LIMITS ON THE REAL LINE OF SYMMETRIC SPACES ON SEGMENTS

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In the same way as the known spaces $M_p, \mathfrak{M}_p,$ and I_p are constructed on the basis of the space $L_p(-1, 1)$, we construct the corresponding "limit" spaces $M_E, \mathfrak{M}_E,$ and I_E on the real line on the basis of a symmetric function space E on a segment and study some of their Banach properties.

In connection with some questions of generalized harmonic analysis, Marcinkiewicz [1] defined the class $\mathfrak{M}_p$, $1 < p < \infty$, as a set of Borel measurable functions $x(t)$ on the real line with

$$\|x\| = \limsup_{T\to\infty} \left[\frac{1}{2T} \int_{-T}^{T} |x(t)|^p dt \right]^{1/p} < \infty.$$

By identifying functions whose difference has norm zero, he proved that $(\mathfrak{M}_p, \|\ \|)$ is a Banach space. Later [2–4], a space similar to $\mathfrak{M}_p$, namely, the space M_p of functions such that

$$\|x\| = \sup_{T\geq 1} \left[\frac{1}{2T} \int_{-T}^{T} |x(t)|^p dt \right]^{1/p} < \infty,$$

and its subspace I_p consisting of functions for which

$$\lim_{T\to\infty} \left[\frac{1}{2T} \int_{-T}^{T} |x(t)|^p dt \right]^{1/p} = 0$$

were investigated. Evidently, $\mathfrak{M}_p = M_p / I_p$. The properties of these spaces having a direct application to some questions of analysis and the usual Banach properties were studied.

In the same way as the spaces M_p, $\mathfrak{M}_p$, and I_p were constructed on the basis of the space $L_p(-1, 1)$, we construct the corresponding "limit" spaces M_E, $\mathfrak{M}_E$, and I_E on the real line on the basis of a symmetric function space E on a segment and study some of their Banach properties. The majority of the properties obtained are known for M_p, but some of them are new. Naturally, the methods of proof are more abstract, it seems, less cumbersome, and more transparent with the point of view of the theory of Banach spaces. First, we consider an abstract construction, which may be called the inductive l_∞-limit of a sequence of Banach spaces.

1. Let X_n be a sequence of linear spaces, $Y_n = X_1 \oplus \ldots \oplus X_n$, and let Y_n be Banach spaces with norms $\|\ \|_n$. Assume also that, for each n and any $y \in Y_n$, we have $\|y\|_{n+1} \leq \|y\|_n$, and the projection of Y_{n+1} onto Y_n along X_{n+1} is bounded in the norm $\|\ \|_{n+1}$. Consider the set

$$X = \{x = (x_1, \ldots, x_n, \ldots) \colon x_n \in X_n, \ \sup_n \|(x_1, \ldots, x_n)\|_n < \infty\}.$$

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As usual, we identify the spaces X_n and Y_n with their natural imbedding into X, which is endowed by the coordinatewise linear operations. For $x = (x_1, \ldots, x_n, \ldots)$, we put $P_n x = (x_1, \ldots, x_n, 0, \ldots)$.

It is easy to see that, for a sequence of Banach spaces X_n and $1 < p < \infty$, the space $X = l_p(X_n)$ satisfies these conditions. However, we shall consider the applications to other spaces [see condition $(*)$ below].

Proposition 1. *The set* X *with norm*

$$\|x\| = \sup_n \|P_n x\|_n$$

is a Banach space.

Proof. It is easy to show that $(X, \|\ \|)$ is a linear normed space. Completeness can be verified as usual. Let x^k, $k = \overline{1, \infty}$, be a Cauchy sequence in the space X, i.e., for any $\varepsilon > 0$ there exists N such that, for each $j, k > N$, we have

$$\|x^j - x^k\| = \sup_n \|P_n x^j - P_n x^k\|_n < \varepsilon.$$

Then, for every n,

$$\|P_n x^j - P_n x^k\|_n < \varepsilon. \tag{1}$$

It is easy to see that every projection Q_n of X onto the subspace X_n along the $\|\ \|$-closed linear span $[X_m \colon m \neq n]$ is bounded in this norm and that the norms $\|\ \|$ and $\|\ \|_n$ coincide on the subspace X_n. Thus, $(X_n, \|\ \|)$ is a complete space. Then, for each n, $Q_n x^k$, $k = \overline{1, \infty}$, is a Cauchy sequence and, therefore, it converges to some element $x_n \in X_n$. Consequently,

$$P_n x^k = \sum_{m=1}^{n} Q_m x^k$$

converges to $\sum_1^n x_m$. Let us show that the sequence x^k, $k = \overline{1, \infty}$, converges to the element $x = (x_1, \ldots, x_n, \ldots)$ in the space X. For any fixed n, we fix k and pass to the limit in inequality (1) as $j \to \infty$. As a result, we obtain $\|P_n x - P_n x^k\|_n < \varepsilon$. This inequality is valid for every n; hence, $\|x - x^k\| \le \varepsilon$. This implies that $\|x - x^k\| \to 0$ as $k \to \infty$. Since

$$\sup_n \|P_n x^k\|_n < \infty \quad \text{and} \quad \sup_n \|P_n x - P_n x^k\|_n \le \varepsilon,$$

we get $\sup_n \|P_n x\| < \infty$. Therefore, $x \in X$. Thus, the space X is complete.

Proposition 2. *The space*

$$X_0 = \left\{ x \in X \colon \lim_n \|P_n x\|_n = 0 \right\}$$

is a closed linear subspace of X.

Proof. It is easy to verify that the set X_0 is linear. Let us show that it is closed. Assume that a sequence $x^k \in X_0$ converges to $x \in X$. Since $\lim_n \| P_n x^k \|_n = 0$ for any k and $\sup_n \| P_n x - P_n x^k \|_n \to 0$ as $k \to \infty$,

$$\limsup_n \| P_n x \|_n \le \limsup_n \| P_n x - P_n x^k \|_n + \limsup_n \| P_n x^k \|_n \le \sup_n \| P_n x - P_n x^k \|_n \to 0$$

as $k \to \infty$. Consequently, $\lim_n \| P_n x \|_n = 0$, i.e., $x \in X_0$.

Consider the space

$$Y = \left\{ y = (y_1, \dots, y_n, \dots) : y_n \in Y_n, \ \sup_n \| y_n \|_n < \infty \right\}$$

with norm $\| y \| = \sup_n \| y_n \|_n$ and its subspace $Y_0 = \left\{ y \in Y : \lim_n \| y_n \|_n = 0 \right\}$, i.e., $Y = l_\infty(Y_n)$ and $Y_0 = c_0(Y_n)$. It is easy to see that the mapping T that associates an element $x = (x_1, \dots, x_n, \dots) \in X$ with the element $Tx = y = (y_1, \dots, y_n, \dots)$, where $y_n = P_n x$, is a linear isometry of X onto some subspace of Y and $TX_0 \subset Y_0$.

Proposition 3. *If, for each n, Y_n is separable, then the same is true for X_0. If the dual spaces Y_n^* are separable, then the same is true for X_0^*.*

Proof. Since X_0 is isometric to a subspace of $c_0(Y_n)$ and X_0^* is isometric to a quotient space of $l_1(Y_n^*)$, this fact is obvious.

We say that *condition (*) is satisfied* if, for any $y_m \in Y_m$, $\| y_m \|_n \to 0$ as $n \to \infty$.

Evidently, if this condition is satisfied, then, for every n, $Y_n \subset X_0$.

Proposition 4. *Let condition (*) be satisfied. Then, for any $\varepsilon > 0$, the space X contains a complemented subspace Z, $(1+\varepsilon)$-isometric to l_∞; moreover, $Z \cap X_0$ contains a subspace $(1+\varepsilon)$-isometric to c_0.*

Proof. Let $\varepsilon_i > 0$, $\sum_1^\infty \varepsilon_i < \varepsilon$. We set $n_1 = 1$. For $i \ge 1$, we choose $x^i = (0, \dots, x_{n_i}, 0, \dots)$, $x_{n_i} \in X_{n_i}$, and n_{i+1} so that $\| x_{n_i} \|_{n_i} = 1$ and $\| x_{n_i} \|_{n_{i+1}} \le \varepsilon_i$. Denote by Z the set

$$\{ x^a = (a_1 x_{n_1}, 0, \dots, 0, a_2 x_{n_2}, 0, \dots, a_i x_{n_i}, 0, \dots) : a = (a_1, \dots, a_i, \dots) \in l_\infty \}.$$

Let us show that Z is a subspace of X, $(1+\varepsilon)$-isometric to l_∞. Obviously, Z is a linear subspace.

On the one hand, for any n, there exists i such that $n_i \le n \le n_{i+1}$ and, by the triangle inequality and the choice of n_i,

$$\| P_n x^a \|_n \le \sum_{k=1}^{i} | a_k | \, \| x^k \|_n + | a_{i+1} | \, \| x^{i+1} \|_n$$

$$\le \sum_{k=1}^{i} | a_k | \varepsilon_k + | a_{i+1} | \le \left(\sum_{k=1}^{i} \varepsilon_k + 1 \right) \sup_k | a_k | \le (1+\varepsilon) \sup_k | a_k |.$$

Hence,

$$\| x^a \| \le (1+\varepsilon) \sup_k | a_k |. \tag{2}$$

On the other hand, let $i \in N$ and let the number $1 \le j \le i$ be such that $\max_{1 \le k \le i} |a_k| = |a_j|$. Then

$$\max_{1\le n\le n_j} \|P_n x^a\|_n \ge \|P_{n_j} x^a\|_{n_j} \ge \|a_j x^j\|_{n_j} - \|P_{n_{j-1}} x^a\|_{n_j} \ge |a_j|\,\|x^j\|_{n_j} - \sum_{k=1}^{j-1} |a_k|\,\|x^k\|_{n_j}$$

$$\ge |a_j| - \sum_{k=1}^{j-1} |a_k|\varepsilon_k \ge |a_j| - \left(\sum_{k=1}^{j-1} \varepsilon_k\right)|a_j| \ge (1-\varepsilon)\max_{1\le k\le i} |a_k|$$

and

$$\|x^a\| \ge (1-\varepsilon)\sup_k |a_k|. \tag{3}$$

If follows from inequalities (2) and (3) that Z is $(1+\varepsilon)$-isometric to l_∞.

If $a = (a_1, \dots, a_i, \dots) \in c_0$, then, for every $\delta > 0$, there exists a number N such that $|a_j| < \delta$ for $i > N$ and

$$\lim_i \|P_{n_i} x^a\|_{n_i} \le \lim_i \|P_N x^a\|_{n_i} + \lim_i \|(P_{n_i} - P_N)x^a\|_{n_i} \le (1+\varepsilon)\sup_{k>n} |a_k| \le (1+\varepsilon)\delta.$$

Consequently, $X_0 \cap Z$ contains the subspace $Z_0 = \{x^a \in Z:\ a \in c_0\}$, $(1+\varepsilon)$- isometric to c_0. Recall that a set of elements $(x_i:\ i \in I)$ of a Banach space X is called a complete minimal system if the closed linear span $[x_i:\ i \in I] = X$ and, for every $j \in I$, $x_j \notin [x_i:\ i \ne j]$. The dimension $\dim X$ of a Banach space X is defined as a minimal cardinality of its subsets, the linear span of which is dense in X.

Corollary. *Let* $\dim X \le c$ *and let* X *satisfy condition (∗). Then the space* X *has a complete minimal system.*

Proof. According to Proposition 4, X has a closed subspace V, which is a complement to $Z \cong l_\infty$. Therefore, there exists a closed subspace $W \subset Z$ such that $X/(V \oplus W)$ is isomorphic to a Hilbert space and [5]

$$\dim X/(V \oplus W) = \dim X.$$

Since a Hilbert space has a complete minimal system, X also has one [6].

Proposition 5. *Let condition (∗) be satisfied and let, for all* n, Y_n *be a reflexive space. Then* $X = X_0^{**}$.

Proof. Consider the spaces Y_0 and Y and the map T defined in the proof of Proposition 2. Since Y_n are reflexive, the space dual to $Y_0 = c_0(Y_n)$ is $l_1(Y_n^*)$ and the second dual is $Y = l_\infty(Y_n)$. It is also known that the second dual to the subspace $TX_0 \subset Y_0$ is its weak* closure $\mathrm{cl}^*(TX_0)$ in Y_0^{**}. Consequently, it suffices to prove that $\mathrm{cl}^*(TX_0) = TX$. If $y = (y_1, \dots, y_n, \dots) = Tx = T(x_1, \dots, x_n, \dots) \in TX$ and $y \notin \mathrm{cl}^*(TX_0)$, then, by the Hahn–Banach theorem, there exists a functional $f \in Y_0^*$ such that $f(\mathrm{cl}^* TX_0) \equiv 0$ and $f(y) = 1$. Since $Y_0^* = l_1(Y_n^*)$, we have

$$f = (f_1, \dots, f_n, \dots), \quad f_n \in Y_n^*, \quad \text{and} \quad f(y) = \sum_{n=1}^{\infty} f_n(y_n).$$

By condition (*), the element $y_n \in TX_0$; therefore, for any n, $f(y_n) = f_n(y_n) = 0$; hence, $f(y) = 0$. Thus, $TX \subset \mathrm{cl}^*(TX_0)$.

We now show the converse inclusion. Suppose that a net

$$(y^\alpha\colon \alpha \in A),\quad y^\alpha = \left(y_1^\alpha, \ldots, y_n^\alpha, \ldots\right) = Tx^\alpha = T\left(x_1^\alpha, \ldots, x_n^\alpha, \ldots\right) \subset TX_0$$

weakly* converges to some element $y = (y_1, \ldots, y_n, \ldots) \in Y$. We need to show that there exists an element $x \in X$ such that $y = Tx$. Since, for every n, the space Y_n is reflexive, the net y_n^α, $\alpha \in A$, weakly converges to an element y_n. The net $T^{-1}\left(y_n^\alpha\right) = \left(x_1^\alpha, \ldots, x_n^\alpha, 0, \ldots\right)$, $\alpha \in A$, is also weakly convergent because T is an isometry. Finally, by continuity of the projection Q_n for any n, the net $\left(x_n^\alpha\right)$ weakly converges to an element x_n. Certainly, $T(x_1, \ldots, x_n, 0, \ldots) = y_n$. Since $y \in Y$, we have $\sup_n \|y_n\|_n < \infty$, i.e., the element $x \in X$ and, certainly, $Tx = y$. The proposition is proved.

Proposition 6. *Suppose that there exists a constant $c < 1$ such that, for every $n > 1$ and every $y \in Y_{n-1}$, the condition $\|y\|_n \le c\|y\|_{n-1}$ holds. For an element $x = (x_1, \ldots, x_n, \ldots) \in X$, we put*

$$\|x\|_0 = \sup_n \|x_n\|_n.$$

Then the norm $\|x\|_0$ is equivalent to the initial norm $\|x\|$; therefore, the spaces $(X, \|\,\|_0)$ and $(X_0, \|\,\|_0)$ are equal to $l_\infty(X_n)$ and $c_0(X_n)$, respectively.

Proof. First, we show that $\|x\| \le (1-c)^{-1}\|x\|_0$. Indeed, assume to the contrary that, for some $a > 1$, $\|x\| > (1-c)^{-1}a\|x\|_0$. Hence, for any $\varepsilon > 0$, there exists a number n such that, simultaneously, $\|P_n x\|_n > (1-\varepsilon)\|x\|$ and

$$\|P_{n-1}x + x_n\|_n = \|P_n x\|_n > (1-c)^{-1}a\|x\|_0 > (1-c)^{-1}a\|x_n\|_n.$$

From the last relation, we get

$$\|P_{n-1}x\|_n \ge \|P_n x\|_n - \|x_n\|_n \ge (1-(1-c)a^{-1})\,\|P_n x\|_n.$$

Taking into account the assumption of the proposition, we have

$$(1-\varepsilon)\|x\| < \|P_n x\|_n \le a(a-1+c)^{-1}\|P_{n-1}x\|_n$$

$$\le ac(a-1+c)^{-1}\|P_{n-1}x\|_{n-1} \le ac(a-1+c)^{-1}\|x\|.$$

Since ε is arbitrary, this leads to a contradiction.

On the other hand,

$$\|x_n\|_n \le \|P_n x\|_n + \|P_{n-1}x\|_n \le \|P_n x\|_n + c\|P_{n-1}x\|_{n-1}.$$

By taking the supremum over all n on both sides of the inequality, we obtain $\|x\|_0 \le (1+c)\|x\|$. The inequalities $(1-c)\|x\| \le \|x\|_0 \le (1+c)\|x\|$ mean the equivalence of the norms, and the space X_0 is equal to $c_0(X_n)$ and X is equal to $l_\infty(X_n)$ in the norm $\| \|_0$.

Recall that a Banach space is called weakly compactly generated (WCG) if it is a closed linear span of its weakly compact subset. It is easy to see that separable and reflexive spaces are WCG spaces and if Y_n is a WCG space, then the space Y_0 has the same property.

Corollary 1. *Let at least one of the following conditions be satisfied:*

(i) for all n, Y_n is weakly compactly generated and $(*)$ holds;

(ii) there exists $c<1$ such that, for every n and any $y \in Y_{n-1}$, we have $\|y\|_n \le c\|y\|_{n-1}$.

Then the space X_0 contains a complemented subspace isomorphic to c_0 and X_0 is uncomplemented in X.

Proof. To prove the first part of Corollary 1, we note that if condition (i) is satisfied, then X_0 is a WCG space. By Proposition 4, it contains a subspace isomorphic to c_0, which, by [7, p. 115] and [8, p. 106], is complemented there. Under condition (ii), by Proposition 6, the space X_0 is isomorphic to $c_0(X_n)$ and, certainly, contains a complemented subspace isomorphic to c_0.

We now show that the subspace X_0 is uncomplemented in X. If condition (i) [condition (ii)] is satisfied, then it follows from Proposition 4 (Proposition 6, respectively) that X contains a subspace Z isomorphic to l_∞ and $Z \cap X_0$ contains a subspace Z_0 isomorphic to c_0. Suppose that X_0 is complemented in X. Then, by the first part of this corollary, this is Z_0 and, in particular, Z_0 is complemented in Z. However, every infinite-dimensional complemented subspace of l_∞ is isomorphic to l_∞ [8, p. 57]. We arrive at contradiction. Corollary 1 is proved.

Corollary 2. *Under the assumptions of Corollary 1, X_0 is not isomorphic to a dual space.*

Proof. In the first case, X_0 is a WCG space and, by Proposition 4, X_0 contains a subspace isomorphic to c_0. Suppose that X_0 is isomorphic to the dual space. Then it contains a subspace isomorphic to l_∞ [8, p. 103]. But a WCG space does not contain a subspace of this sort; this can be easily deduced, for example, from Corollary 3 in [7, p. 114].

If the second condition is satisfied and X_0 is isomorphic to the dual space, then it is complemented in the second dual X_0^{**}. But $X \subset X_0^{**}$ and, therefore, X_0 is complemented in X. This contradicts Corollary 1.

Definition 1. *We say that a sequence of closed subspaces X_n, $n=\overline{1,\infty}$, of a Banach space X_0 forms a basic decomposition if $[X_n\colon n=\overline{1,\infty}]=X_0$ and there exist projections $P_n\colon X_0 \to [X_i]_1^n$ along $[X_i\colon i=\overline{n+1,\infty}]$, $n=\overline{1,\infty}$, which are uniformly bounded. Moreover, if there exists a constant $k\ge 1$ such that, for every finite collection $(x_i)_1^n$, $x_i \in X_i$, and every collection of signs $(\theta_i)_1^n$, the relation $\|\sum \theta_i x_i\| \le K\|\sum x_i\|$ holds, then a basic decomposition is called unconditional and the minimal number K is called an unconditional constant of a decomposition (X_n). In addition, if there exists a number $c\ge 1$ such that, for every finite collection $x_i, y_i, i=\overline{1,n}$, $x_i, y_i, \in X_i$, the inequality $\|y_i\| \le \|x_i\|$ implies $\|\sum y_i\| \le c\|\sum x_i\|$, then the sequence (X_n) is called a strong unconditional decomposition of X_0.*

It follows immediately from this definition that, in the case under consideration, the subspaces $(X_n, \| \|)$ form a basic decomposition of the space X_0 and, under the conditions of Proposition 6, they form a strong unconditional decomposition. It is easy to see that if (X_n) form a strong unconditional decomposition, for each n, the subspace X_n has an unconditional basis $(X_n^m)_{m=1}^{\infty}$, and, moreover, their unconditional constants are uniformly bounded, then the system $(X_n^m)_{m,n=1}^{\infty}$ is an unconditional basis of X_0.

2. ***Definition 2*** [10, p. 21]. *Let* (Ω, Σ, μ) *be a measure space with a positive measure* μ. *A Banach space* E *of (classes of) measurable functions on* Ω *is called symmetric if*

(i) *the facts that* $y \in E$ *and* $|x(\omega)| \le |y(\omega)|$ *for almost all* $\omega \in \Omega$ *imply that* $x \in E$ *and* $\|x\| \le \|y\|$;

(ii) *the facts that* $y \in E$ *and* $d_{|x|}(t) = d_{|y|}(t)$ *for all* $t > 0$ *imply that* $x \in E$ *and* $\|x\| = \|y\|$, *where* $d_{|x|}(t) = \mu\{\omega : |x(\omega)| > t\}$ *is the distribution function of* $|x(\omega)|$.

For a number $T > 0$, denote by φ_T the linear map of $[-T, T]$ onto $[-1, 1]$ with $\varphi(-T) = -1$, $\varphi(T) = 1$. Let E be a symmetric space on $[-1, 1]$ with normalized Lebesgue measure $\lambda(-1, 1) = 1$. Then all functions $x(\varphi_T(t))$, where x runs through E, form a symmetric space E_T on $[-T, T]$ with norm $\|x \circ \varphi_T\|_T = \|x\|_E$. We denote the composition of functions by the sign $\circ$. Every function on the segment $[-T, T]$ is identified with a function on the real line by defining it to be zero outside $[-T, T]$. Denote the set of measurable functions on the real line for which the number $\|x\|_{M_E} = \sup_{T \ge 1} \|x\|_T$ is finite by M_E and the subspace of M_E consisting of functions for which $\lim_{T \to \infty} \|x\|_T = 0$ by I_E. It is easy to see that $(M_E, \|\|_{M_E})$ is a linear normed space and I_E is its linear subspace. Certainly, for $E = L_p(-1, 1)$, $\lambda(-1, 1) = 1$, our construction gives the spaces M_p and I_p defined at the beginning of this paper. It is also evident that the spaces M_E and I_E are normed lattices with a natural pointwise order. Even the spaces M_p and I_p are not symmetric function spaces on the real line. Some weak property of symmetry for the spaces M_E and I_E will be mentioned below (see the proof of Proposition 8).

Proposition 7. *Let* $T_n \ge 1$, $T_1 = 1$, $T_n \to \infty$, *and* $\sup_n T_{n+1}/T_n = a$ *for some* $1 \le a < \infty$. *Then*

$$\|x\|_{M_E} \le 2a \sup_n \|x\|_{T_n}$$

for any $x \in M_E$ *and, therefore,* $\sup_n \|x\|_{T_n}$ *is the norm on the space* M_E *equivalent to the norm* $\|x\|_{M_E}$.

To prove this proposition, we need the following lemma:

Lemma. *If* $1 \le S \le T$, *then* $E_S \subset E_T$ *and, moreover,* $(S/T)\|y\|_S \le \|y\|_T \le \|y\|_S$ *for every* $y \in E_S$.

Proof. Let $y \in E_S$, $y = x \circ \varphi_S$ where $x \in E$. It is necessary to find a function $z \in E$ such that $x \circ \varphi_S = z \circ \varphi_T$ and $(S/T)\|x\|_E \le \|z\|_E \le \|x\|_E$. We set

$$\psi(t) = \begin{cases} \frac{T}{S} t & \text{if } |t| \le S/T, \\ 0 & \text{if } S/T < |t| \le 1 \end{cases}$$

and $z = x \circ \psi$. Since $\varphi_T(t) = t/T$ and $\varphi_S(t) = t/S$, $\psi(\varphi(t)) = (T/S)(t/T) = \varphi_S(t)$. Thus, $x \circ \varphi_S = z \circ \varphi_T$. The operator

$$D_{S/T}x(t) = \begin{cases} x\left(\frac{T}{S}t\right) & \text{if } |t| \le S/T, \\ 0 & \text{if } S/T < |t| \le 1 \end{cases} \tag{4}$$

associates a function $z(t)$ with a function $x(t)$. As is known [9, p. 130], $D_{S/T}$ acts in the space E, its norm is at most one, $\|D_{T/S}\| \le \max(1, T/S) = T/S$, and $D_{S/T}D_{T/S}x = \chi_{[0,\,S/T]}x$. Then $D_{S/T}D_{T/S}z = z$ and

$$\|x\|_E = \| D^{-1}_{S/T}z \|_E = \|D_{T/S}z\|_E = \|D_{T/S}z\|_E \le (T/S)\|z\|_E.$$

Proof of Proposition 7. Let us take an arbitrary number $1 < T < \infty$; for some n, $T_n \le T \le T_{n+1}$. Denote by χ_n the characteristic function of the set $\{t\colon T_n < |t| \le T_{n+1}\}$. Then $\|x\|_T = \|x\chi_{[-T_n,\,T_n]} + x\chi_n\|_T$. Consider two possible cases.

I. $\|x\chi_{[-T_n,\,T_n]}\|_T \ge (1/2)\|x\|_T$. Then $\|x\|_T \le 2\|x\|_{T_n} \le 2a \sup_n \|x\|_{T_n}$.

II. $\|x\chi_n\|_T \ge (1/2)\|x\|_T$. Then, by the lemma, $\|x\chi_n\|_T \le (T_{n+1}/T)\|x\chi_n\|_{T_{n+1}} \le (T_{n+1}/T_n)\|x\chi_n\|_{T_{n+1}}$.

Consequently,

$$\|x\|_T \le 2\|x\chi_n\|_T \le (2T_{n+1}/T_n)\|x\chi_n\|_{T_{n+1}} \le (2T_{n+1}/T_n)\|x\|_{T_{n+1}} \le 2a \sup_n \|x\|_{T_n}.$$

Since T is arbitrary, we obtain

$$\|x\|_{M_E} = \sup_{T \ge 1} \|x\|_T \le 2a \sup_n \|x\|_{T_n}.$$

Note that, for the spaces

$$Y_n = E_{T_n}, \quad X_n = \{x\chi_{n-1} : x \in E_{T_n}\}, \quad X = (M_E, \sup_n \|x\|_{T_n}), \quad \text{and} \quad X_0 = (I_E, \sup_n \|x\|_{T_n}),$$

the conditions from Sec. 1 of the present paper are satisfied. Therefore, Propositions 1, 2, and 7 yield the following statement:

Corollary 1. *M_E is a Banach space and I_E is its closed subspace.*

Recall that the norm $\| \;\|$ of a symmetric space E is called absolutely continuous if, for every $x \in E$ and every decreasing sequence Ω_n of measurable subsets of Ω with empty intersection, $\|x\chi_{\Omega_n}\| \to 0$ as $n \to 0$. Note also that a symmetric Banach space on $(-1, 1)$ with an absolutely continuous norm is a rearrangement invariant in the sense of [9], and the Haar system forms a basis in $E(-1, 1)$ [9, p. 150].

Further, we consider symmetric spaces E with an absolutely continuous norm only. It is easy to see that if E is a symmetric space with an absolutely continuous norm, then condition $(*)$ is satisfied for the spaces $Y_n = E_{T_n}$, $T_n \to \infty$. Therefore, the reasoning presented after Definitions 1 and 2 and Proposition 7 yields the following assertion:

Corollary 2. *The subspaces $E^n = \{x\chi_{n-1} : x \in I_E\}$ form an unconditional decomposition of the space I_E*

with the unconditional constant equal to one.

The next corollary is a consequence of Propositions 3–5, 7 and Corollaries 1, 2 of Proposition 6.

Corollary 3. *The space I_E is separable, not isomorphic to a dual space, and uncomplemented in M_E and contains a complemented subspace isomorphic to c_0. If E is a reflexive space, then $I_E^{**} = M_E$.*

Since a space E on $(-1, 1)$ with an absolutely continuous norm is separable, $\dim M_E = c$ and, by Proposition 4, the corollary of Proposition 4, and Proposition 7, we obtain the following corollary:

Corollary 4. *The space M_E contains a complemented subspace isomorphic to l_∞ and has a complete minimal system.*

Denote by $\mathfrak{M}_E$ the set of (classes of) measurable functions $x(t)$ on the real line for which the following norm is finite:

$$\|x\|_{\mathfrak{M}_E} = \limsup_{T\to\infty} \|x\|_T.$$

Thus, the functions $x, y \in M_E$, for which $x-y \in I_E$, are identified in the space $\mathfrak{M}_E$ and $\mathfrak{M}_E = M_E/I_E$. Note that the symmetric space of Bezikovich almost periodic functions E_{AP} considered in [10] is a subspace of $\mathfrak{M}_E$.

Corollary 5. *If a space E is reflexive, then $\mathfrak{M}_E^* = I_E^\perp$, where $I_E^\perp$ is the annihilator of $I_E \subset M_E$ in the dual space M_E^*.*

Indeed, by Corollary 2, $I_E^{**} = M_E$ and, consequently, $M_E^* = I_E^\perp \oplus I_E^*$. However, we have $\mathfrak{M}_E = M_E/I_E$.

Corollary 6. *If a space E is reflexive, then $\mathfrak{M}_E$ contains a subspace isomorphic to l_∞/c_0 and, hence, $\mathfrak{M}_E$ has no equivalent strictly convex norm [11].*

Indeed, by Corollary 3, $I_E = U \oplus V$, U is isomorphic to c_0, and $M_E = U^{**} \oplus V^{**}$, U^{**} is isomorphic to l_∞. Therefore, $\mathfrak{M}_E = M_E/I_E$ contains a subspace isomorphic to U^{**}/U, i.e., l_∞/c_0.

Recall that the lower and upper Boyd indices of a symmetric space E are defined by

$$p_E = \sup_{s>1} (\log s)/\log\|D_s\| \quad \text{and} \quad q_E = \inf_{0<s<1} (\log s)/\log\|D_s\|,$$

respectively, where D_s is the operator defined by (4).

Corollary 7. *Let E be a symmetric space with $q_E < \infty$. Then the space M_E is isomorphic to $l_\infty(E)$ and I_E is isomorphic to $c_0(E)$.*

To prove this corollary, we need two additional statements.

Lemma 1. *Let E be a symmetric space with $q_E < \infty$. Then, for every S and T, $S < T$, we have $\|y\|_T \le C\|y\|_S$ for every $y \in E_S$, where $C = (S/T)^{1/q_E} < 1$.*

Proof. Since $q_E \le (\log s)/\log \|D_s\|$ for any $0 < s < 1$, we have

$$\log \|D_s\| \le (\log s)/q_E = \log s^{1/q_E}.$$

Hence, $\|D_s\| \le s^{1/q_E}$. Further, by analogy with the proof of the lemma after Proposition 7, by putting $s = S/T$, we get

$$\|y\|_T = \|z \circ \varphi_T\|_T = \|z\|_E = \|D_s x\|_E \le \|D_s\|\|x\|_E \le C\|x \circ \varphi_S\|_S = C\|y\|_S,$$

where $C = (S/T)^{1/q_E} < 1$.

Lemma 2. *Let a sequence* $T_n \ge 1$, $T_n \to \infty$, *and*

$$\inf_n T_{n+1}/T_n > 1, \qquad \sup_n T_{n+1}/T_n < \infty.$$

Then the spaces E^n *constructed by this sequence are uniformly isomorphic to* E.

Proof. Let $E[-S_n, S_n]$ be a subspace of E consisting of functions $x\chi_{[-S_n, S_n]}$, where $S_n = 1 - T_{n-1}/T_n$, $\inf S_n > 0$. By using the definition of E_{T_n} and the symmetry, we see that the space E_{T_n} is isometric to E and E^n is isometric to $E[-S_n, S_n]$. Then the operator D_{S_n} defined by (4) acts from E into its subspace $E[-S_n, S_n]$ and has norm at most 1, and the norm of the inverse operator is at most $1/S_n$.

Proof of Corollary 7. Choose a sequence $T_n \ge 1$, $T_n \to \infty$, such that

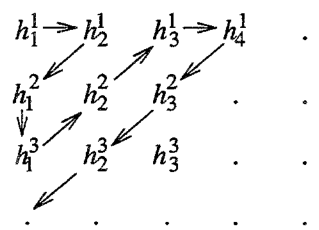
$$1 < \inf_n T_{n+1}/T_n \quad \text{and} \quad \sup_n T_{n+1}/T_n < \infty.$$

Then, by Proposition 6 and Lemma 1, we find that, for the subspaces E^n constructed by this sequence, $M_E = l_\infty(E^n)$ and $I_E = c_0(E^n)$ in the equivalent norm. To complete the proof, we apply Lemma 2.

Remark. It follows from the proof of Corollary 7 that if $q_E < \infty$, then the subspaces E^n form a strong unconditional decomposition of I_E.

Proposition 8. *The space* I_E *has a (Schauder) basis.*

Proof. Denote by h_m^n, $m = \overline{1, \infty}$, the Haar system in the space E^n and enumerate the Haar functions by one index as is shown in the scheme below:



Let $(h_i)_1^\infty$ be the system obtained. It is clear that its linear span is dense in I_E. Therefore, by the sufficient condition of basisness [8, p. 2], it suffices to verify that, for any finite collection of scalars a_i, $i=\overline{1,k+1}$, the inequality $|||x||| \le |||y|||$ holds for

$$x = \sum_1^k a_i h_i, \quad y = \sum_1^{k+1} a_i h_i,$$

where

$$T_n \ge 1, \quad T_n \to \infty, \quad \sup_n T_{n+1}/T_n < \infty, \quad \text{and} \quad |||x||| = \sup_n \|x\|_{T_n}$$

is the norm equivalent, by Proposition 7, to the initial norm of I_E. Consider two cases.

I. If $\operatorname{supp} x \ne \operatorname{supp} y$, where $\operatorname{supp} x = \{t: x(t) \ne 0\}$, i.e., there exists j such that $h_{k+1} = h_1^j$, $\operatorname{supp} h_{k+1} \cap \operatorname{supp} x = \varnothing$, then, for all t, $|y(t)| \ge |x(t)|$ and, consequently, for every n, $\|y\|_{T_n} \ge \|x\|_{T_n}$ and $|||y||| \ge |||x|||$.

II. The functions x and y coincide everywhere with the exception of an interval $A \subseteq \{t: T_n < |t| \le T_{n+1}\}$ on which x is a constant, say, it takes a value b there, and $y(t)$ is equal to $b + a_{k+1}$ on the first half of A and to $b - a_{k+1}$ on the second half of A. Let τ be an automorphism of the real line which permutes the first half of A with its second half and leaves invariant every point outside A. It is easy to see that, for this automorphism, $|||y||| = |||y \circ \tau|||$ and $x(t) = (y(t) + y(\tau(t)))/2$ for every $t \in \mathbb{R}$. Therefore, $|||x||| \le |||y|||$.

Proposition 9. *The system $(h_i)_1^\infty$ from the preceding proposition is an unconditional basis of I_E if and only if $p_E > 1$ and $q_E < \infty$.*

Proof. Since the Haar system h_m^n, $m = \overline{1, \infty}$, forms an unconditional basis of E_n if and only if $p_{E^n} > 1$ and $q_{E^n} < \infty$ [9, p. 156], these conditions are necessary.

By Lemma 2 of Corollary 7, the spaces E^n are uniformly isomorphic to E. Moreover, under the assumptions of Proposition 9, the subspaces E^n form a strong unconditional decomposition of I_E (Corollary 7). Applying the remark after Definition 1, we conclude that the system $(h_i)_1^\infty$ forms an unconditional basis of this space.

Definition 3. *Let K be a convex subset in a linear space X. An element $x \in K$ is called an extreme point of K if, for any $y \in X$, $x \mp y \in K$ implies that $y = 0$.*

Proposition 10. *Let*

$$x \in M_E, \ \|x\| = 1, \ \ \textit{and} \ \ \limsup \|x\|_T < 1 \ \ \textit{as} \ \ T \to \infty. \tag{5}$$

Then there exists an element $y \in I_E$, $\|y\| \ne 0$, such that $\|x \mp y\| \le 1$. Thus, any point with condition (5) is not an extreme point of the unit ball of M_E, and the unit ball $B(I_E)$ of I_E contains no extreme point.

Proof. Let $\sup_{T \ge 1} \|x\|_T = 1$ and, for some $a < 1$, there exist a number S such that $\|x\|_T \le a$ as $T > S$.

Let $y \in I_E$, $\|y\| < 1-a, y \neq 0$, $\operatorname{supp} y \in [S, \infty)$. For $T \leq S$, $\|x \mp y\|_T = \|x\|_T$, and, for $T > S$,

$$\|x \mp y\|_T \leq \|x\|_T + \|y\|_T \leq a + 1 - a \leq 1.$$

The proposition is proved.

Definition 4. *A Banach space X is called uniformly convex [7, p. 34] if*

$$\delta(\varepsilon) = \inf\{1 - \|x+y\|/2 : \|x-y\| \geq \varepsilon,\ x, y \in B(X)\}, \quad \varepsilon > 0,$$

is a strictly positive function on $\mathbb{R}^+$; $\delta(\cdot)$ is called the modulus of convexity of X.

Proposition 11. *Let E be a uniformly convex symmetric space and let $x \in B(M_E)$. Suppose that there exists $C > 0$ and a sequence $T_n \to \infty$ such that $\|x\|_{T_n} > 1 - \delta(C/T_n)$. Then x is an extreme point of $B(M_E)$.*

Remark. The condition of Proposition 11 states that if there exists a sequence T_n such that $\|x\|_{T_n} \to 1$ sufficiently fast, then x is an extreme point of $B(M_E)$.

Proof. Suppose that there exists an element $y \in M_E$ such that $\|x \mp y\| \leq 1$ and $y(t) \not\equiv 0$ on $[-S, S]$ for some $S > 0$. We set $C = \|y\|_S$, $u = x + y$, $v = x - y$. Then, for $T > S$, $\|u - v\|_T = 2\|y\|_T \geq 2ST^{-1}\|y\|_S = 2ST^{-1}C > C/T$. The uniform convexity of E yields $\delta(C/T) \leq 1 + \|u+v\|/2 = 1 - \|x\|_T$, i.e., $\|x\|_T \leq 1 - \delta(C/T)$. We arrive at a contradiction.

Proposition 12. *Let E be a uniformly convex symmetric space and $u \in B(\mathfrak{M}_E)$. Suppose that there exists a sequence $T_n \to \infty$ such that $\sup_n T_{n+1}/T_n < \infty$ and $\lim \|u\|_{T_n} = 1$ as $n \to \infty$. Then u is an extreme point of $B(\mathfrak{M}_E)$.*

Proof. Let $v \in \mathfrak{M}_E$ be a point such that

$$\limsup_{T \to \infty} \|u \mp v\|_T \leq 1.$$

Let us show that

$$\limsup_{T \to \infty} \|v\|_{T_n} = 0.$$

Assume the contrary. Then, by passing to a subsequence if necessary, we can assume that $\|v\|_{T_n} \geq \varepsilon$ for some $\varepsilon > 0$. For each n, we consider u and $u \mp v$ as elements of E_{T_n}. Since the norms $\|\|_{T_n}$ are uniformly convex, by putting $u = x$ and $u + v = y$ in Definition 4, we can find $\delta(\varepsilon) > 0$ such that $\delta < 1 - \|u + u + v\|_{T_n}/2 < 1 - \|u\|_{T_n}$. This contradicts the hypothesis that $\lim\|u\|_{T_n} = 1$ as $n \to \infty$. Thus, $\lim_{n \to \infty}\|v\|_{T_n} = 0$. For an arbitrary $T > 0$, there exists n: $T_n \leq T \leq T_{n+1}$. Then

$$\|v\|_T \leq T_{n+1}T^{-1}\|v\|_{T_{n+1}} \leq T_{n+1}T_n^{-1}\|v\|_{T_{n+1}}.$$

The boundedness of $\{T_{n+1}/T_n\}$ implies that the last term tends to 0 as $T \to \infty$. Therefore, $\|v\|_{\mathfrak{M}_E} = 0$ and u is an extreme point of $B(\mathfrak{M}_E)$.

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