X \in \mathcal{M}_2 \setminus \mathcal{M}_1$ and X is a Baire space.*

The following lemma is a consequence of Theorem 2 of [15, §37] stating that every separable absolute Borel space is a continuous bijective image of a Polish noncompact space and a result of [6] claiming that every separable non-σ-compact Borel space condenses onto a compactum.

Lemma 6. *There are condensations*

(1) $f_1\colon X_1 \to Y_1$, *where* $X_1 \in \mathcal{M}_1 \setminus \mathcal{M}_0$, $Y_1 \in \mathcal{M}_0$;
(2) $f_2\colon X_2 \to Y_2$, *where* $X_2 \in \mathcal{M}_1 \setminus \mathcal{M}_0$, $Y_2 \in \mathcal{M}_2 \setminus \mathcal{M}_1$, *and* Y_2 *is not a Baire space;*
(3) $f_3\colon X_3 \to Y_3$, *where* $X_3 \in \mathcal{M}_1 \setminus \mathcal{M}_0$, $Y_3 \in \mathcal{M}_2 \setminus \mathcal{M}_1$, *and* Y_3 *is a Baire space;*
(4) $f_4\colon X_4 \to Y_4$, *where* $Y_4 \in \mathcal{M}_0$, $X_4 \in \mathcal{M}_2 \setminus \mathcal{M}_1$, *and* X_4 *is not a Baire space;*
(5) $f_5\colon X_5 \to Y_5$, *where* $Y_5 \in \mathcal{M}_0$, $X_5 \in \mathcal{M}_2 \setminus \mathcal{M}_1$, *and* X_5 *is a Baire space;*

Applying the construction P to the condensations f_1–f_5 and using Lemmas 2 and 5 we get

Lemma 7. *There are condensations* $s \to Q$, $s \to \Omega_2$, $s \to Q \setminus \Lambda_2$, $\Omega_2 \to Q$, *and* $Q \setminus \Lambda_2 \to Q$.

3. Proof of Main Theorem

Let K be an uncountable compactum. According to Corollary 4 of [15, §39.II], K can be expressed as a union $K = \bigcup_{i\in\omega_1} K_i$ of ω_1 disjoint Borel subsets of K. Without loss of generality, each K_i is infinite. It follows from Theorem 2 of [15, §37] that for every $i \in \omega_1$ there is a condensation $f_i\colon X_i \to K_i$ of a Polish noncompact space X_i onto K_i. Then the discrete sum $X = \bigsqcup_{i\in\omega_1} f_i\colon X = \bigsqcup_{i\in\omega_1} X_i \to \bigcup_{i\in\omega_1} K_i = K$ is a condensation. By Lemma 3, the space $P(X)$ is homeomorphic to $l^2(\omega_1)$ and by Lemma 1, the map $P_d(f)\colon P_d(X) \to P_d(K)$ is a condensation. According to Lemmas 4 and 7 the space $P_d(K)$ admits a condensation onto Q. Thus, to prove our theorem, it is enough to construct a condensation of $P(X)$ onto $P_d(X)$.

It follows from Lemmas 4, 5, and 7 that for every $i \in \omega_1$ there is a condensation $g_i\colon P(X_i) \to P_d(X_i)$. We define a condensation $g\colon P(X) \to P_d(X)$ as follows. Given a measure $\mu \in P(X) = P(\bigsqcup_{i\in\omega_1} X_i)$ let $\mathcal{I} = \{i \in \omega_1 \mid \mu(X_i) > 0\}$. Then μ can be written in a unique way as $\mu = \sum_{i\in\mathcal{I}} \mu(X_i)\mu_i$, where $\mu_i \in P(X_i)$ for $i \in \omega_1$. Finally, letting $g(\mu) = \sum_{i\in\mathcal{I}} \mu(X_i) g_i(\mu_i)$ we define a required condensation $g\colon P(X) \to P_d(X)$. □

4. Concluding remarks and questions

It follows from the Pytkeev result [6] that every separable Fréchet (= locally convex linear complete metric) space condenses onto a compactum. In fact, we can say more.

Proposition. *For every separable infinite-dimensional Fréchet space* X *there is a condensation* $f\colon X \to Q$ *such that* f^{-1} *is of the first Baire class, or following the terminology of* [15], f *is a generalized homeomorphism of the class* $(0,1)$.

Proof. It is well known [9, p.189, 190] that X is homeomorphic to the closed unit ball B of the separable Hilbert space l^2. The ball B equipped with the weak topology is an infinite-dimensional convex compactum and thus is homeomorphic to the Hilbert cube according to Keller Theorem [9, p.100]. Finally, the remark that the identity map $(B, weak) \to (B, norm)$ is of the first Baire class, see [16, p.187], completes the proof. □

In light of this proposition the following question seems to be natural.

Question 2. *Does every linear separable complete metric space X admit a condensation $f\colon X \to K$ onto a compactum such that f^{-1} is of the first Baire class?*

Using the terminology of [15] we may pose the question more generally.

Question 3. *Let X be a separable absolute Borel space of the multiplicative class α, α a countable ordinal, such that X is not σ-compact. Does then X admit a condensation $f\colon X \to K$ onto a compactum such that f is a generalized homeomorphism of the class $(0,\alpha)$?*

Notice that the condensation of the space of irrationals onto [0,1], constructed in [2], is actually a generalized homeomorphism of the class $(0,1)$.

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