EXTENSION OF QUASI-COMPLEMENTARITY IN BANACH SPACE

A. N. Plichko

Closed subspaces X and Y of the Banach space E are said to be quasi-complementary if $X \cap Y = \theta$ and $\overline{X + Y} = E$. The bar denotes closure in the norm of the space E. We also denote by $[z_i]_1^n$ the linear hull of the elements $\{z_i\}_1^n$.

The following was proved in [1] under the assumption that the space E is reflexive.

THEOREM 1. Let X and Y be quasi-complementary but not complementary closed subspaces of separable Banach space E. Then a closed subspace $\tilde{Y} \subset E$ can be found such that $\tilde{Y} \supset Y$, $\dim \tilde{Y}/Y = \infty$ and that the subspaces X and $\tilde{Y}$ are quasi-complementary.

There the following question was raised: does the theorem remain valid if the space E is not reflexive. In the present article Theorem 1 is proved without the assumption that the space E is reflexive and other results close to it are obtained.

LEMMA. Let X and Y be quasi-complementary but not complementary closed subspaces of Banach space E. Then an infinite-dimensional vector subspace $Z \subset E$ can be found such that for any $z \in Z$, $z \neq \theta$, no bounded sequences $\{x_n\}_1^\infty \subset X$ and $\{y_n\}_1^\infty \subset Y$ exist for which $\|x_n + y_n - z\| \to 0$ if $n \to \infty$.

Proof of Theorem 1. Let

$$D_i = \{x \in X: \inf\{\|x - y\|: y \in Y\} > 1/i\}.$$

Of course, one has $X \setminus \theta = \bigcup_{i=1}^{\infty} D_i$. Since the space E is separable, a set D_i can be covered by a countable set of balls $W_1^i, \ldots, W_j^i, \ldots$ belonging to the space E with centers $x_1^i, \ldots, x_j^i, \ldots$ in D_i of radius $1/2i$. The balls W_j^i and their centers x_j^i are relabelled using a single index. One obtains a sequence of balls $V_n \subset E$ with centers $x_n \in X$, which cover the set $X \setminus \theta$ and the distance of each of which from the subspace Y is greater than zero.

Let Z be a subspace whose existence was established by the lemma. A sequence of elements $\{z_n\}_1^\infty$ and closed hyperplanes $H_n \subset E$, $n = 1, \ldots, \infty$, is now defined inductively which satisfy the following conditions:

$$\left.\begin{array}{l} \text{a) } z_i \in Z,\ z_i \notin [z_j]_1^{i-1}, \\ \text{b) } (V_i \cap X) \cap H_i = \phi, \\ \text{c) } H_i \supset Y + [z_j]_1^n, \end{array}\right\} \quad i = 1, \ldots, n.$$

Since the distance between a ball V_1 and the subspace Y is greater than zero (see [2], Chap. 4, Sec. 1, Corollary 2), a closed hyperplane $H_1 \supset Y$ can be found such that $H_1 \cap V_1 = \phi$. But H_1 is a hyperplane and the subspace Z is infinite-dimensional; therefore, an element $z_1 \in H_1 \cap Z$, $z_1 \neq \theta$ can be found. Of course, the element z_1 and the hyperplane H_1 satisfy conditions a, b, and c. Let us suppose now that the elements $\{z_i\}_1^n$ and hyperplanes $\{H_i\}_1^n$ with properties a, b, c have already been selected. It will be shown that the distance between the sets $V_{n+1} \cap X$ and $Y + [z_i]_1^n$ is greater than zero. Let us suppose the opposite.

Let there exist sequences $\{x_i\}_1^\infty \subset V_{n+1} \cap X$ and $\left\{y_i + \sum_{j=1}^n \alpha_i^j z_j\right\}_{i=1}^\infty \subset Y + [z_i]_1^n$ such that $\left\| x_i - \left(y_i + \sum_{j=1}^n \alpha_i^j z_j\right) \right\| \to 0$ for $n \to \infty$. The sequence $\{x_i\}_1^\infty$ is bounded, and the same applies to $\left\{y_i + \sum_{j=1}^n \alpha_i^j z_j\right\}_{i=1}^\infty$. Since $Y \cap Z = \theta$ and

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$[z_i]_1^n$ is finite-dimensional, the sequence $\{y_i\}_1^\infty$ is bounded and the sequence of n-dimensional vectors $\{(\alpha_i^1, \ldots, \alpha_i^n)\}_{i=1}^\infty$ is bounded in every coordinate. A subsequence $\{(\alpha_{i_k}^1, \ldots, \alpha_{i_k}^n)\}_{k=1}^\infty$ is now selected which converges coordinatewise to a vector $(\alpha^1, \ldots, \alpha^n)$. The distance between the set V_{n+1} and the subspace Y is greater than zero; hence, $(\alpha^1, \ldots, \alpha^n) \neq (0, 0, \ldots, 0)$. Thus, one has obtained $\left\| x_{i_k} - y_{i_k} - \sum_{j=1}^n \alpha^j z_j \right\| \to 0$ for $k \to \infty$ and the sequences $\{x_{i_k}\}_{k=1}^\infty$ and $\{-y_{i_k}\}_{k=1}^\infty$ are bounded. This, however, is inconsistent with the manner in which the subspace Z was selected. Hence, the distance between the sets $V_{n+1} \cap X$ and $Y + [z_i]_1^n$ is greater than zero. According to [2] (Chap. 4, Sec. 1, Corollary 2) one can always find a hyperplane $H_{n+1} \supset Y + [z_i]_1^n$, such that $H_{n+1} \cap V_{n+1} = \phi$. However, $\bigcap_{i=1}^n H_i$ is a subspace of finite defect and Z is infinite-dimensional; therefore, an element $z_{n+1} \in Z \cap \left(\bigcap_{i=1}^n H_i\right)$, $z_{n+1} \notin [z_i]_1^n$ can be found. Of course, the elements $\{z_i\}_1^{n+1}$ and the hyperplanes $\{H_i\}_1^{n+1}$ satisfy conditions a, b, and c.

Let us now set $\tilde{Y} = \overline{Y + [z_i]_1^\infty}$. Since $Z \cap Y = \theta$ and $z_{n+1} \notin [z_i]_1^n$, then $\dim \tilde{Y}/Y = \infty$. To prove that the subspaces X and Y are quasi-complementary it suffices to check whether $X \cap \tilde{Y} = \theta$. Let us suppose the opposite, that is, that $x \neq \theta$, $x \in \tilde{X} \cap \tilde{Y}$. Then a ball $V_n \ni x$ and a hyperplane $H_n \supset Y + [z_i]_1^\infty$ exist such that $H_n \cap V_n = \phi$. An inconsistency has been arrived at.

COROLLARY. Let X and Y be quasi-complementary but not complementary closed subspaces of separable Banach space E. Then a closed infinitely dimensional subspace $\tilde{Z} \subset E$ can be found such that $\tilde{Z} \cap (X+Y) = \theta$.

Indeed, the required space $\tilde{Z}$ can be given by the quasi-complement of the subspace Y in the space $\tilde{Y}$.

THEOREM 2. In every infinitely dimensional Banach space E there exists a vector subspace M of infinite defect for which no closed infinitely dimensional subspace Z can be found such that $Z \cap M = \theta$.

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