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BANACH SPACES WITHOUT THE KADEC H-PROPERTY (SOLUTION OF A PROBLEM FROM THE "SCOTTISH BOOK")

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Dedicated to Professor V. Lyantse on his 75th birthday

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It is proved that if a separable Banach space X has not the Schur property then there exists an equivalent strictly convex norm on X without H-property.

In the book [2] S. Mazur rose the following question (Problem 89). "Let W be a convex body, located in the space (L^2), and such that its boundary W_b does not contain any interval; let $x_n \in W$, $(n = 1, 2, ...)$, $x_0 \in W_b$ and in addition let the sequence (x_n) converge weakly to x_0. Does then the sequence (x_n) converge strongly to x_0 ? It is known that this statement is true in the case where W is a sphere. Examine this problem for the case of other spaces."

A norm $\| \ \|$ of a Banach space X is called strictly convex if $\|x + y\| < \|x\| + \|y\|$ for any linearly independent elements $x, y \in X$. It has the Kadec H-property if $x_n \to 0$ weakly and $\|x_n\| \to \|x\|$ implies $\|x_n - x\| \to 0$ as $n \to \infty$ [1]. The Problem 89 is close to the following question: Has every equivalent strictly convex norm on L^2 H-property? It was well known that original norm on the space L^p, $1 < p < \infty$ has the H-property [4, 5]; it is easy to see that any locally uniformly convex norm on a Banach space has this property. The following proposition gives a negative solution of the Mazur problem.

Proposition. *Suppose a separable Banach space $(X, \| \ \|)$ has not the Schur property, i.e. it has a weakly converging sequence (x_n) which does not converge in norm. Then there exists an equivalent strictly convex norm on X without H-property.*

Proof. Let y_0 be an element of X having the unit norm and let $H \subset X$ be a hyperplane such that $\operatorname{dist}(y_0, H) = 1$. Without loss of generality we can suppose that $x_n \in H$, $n = \overline{1, \infty}$ and $\inf_{m \neq n} \|x_n - x_m\| > 0$. Then the sequence $y_n = (x_{n+1} - x_n)/\|x_{n+1} - x_n\|$, $n = \overline{1, \infty}$ converges to 0 weakly but no in norm. As it is well known, it contains a basic subsequence (which we denote by the same symbol (y_n))[3, p.48]. Let $\| \ \|_0$ be the norm generated by the closed covex hull of the unit ball of the space X and the elements $\pm(y_0 + y_n)$, $n = \overline{1, \infty}$. Then the norm $\| \ \|_0$ is equivalent to the norm $\| \ \|$ and $\|y_0\|_0 = \|y_0 + y_n\|_0 = 1$, $n = \overline{1, \infty}$. Extend the

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sequence $(y_n)_0^\infty$ to a Markushevich basis $(z_n)_0^\infty$ of the space X, $z_0 = y_0$ [3, p.231]. (We recall, that a sequence (z_n) is called a Markushevich basis provided $[z_n]_1^\infty = X$ and there exist linear continuous functionals (f_n) on X such that $f_m(z_n) = 1$ for $m = n$ and $= 0$ for $m \neq n$, and for every $x \neq 0$ there exists n such that $f_n(x) \neq 0$). Define the norm

$$\|x\|_1 = \left(\sum_0^\infty |f_n(x)|^2 z_n / (2^n \|z_n\| \; \|f_n\|^2) \right)^{1/2}$$

which is weaker than the norm $\| \; \|$ and strictly convex. Put finally $|||x||| = \|x\|_0 + \|x\|_1$. This norm is equivalent to the norm $\| \; \|$ and strictly convex. Of course, the sequence $u_n = y_0 + y_n$ converges to y_0 weakly, $|||u_n||| \to |||y_0|||$ but $|||u_n - y_0||| \not\to 0$ as $n \to \infty$. Therefore the strictly convex equivalent norm $||| \; |||$ has not the H-property.

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