# ON THE LAW OF THE ITERATED LOGARITHM FOR WEIGHTED SUMS OF INDEPENDENT RANDOM VARIABLES IN A BANACH SPACE

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UDC 519.21

Assume that $(X_n)$ are independent random variables in a Banach space, $(b_n)$ is a sequence of real numbers, $S_n = \sum_1^n b_i X_i$, and $B_n = \sum_1^n b_i^2$. Under certain moment restrictions imposed on the variables $X_n$, the conditions for the growth of the sequence $(b_n)$ are established, which are sufficient for the almost sure boundedness and precompactness of the sequence $(S_n / (B_n \ln\ln B_n)^{1/2})$.

## 1. Introduction and Principal Results

Let $E$ be a separable Banach space with a norm $\|\cdot\|$ and let $E^*$ be its dual space. Denote by $X_n,\ n = \overline{1, \infty}$, $E$-valued independent random variables defined on a probability space $(\Omega, \mathcal{B}, P)$ [1, p. 201; 2]. Let $b_n,\ n = \overline{1, \infty}$, be a sequence of real numbers, $S_n = \sum_1^n b_i X_i$, $B_n = \sum_1^n b_i^2$, and let $L(t) = \ln t$ for $t > e$ and $L(t) = 1$ for $t \le e$. We also set $L_2(t) = L(L(t))$ and $\chi(t) = (2tL_2(t))^{1/2}$.

We say that a sequence $b_n X_n$ satisfies the law of the iterated logarithm if

$$\Lambda(b, X) = \limsup_n \frac{\|S\|}{\chi(B_n)} < \infty \tag{1}$$

almost surely (a.s.) and

$$\left\{ \frac{S_n}{\chi(B_n)},\ n \ge 1 \right\} \tag{2}$$

is precompact in $E$ almost surely.

Under the conditions imposed in the next paragraph, it follows from the 0 or 1 law that $\Lambda(b, X)$ is a nonrandom variable.

A survey of the results and a fairly complete bibliography on the law of the iterated logarithm in Banach spaces can be found in [3–6]. In this paper, we extend to weighted sums some well-known results related to the law of the iterated logarithm for equally distributed random variables in a Banach space. In what follows, we suppose that $MX_n = 0$, $DX_n = (M\|X_n\|^2)^{1/2} < \infty$, $B_n \uparrow \infty$ and $B_n / b_n^2 \to \infty$ as $n \to \infty$. Let

$$\Gamma(b, X) = \limsup_n M\|S_n\| / \chi(B_n)$$

and let $\varepsilon_n$ be symmetric Bernoulli independent random variables, $P(\varepsilon_n = \pm 1) = 1/2$.

**Theorem 1.** *Assume that*

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Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 45, No. 9, pp. 1225–1231, September, 1993. Original article submitted April 2, 1991.



0041-5995/93/4509-1372 $12.50 

$$b_n^2 = O(B_n / (L(B_n))^{(1+\delta)/(p-1)}), \tag{3}$$

$$\sup_n M \| X_n \|^{2p} < \infty \tag{4}$$

*for some* $\delta > 0$ *and* $1 < p < \infty$. *Then*

$$\Gamma(b, X) \le \Lambda(b, X) \le d + \Gamma(b, X) \tag{5}$$

*for* $d = \sup_n D X_n$.

***Corollary 1.*** *Let* $\Pi_N$ *be a sequence of linear bounded finite-dimensional operators in the space* $E$ *and let* $Q_N x = x - \Pi_N x$. *If* $\Gamma(b, X) = 0$ *and*

$$\sup_n D(Q_n X_n) \to 0 \quad as \quad N \to \infty, \tag{6}$$

*then, under the conditions of Theorem 1, the law of the iterated logarithm (1), (2) is valid.*

Note an important specific case where condition (6) is satisfied. Let $R: E^* \to E$ be the covariance operator of a random variable $Y$, $MY = 0$, $DY < \infty$, and let $H_R$ be the Hilbert subspace of the space $E$ associated with $R$ [1, p. 126], i.e., the completion of $R(E^*)$ in the norm of the scalar product $\langle Rf, Rg \rangle = Mf(Y)g(Y)$, $f, g \in E^*$. Assume that $(f_k) \subset E^*$ is a sequence, which is total in $E$ and orthogonal with respect to the scalar product indicated above. If we set $\Pi_n x = \sum_{k=1}^N f_k(x) Rf_k$, then $D(Q_n Y) \to 0$ as $N \to \infty$. Therefore, if the variables $X_n$ are equally distributed (and, hence, have the same covariance operator $R$), then the sequence $(f_k)$ satisfies condition (6).

For a random variable $Y$ with a covariance operator $R$, we set

$$\sigma(R) = \sup\{ |\langle Rf, f \rangle|^{1/2} : \|f\| = 1 \} = \sup\{ (M|f(Y)^2|)^{1/2} : \|f\| = 1 \}.$$

**Theorem 2.** *Under the conditions of Theorem 1, assume that the variables* $X_n$ *have the same covariance operator* $R$ *and that condition (6) is satisfied for the variables* $Q_N X_n = X_n - \sum_{k=1}^N f_k(X_n) Rf_k$, *where* $(f_k)$ *is an orthogonal sequence total in* $E$. *Then*

$$\max(\sigma(R), \Gamma(b, X)) \le \Lambda(b, X) \le \sigma(R) + \Gamma(b, X).$$

Recall that $E$ is called a space of type 2 if, for any sequence $(x_n) \in E$, the condition $\sum_1^\infty \|x_n\|^2 < \infty$ implies the almost sure convergence of the series $\sum_1^\infty \varepsilon_n x_n$; it is called a space of cotype 2 if it follows from the almost sure convergence of the series $\sum_1^\infty \varepsilon_n x_n$ that $\sum_1^\infty \|x_n\|^2 < \infty$ [1, p. 251].

***Corollary 2.*** *If* $E$ *is a space of type 2, then, under the conditions of Theorem 1, relation (1) is satisfied or, more precisely,* $\Lambda(b, X) \le d$. *Under the conditions of Theorem 2, the law of the iterated logarithm (1), (2) holds and* $\Lambda(b, X) = \sigma(R)$.

Recall that a covariance operator $R$ is called Gaussian if there exists a Gaussian random variable $G(R)$ for which $R$ is a covariance operator.

**Corollary 3.** *If $E$ is a space of cotype 2 and the operator $R$ is Gaussian under the conditions of Theorem 2, then the law of the iterated logarithm (1), (2) holds and* $\Lambda(b, X) = \sigma(R)$.

For equally distributed independent random variables $X_n$, condition (3) can be slightly weakened. We set $\psi(t) = t / L_2(t)$.

**Theorem 3.** *Assume that $R$ is the covariance operator of $X_1$, the independent random variables $X_n$ are equally distributed, condition (4) is satisfied, and*

$$b_n^2 = O(\psi(B_n)\, n^{-1/p}). \tag{7}$$

*Then Theorem 2 is true and the law of the iterated logarithm (1), (2) holds for* $\Gamma(b, X) = 0$.

**Theorem 4.** *Assume that the independent random variables $X_n$ are symmetric and equally distributed. Let $E$ be a space of type 2. In order that the law of the iterated logarithm (1), (2) and the equality $\Lambda(b, X) = \sigma(R)$ be valid, it is sufficient that any of the following groups of conditions be satisfied:*

*(i)* $b_n^2 \downarrow$ *as* $n \uparrow \infty$ *and*

$$b_n^2 = o(\psi(B_n)); \tag{8}$$

*(ii) For some* $1 < p < \infty$, *condition (4) is satisfied,*

$$b_n^2 \uparrow \text{ and } B_n / b_n^2 \uparrow \text{ as } n \to \infty, \tag{9}$$

*and*

$$b_n^2 = O(B_n / (L(B_n))^{1/(p-1)}); \tag{10}$$

*(iii) Conditions (8) and (9) are satisfied and there exists* $h > 0$ *such that*

$$M \exp(h \| X_1 \|) < \infty.$$

## 2. The Law of the Iterated Logarithm in $R^1$

Assume that $(\xi_n)$ is a sequence of independent random variables in $R^1$, $M\xi_n = 0$, $D\xi_n = 1$, and $Z_n = \sum_1^n b_i \xi_i$. In $R^1$, the law of the iterated logarithm is usually understood in the sense that the equalities

$$\limsup_n \frac{Z_n}{\chi(B_n)} = 1, \tag{11}$$

$$\limsup_n \frac{Z_n}{\chi(B_n)} = -1 \tag{12}$$

should be satisfied almost surely.

**Proposition 1.** *In order that the law of the iterated logarithm (11), (12) be valid, it is sufficient that conditions (3) and (4) be satisfied for* $E = R^1$ *and* $X_n = \xi_n$.

**Proposition 2.** *Assume that the independent random variables* $(\xi_n)$ *are equally distributed and conditions (4) and (7) are satisfied (for* $E = R^1$ *and* $X_n = \xi_n$*). Then the law of the iterated logarithm (11), (12) holds.*

**Proposition 3.** *Assume that the independent random variables* $(\xi_n)$ *are equally distributed and symmetric. In order that the law of the iterated logarithm (11), (12) be valid, it is sufficient that any group of conditions (i)–(iii) in Theorem 4 be satisfied (for* $E = R^1$ *and* $X_n = \xi_n$*).*

***Remark 1.*** Conditions ensuring the validity of the law of the iterated logarithm in $R^1$ were studied in many papers (see, e.g., [6–8]). In [9], the weighted sums of equally distributed independent random variables in $R^1$ were investigated and a condition slightly weaker than (7) was imposed for $1 \le p < 3/2$.

**Lemma 1.** *Let* $I\{A\}$ *be the characteristic function of a set* $A$. *If*

$$\sum_n P(|b_n \xi_n| > \tau \psi(B_n)^{1/2}) < \infty \tag{13}$$

*for any* $\tau > 0$ *and*

$$\lim_{n \to \infty} B_n^{-1} \sum_{i=1}^{n} b_i^2 M(\xi_i^2 I\{|b_i \xi_i| > \tau \psi(B_i)^{1/2}\}) = 0, \tag{14}$$

*then the sequence* $Z_n$ *satisfies the law of the iterated logarithm (11), (12).*

**Lemma 2.** *Assume that the conditions in Statement 3 and any group of conditions (i)–(iii) in Theorem 4 are satisfied. Then* $\lim_{n \to \infty} B_n^{-1} \sum_{i=1}^{n} b_i^2 \xi_i^2 = 1$ *almost surely.*

Lemma 1 is a corollary of the classical Kolmogorov law of the iterated logarithm [7, 10]. Lemma 2 follows from the results in [11].

***Proof of Proposition 1.*** It suffices to establish relations (13) and (14). We have

$$P(|b_n \xi_n| > \tau \psi(B_n)^{1/2}) \le \frac{M|\xi_n b_n|^{2p}}{\tau^{2p}} \psi(B_n)^p \le \frac{C b_n^{2p}}{\tau^{2p}} \psi(B_n)^p.$$

It follows from condition (3) that $b_n^{2p} / \psi(B_n)^p \le C_1 b_n^2 / B_n L(B_n)^{1+\delta/2}$ for sufficiently large $n$. Since the series $\sum_n b_n^2 / B_n L(B_n)^{1+\delta/2}$ is convergent [7, p. 339], condition (13) is satisfied. By using Hölder's inequality, we obtain

$$M(\xi_i^2 I\{|b_i \xi_i| > \tau \psi(B_n)^{1/2}\}) \le (M|\xi_i|^{2p})^{1/p} (P(|b_i \xi_i| > \tau \psi(B_n)^{1/2}))^{1/q},$$

where $1/p + 1/q = 1$. This and relations (4) and (13) imply (14).

***Proof of Proposition 2.*** Proposition 2 can be proved similarly. We have

$$\sum_n P(|b_n \xi_n| > \tau\psi(B_n)^{1/2}) \le \sum_n P(|\xi_n| > C\tau n^{1/p}) < \infty.$$

It is well known that the second series is convergent provided that condition (4) is satisfied and the variables are equally distributed.

***Proof of Proposition 3.*** We use the following well-known inequality [12]:

$$\limsup_n \sum_1^n a_i \frac{\varepsilon_i}{\chi(A_n)} \le 1 \quad \text{a.s.}$$

for $A_n = \sum_1^n a_i^2 \to \infty$ as $n \to \infty$. For symmetric independent random variables, this inequality implies that

$$\limsup_n \sum_1^n b_i \frac{\xi_i}{\chi(A_n(\xi))} \le 1 \quad \text{a.s} \tag{15}$$

if

$$A_n(\xi) = \sum_1^n b_i^2 \xi_i^2 \to \infty \quad \text{as} \quad n \to \infty$$

almost surely.

The divergence of the series $\sum_n b_n^2 \xi_n^2$ under the conditions of Proposition 3 follows from the equalities

$$\sum_n M \min(1, b_n^2 \xi_n^2) = \sum_n (P(b_n^2 \xi_n^2 < 1) + b_n^2 M(\xi_n^2 I\{b_n^2 \xi_n^2 \ge 1\})) = \infty$$

(see [13, p. 53]).

Inequality (15) and Lemma 2 imply the estimate

$$\limsup_n \frac{Z_n}{\chi(B_n)} \le 1 \quad \text{a.s.} \tag{16}$$

In Proposition 3, condition (8) is assumed to be satisfied, whence we immediately get (14) for equally distributed variables $\xi_n$. It is known that (14) implies the inequality $\limsup_n Z_n/\chi(B_n) \ge 1$ a.s. [14]. This and (16) yield equality (11). Equality (12) can be obtained from (11) by passing to the variables $-\xi_n$.

The following auxiliary statement was presented in the proof of Lemma 3 in [15].

**Lemma 3.** *Let* $(\zeta_n)$ *be a sequence of random variables in* $R^1$ *and let* $A_n$ *be a real-valued sequence,* $A_n \uparrow \infty$ *as* $n \to \infty$. *If, for some* $C > 0$, $\beta > 1$, $n_0$, *and all* $t \in (1, \beta)$, *we have* $P(\max_{1\le k\le n} \zeta_k \ge t\chi(A_n)) \le CL(A_n)^{-t^2}$ *for* $n > n_0$, *then* $\limsup_n \zeta_n/\chi(A_n) \le 1$ *almost surely.*

## 3. Proof of Theorems 1–4

***Proof of Theorem 1.*** Fix an arbitrary number $0 < \tau < 1/2$ and set

$$X'_n = I\{\|b_n X_n\| \le \tau d \psi(B_n)^{1/2}\} X_n, \quad X''_n = X_n - X'_n; \quad S'_n = \sum_1^n b_i X'_i, \quad S''_n = S_n - S'_n.$$

To estimate the value $\|S'_n\|$ we need an infinite-dimensional analog of the well-known Bernshtein's exponential inequality [16],

$$P\left( \left\| \sum_1^n Y_i \right\| - M \left\| \sum_1^n Y_i \right\| \ge u \right) \le \exp\left( -\frac{u^2}{2B + 2uV} \right), \tag{17}$$

where $(Y_i)$ are independent random variables in $E$, $u > 0$, $B \ge \sum_1^n M\|Y_i\|^2$, and $V > 0$ is such that $M\|Y_i\|^m \le m!\, M\|Y_i\|^2 V^{m-2}/2$ for $m = 2, 3, \ldots$.

In addition, we use the following inequalities for the independent random variables $(Y_i)$ in $E$ [16, 17]:

$$P\left( \sup_{1 \le k \le n} \left\| \sum_{i=1}^k Y_i \right\| > t \right) \le 2P\left( \left\| \sum_{i=1}^n Y_i \right\| > t - \left( \sum_{i=1}^n M\|Y_i\|^2 \right)^{1/2} \right) \tag{18}$$

for $MY_i = 0$ and

$$M\left| \left\| \sum_{i=1}^n Y_i \right\| - M \left\| \sum_{i=1}^n Y_i \right\| \right|^2 \le \sum_{i=1}^n M\|Y_i\|^2. \tag{19}$$

By the definition of $X''_n$, we have

$$\sum_n P(X''_n \ne 0) \le \sum_n (b_n/\tau d)^{2p} M\|X_n\|^{2p} \psi(B_n)^{-p} < \infty.$$

Under condition (4), the last inequality was established in the proof of Proposition 1. Thus, by virtue of the Borel–Cantelli lemma, we get

$$\sup_n \|S''_n\| = S < \infty \quad \text{a.s.} \tag{20}$$

Hence, to prove the inequality on the right-hand side of (5), it suffices to establish the estimate

$$\limsup_n \frac{\|S'_n\|}{\chi(B_n)} \le d + \Gamma(b, X). \tag{21}$$

By using estimates (19) and (20), we obtain

$$\frac{1}{2} \ge P(|\|S''_n\| - M\|S''_n\|| \ge \sqrt{2} B_n^{1/2}) \ge P(M\|S''_n\| B_n^{-1/2} \ge \|S''_n\| B_n^{-1/2} + \sqrt{2})$$

$$\ge P(M\|S''_n\| B_n^{-1/2} \ge S B_n^{-1/2} + \sqrt{2}).$$

This and (20) imply that

$$M\| S_n'' \| \leq C B_n^{1/2}. \tag{22}$$

Further, we set $Y_i = b_i X_i'$, $B = d^2 B_n$, and $V = \tau d \psi(B_n)^{1/2}$ in (17). Then

$$P(\| S_n' \| - M \| S_n' \| > u) \leq \exp(-u^2 / [2B_n d^2 + 2u\tau d\psi(B_n)^{1/2}]).$$

By inserting in this inequality $u = vd(1+2\tau)^{1/2}\chi(B_n)$ for $v \in (1+\beta)$, $\beta = (2/(1+2\tau))^{1/2}$, we get

$$P(\| S_n' \| - M \| S_n' \| \geq vd(1+2\tau)^{1/2}\chi(B_n)) \leq L(B_n)^{-v^2}. \tag{23}$$

For fixed $\tau$ and sufficiently large $n$, we obtain

$$M\| S_n \| \leq (\Gamma(b, X) + \tau)\chi(B_n). \tag{24}$$

Since $| M\| S_n \| - M \| S_n' \| | \leq M \| S_n'' \|$, it follows from estimates (22)–(24) that

$$P(\| S_n' \| \geq [\Gamma(b, X) + 2\tau + vd(1+2\tau)^{1/2}]\chi(B_n)) \leq L(B_n)^{-v^2} \tag{25}$$

for $v \in (1, \beta)$ and sufficiently large $n$. In view of the equality $M S_n' + M S_n'' = M S_n = 0$, we have

$$M\| S_n' \| \leq M\| S_n'' \| \leq C B_n^{1/2}. \tag{26}$$

It follows from estimates (18), (25), and (26) that

$$P(\sup_{1 \leq k \leq n} \| S_k' - M S_k' \| \geq v[\Gamma(b, X) + 4\tau + d(1+2\tau)^{1/2}]\chi(B_n)) \leq L(B_n)^{-v^2}$$

for $v \in (1, \beta)$ and sufficiently large $n$. By using Lemma 3, we obtain the inequality

$$\limsup_n \frac{\| S_n' - M S_n' \|}{\chi(B_n)} \leq \Gamma(b, X) + 4\tau + d(1+2\tau)^{1/2}$$

almost surely. Since $\tau$ is an arbitrary number from the interval $(0, 1/2)$, we get

$$\limsup_n \frac{\| S_n' - M S_n' \|}{\chi(B_n)} \leq d + \Gamma(b, X) \quad \text{a.s.}$$

This and (26) yield (21), i.e., the right inequality in (5) is established. The left inequality in (5) follows from the Fatou lemma (see [5]).

***Proof of Corollary 1.*** The equality $\Gamma(b, X) = 0$ and (5) imply (1); moreover, if we take into account that

$$\Gamma(b, \Pi_N X) = 0 \tag{27}$$

in the finite-dimensional space $\Pi_N E$ and, hence, $\Gamma(b, Q_N X) = 0$, then we obtain

$$\Lambda(b, X) \le \Lambda(b, \Pi_N X) + \Lambda(b, Q_N X) \le \sup_n D(\Pi_N X_n) + \sup_n D(Q_N X_n).$$

By virtue of (6), the second term tends to zero as $N \to \infty$, while the first one, in view of (4), is less than $\infty$ for all $N$. Hence, for any $\varepsilon > 0$, the set $\{S_n / \chi(B_n), n \ge 1\}$ can be covered (almost surely) by finitely many balls of radius $\varepsilon$ and, thus, it is precompact.

***Proof of Theorem 2.*** The left inequality of the theorem is a consequence of the law of the iterated logarithm in $R^1$ (Proposition 1) and the Fatou lemma. Before proving the right inequality, let us establish the equality $\Lambda(b, X) = \sigma(R)$ in the finite-dimensional case $(E = R^m)$.

Due to the left inequality in Theorem 2, it suffices to show that $T = \Lambda(b, X) \le \sigma(R)$. By Proposition 1, we have $T < \infty$. In our case, it follows from the 0 or 1 law that there exists a measurable subset $\Omega_c \subset \Omega$ such that $P(\Omega_c) = 1$ and $\limsup_n \| S_n(\omega) \| / \chi(B_n) = T$ for $\omega \in \Omega_c$ (see [3–5]). Since any bounded set in $R^m$ is precompact, we conclude that, for every $\omega \in \Omega_c$, there exist a (random) sequence $n_k$ and a (random) point $x$ such that

$$\lim_k \| S_{n_k}(\omega) \| / \chi(B_{n_k}) = T \quad \text{and} \quad \lim_k \| S_{n_k}(\omega) / \chi(B_{n_k}) - x \| = 0.$$

Thus, $\|x\| = T$. According to Proposition 1, for any $f \in E^*$, $\|f\| = 1$, we have

$$(M|f(X_1)|^2)^{1/2} = \limsup_n f(S_n) / \chi(B_n) \ge \lim_k f(S_{n_k}) / \chi(B_{n_k}) \ge f(x).$$

Then

$$\sigma(R) = \sup\{(M|f(X_1)|^2)^{1/2} : \|f\| = 1\} \ge \sup\{|f(x)| : \|f\| = 1\} \ge \|x\| = T.$$

Thus, $\Lambda(b, \Pi_N X) = \sigma(R_N)$ for any $N$, where $R_N$ is a (common) covariance operator of the variables $\Pi_N X_n$. It follows from the last equality and (5) that

$$\Lambda(b, X) \le \Lambda(b, \Pi_N X) + \Lambda(b, Q_N X) \le \sigma(R_N) + \sup_n D(Q_N X_n) + \Gamma(b, Q_N X).$$

Since $\sigma(R_N) \le \sigma(R)$ (see [5]) and $\Gamma(b, Q_N X) = \Gamma(b, X)$ [by virtue of (27)], we can pass to the limit with respect to $N$; as a result, by using (6), we arrive at the right inequality of Theorem 2.

We have $M \| S_n \|^2 \le C(E)\, d^2 B_n$ in a space of type 2 and

$$M \| S_n \|^2 \le C(E)\, M \| G(R) \|^2 B_n$$

in a space of cotype 2 [2]. Thus, $\Gamma(b, N) = 0$ in both these cases. This equality guarantees the validity of Corollaries 2 and 3.

***Proof of Theorem 3.*** The proof of Theorem 3 is similar to the proof of Theorem 2 (the transition from conditions (3) to conditions (7) is justified in the proof of Proposition 2).

***Proof of Theorem 4.*** According to the law of the iterated logarithm in $R^1$ (Proposition 3), we have, for any $f \in E^*$,

$$\limsup_n \frac{f(S_n)}{\chi(B_n)} = (M|f(X_1)|^2)^{1/2} \tag{28}$$

almost surely.

Thus, for any $N$,

$$\Lambda(b, \Pi_N X) = \sigma(R_N) \tag{29}$$

almost surely (see the proof of Theorem 2).

In a space of type 2, for the independent symmetric random variables $(X_n)$, we have

$$\limsup_n \frac{\|S_n\|}{\chi\left(\sum_1^n b_i^2 \|X_i\|^2\right)} \le C < \infty \quad \text{a.s.} \tag{30}$$

for

$$\sum_1^n b_i^2 \|X_i\|^2 \uparrow \infty, \quad n \to \infty \quad \text{a.s.} \tag{31}$$

(see [15]; the validity of condition (31) is established in Proposition 3).

Lemma 2 and estimate (30) yield $\Lambda(b, X) \le Cd$. To complete the proof, it suffices to repeat the argument in the proof of Theorem 2 by using this inequality and (29).

***Remark 2.*** Let $K$ be a unit ball in the Hilbert space $H_R$. It is known [3] that, in the case where relations (2) and (28) are satisfied (they are satisfied, e.g., under the conditions of Theorems 2–4 with $\Gamma(b, X) = 0$), we have $\inf\{\|S_n/\chi(B_n) - x\| : x \in K\} \to 0$ as $n \to \infty$ almost surely; if, in addition, $H_R$ is an infinite-dimensional space, then $C(\{S_n/\chi(B_n)\}) = K$ almost surely, where $C(\{x_n\})$ is the set of limiting points of the sequence $\{x_n\}$.

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