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ON THE VOLUME METHOD IN THE STUDY OF AUERBACH BASES OF FINITE-DIMENSIONAL NORMED SPACES

BY

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In this note we show that if the ratio of the minimal volume V of n-dimensional parallelepipeds containing the unit ball of an n-dimensional real normed space X to the maximal volume v of n-dimensional crosspolytopes inscribed in this ball is equal to $n!$, then the relation of orthogonality in X is symmetric. Hence we deduce the following properties: (i) if $V/v = n!$ and if $n > 2$, then X is an inner product space; (ii) in every finite-dimensional normed space there exist at least two different Auerbach bases and (iii) the finite-dimensional normed space X is an inner product space provided any two Auerbach bases are isometrically equivalent. Property (i) generalizes a result of Lenz [8], and (iii) answers a question of R. J. Knowles and T. A. Cook [7].

An element x of a normed space X is said to be *orthogonal* to an element y if $\|x\| \le \|x + \lambda y\|$ for every real number λ; we then write $x \perp y$. A basis $(e_i)_{i=1}^n$ of an n-dimensional normed space is called an *Auerbach basis* provided $\|e_i\| = 1$ for every i and every e_i is orthogonal to any element of the linear span $\mathrm{lin}(e_j : j \neq i)$. It seems that for the first time the existence of an Auerbach basis in every two-dimensional space was established in [1] in terms of conjugate diameters. Unfortunately, we do not know Auerbach's original proof. The definition of an Auerbach basis and the Auerbach theorem in terms of normed spaces appear in Banach's book [4, Remarks to Ch. VII]. Proofs were given in the notes [5] and [14].

Auerbach bases of an n-dimensional linear normed space X are obtained as the centers of faces of a minimum volume n-dimensional parallelepiped circumscribed about the unit ball $B(X)$ and from the vertices of a maximum volume n-dimensional crosspolytope inscribed in $B(X)$. By an *n-dimensional crosspolytope* we mean the convex hull $\mathrm{conv}(\pm e_i)_{i=1}^n$ of $2n$ elements called its *vertices*. Concrete examples show that the Auerbach bases obtained by the above minimization and maximization are usually distinct.

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Exceptions are inner product spaces and some two-dimensional spaces. That is why the following questions naturally came up. Does every n-dimensional normed space, where $n > 1$, have at least two different Auerbach bases? (Here and in the sequel, we do not distinguish Auerbach bases which differ by ± 1 factors.) Is a finite-dimensional normed space an inner product space if for any two of its Auerbach bases there exists an isometry which transforms one basis to the other?

In [7] the first question was posed for $n = 2$ and the second for $n > 2$. There it is stated that an answer to the first question (for $n = 2$) was given by J. Simons. The existence of at least two different Auerbach bases in a two-dimensional normed space (probably under a smoothness condition) was noticed in [2]. The investigation of Auerbach bases in terms of conjugate diameters was continued in the theory of convex bodies. In particular, under some additional smoothness assumptions on the norm $\| \ \|$ in the finite-dimensional normed space there exist at least two different Auerbach bases (see [8]). There it is also proved that under additional smoothness assumptions we have $V/v \le n!$. Moreover, for $n > 2$ equality holds for an inner product space X only. For applications of Auerbach bases see for example [9; 10; 13; 11, B.4.9].

THEOREM. *If $V/v = n!$ for a real n-dimensional normed space X, then the orthogonality relation in X is symmetric, i.e. $x \perp y$ implies $y \perp x$.*

Proof. We prove the theorem by induction on n.

Let $n = 2$, $\|x\| = \|y\| = 1$, $x \perp y$ and $y \not\perp x$. Let us circumscribe about the ball $B(X)$ a parallelogram P with sides which are formed by the tangents at the points x, y. Moreover, the tangent at x is parallel to the vector y. The area of this parallelogram equals the area of the parallelogram Q whose sides contain the points x, y and are parallel to the vectors y, x respectively. Because of the nonorthogonality, the area of Q is strictly less than $2v = V$. Therefore, the area of the circumscribed parallelogram P is strictly less than V; a contradiction.

Now assume that the theorem is proved for $n-1$. If the orthogonality relation is nonsymmetric, then it is nonsymmetric in some $(n-1)$-dimensional subspace $Y \subset X$. Thus, in Y there exists an Auerbach basis $y_1, \ldots, y_{n-1}$ which spans an $(n-1)$-dimensional crosspolytope of nonmaximal $(n-1)$-dimensional volume. Let y_n be an element of unit norm contained in the intersection of the hyperplanes containing the origin and parallel to the hyperplanes which are tangent to $B(X)$ at the points y_i, where $i = 1, \ldots, n-1$. The hyperplanes tangent to $B(X)$ at the points y_i, where $i = 1, \ldots, n$, form a parallelepiped P circumscribed about $B(X)$. The volume of P is $n!$ times the volume of the crosspolytope $\operatorname{conv}(\pm y_i)_{i=1}^n$. Let $y_1', \ldots, y_{n-1}'$ be elements of $B(Y)$ spanning an $(n-1)$-dimensional crosspolytope of maximal volume.

We have

$$n!v = V \le \mathrm{Vol}(P) = n!\,\mathrm{Vol}(\mathrm{conv}(\pm y_i)_{i=1}^n) \\ < n!\,\mathrm{Vol}[\mathrm{conv}((\pm y_i')_{i=1}^{n-1}, \pm y_n)] \le n!v;$$

a contradiction.

COROLLARY 1. *In every n-dimensional normed space X we have $V/v \le n!$. If $n > 2$, then equality holds true in an inner product space only.*

The first part of the corollary is evident. For $n > 2$, the symmetry of the orthogonality implies that X is an inner product space (see [6]).

Remark. In a two-dimensional space which has a regular hexagon as unit ball we have $V/v = 2$.

COROLLARY 2. *In every n-dimensional normed space there are at least two distinct Auerbach bases.*

Indeed, if the Auerbach bases constructed by the maximization of the volume of inscribed crosspolytopes and by the minimization of the volume of circumscribed parallelepipeds coincide, then $V/v = n!$. By the Theorem, the orthogonality relation is then symmetric. Hence there are infinitely many Auerbach bases.

COROLLARY 3. *An n-dimensional normed space X is an inner product space provided for any two Auerbach bases there is an isometry of X which transforms one basis to the other.*

Proof. By Auerbach's result [3; 12, p. 408], there exists an inner product on X whose isometry group G contains the isometry group of the normed space X. This inner product generates the measure (volume) which is invariant with respect to its isometry group. If $V/v < n!$, then the Auerbach basis formed by the minimization of the volume of inscribed octahedrons cannot be isometrically transformed to the Auerbach basis formed by the maximization of the volume of circumscribed parallelepipeds. Therefore $V/v = n!$. Hence the orthogonality relation is symmetric. For $n > 2$ this finishes the proof (see Corollary 2).

Let $n = 2$. By symmetry of the orthogonality relation any element of the unit sphere is an element of some Auerbach basis. By our assumption, this implies that the isometry group of the normed space X is infinite and that it is a closed subgroup of G. It is easily seen that every infinite closed subgroup of G contains the subgroup of all rotations. Therefore, the norm of X coincides with the norm generated by the inner product.

QUESTION (for students). Suppose the orthogonality relation in a two-dimensional space is symmetric. Is $V/v = 2$?

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