AUTOMATIC CONTINUITY, BASES, AND RADICALS IN METRIZABLE ALGEGBRAS

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The automatic continuity of a linear multiplicative operator T: $X \to Y$, where X and Y are real complete metrizable algebras and Y semi-simple, is proved. It is shown that a complex Frechét algebra with absolute orthogonal basis (x_i) (orthogonal in the sense that $x_i x_j = 0$ if $i \neq j$) is a commutative symmetric involution algebra. Hence, we are able to derive the well-known result that every multiplicative linear functional defined on such an algebra is continuous. The concept of an orthogonal Markushevich basis in a topological algebra is introduced and is applied to show that, given an arbitrary closed subspace Y of a separable Banach space X, a commutative multiplicative operation whose radical is Y may be introduced on X. A theorem demonstrating the automatic continuity of positive functionals is proved.

The subject of automatic continuity of linear multiplicative operators and linear multiplicative functionals defined on algebras with involution possesses a rich history, having been initiated by a still unsolved problem posed by S. Mazur concerning the continuity of a linear multiplicative functional on a complete metrizable complex-valued algebra [1, p. 90]. Many results on the topic may be found in [1]. The definitions and notation used in the present article are taken from [1].

THEOREM 1. Let X and Y be real-valued complete metrizable algebras, Y semi-simple. Then every linear multiplicative operator T: $X \to Y$ is continuous.

Proof. We use the closed map theorem. Let $x_n \to x$, $Tx_n \to y$. If $Tx \neq y$, then, from the fact that Y is semi-simple, there exists a linear multiplicative functional g on Y such that $g(Tx) \neq g(y)$. Since the algebra R of real numbers satisfies the following condition:

(C) for any sequence $y_n \in R$, $|y_n| > a > 0$, there exists a sequence f_n of real multiplicative linear functionals with $\inf_{mn}|f_m \times (y_n)| = \varepsilon > 0$, then, by Theorem 3.5 of [1] the functionals $g(y)$ and $gT(x)$ are continuous on Y and X, respectively. Thus, $gT(x_n) \longrightarrow g(Tx)$ and $g(Tx_n) \longrightarrow g(y)$. Contradiction.

Remark. Theorem 1 generalizes Theorem 3.5 of [1] and Theorem 1 of [2]; moreover, the proof as a whole is less tedious, since it is only necessary to derive Theorem 3.5 of [1] for the case of multiplicative functionals. As in Theorem 1 of [3], in Theorem 1 in place of semi-simplicity we need only require that TX intersect the radical of Y in zero.

Let us recall some notation. A basis $(x_i)_1^\infty$ of a Frechét algebra is said to be orthogonal if $x_i x_j = 0$ whenever $i \neq j$. We say that an element x of a commutative algebra X is quasi-regular if there exists an element $y \in X$ such that $xy + x + y = 0$. A commutative involution algebra X is said to be symmetric if for every $x \in X$ the element xx^* is quasi-regular.

THEOREM 2. Any complex-valued Frechét algebra with absolute orthogonal basis (x_i) is a commutative symmetric involution algebra.

Proof. That an algebra with orthogonal basis is commutative is proved in [1, p. 63].

Let us introduce an involution on X in the following way: if $x = \sum_1^\infty a_i x_i$, we set $x^* = \sum_1^\infty \bar{a}_i x_i$.

From the fact that the basis is absolute it is clear that the remainder of the series con-

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verges [1, p. 61]. Hence, it follows that the involution operation is continuous. It remains for us to verify that every element $xx^* \in X$ is quasi-regular, that is, to establish the existence of an element $y \in X$ such that $yxx^* + xx^* + y = 0$. Let $x = \sum_1^\infty a_i x_i$. The element y is found in the form $y = \sum_1^\infty b_i x_i^2$. We write

$$\sum_1^\infty b_i |a_i|^2 x_i^4 + \sum_1^\infty |a_i|^2 x_i^2 + \sum_1^\infty b_i x_i^2 = 0.$$

If $x_i^2 = 0$, we have $x_i^4 = 0$ and we may set $b_i = 0$. If $x_i^2 \neq 0$, but $x_i^4 = 0$, we set $b_i = -|a_i|^2$. But if $x_i^4 \neq 0$ (whence $x_i^2 \neq 0$ as well), from the representation $x_i^3 = \sum_1^\infty c_k x_k$ we have that $x_i^4 = c_i x_i^2$. Therefore, we may suppose that $x_i^4 = x_i^2$, or else make the substitution $x_i' = x_i/\sqrt{c_i}$. Thus, for this i we have the equality

$$b_i |a_i|^2 x_i^2 + |a_i|^2 x_i^2 + b_i x_i^2 = 0,$$

whence $b_i = -|a_i|^2/(1 + |a_i|^2)$.

Since in all three cases $\sqrt{|b_i|} \leq |a_i|$, the series $\sum_1^\infty \sqrt{|b_i|}\, x_i$ converges. Therefore, the series $\sum_1^\infty |b_i| x_i = \left(\sum_i^\infty \sqrt{|b_i|}\, x_i\right)\left(\sum_1^\infty \sqrt{|b_i|}\, x_i\right)$ converges, and hence the series $\sum_1^\infty b_i x_i^2$ also converges.

<u>COROLLARY.</u> Every linear multiplicative functional defined on a complex-valued Frechét algebra with absolute orthogonal basis is continuous.

The <u>proof</u> is a simple combination of Theorem 2 and Mackley's theorem [1, p. 37].

<u>Remark.</u> A straightforward proof of this corollary is given in [1, p. 66], though it is lengthier.

<u>Definition.</u> A system x_i, f_i, $i = 1, \infty$, $x_i \in X$, $f_i \in X^*$ (X is a topological algebra and X^* a directed space) is said to be an orthogonal Markushevich basis (briefly, an orthogonal M-basis) if the linear hull $[x_i]_1^\infty = X$ is closed, $f_i(x_j) = \delta_{ij}$ (δ_{ij} is the Kronecker symbol), $\forall x \in X$, $x \neq 0$ $\exists i: f_i(x) \neq 0$ and $x_i x_j = 0$ for $i \neq j$.

The trigonometric system of the algebra $L_1(0, 2\pi)$ is an example of an orthogonal M-basis that is not an orthogonal basis. Many of the results found by Husain and his coworkers on orthogonal bases may be carried over to orthogonal M-bases. We will not make this transition, but instead will use orthogonal M-bases to determine whether radicals are complementable in a Banach algebra. Determining under what conditions a radical will possess a closed or open subalgebra as complement has been investigated in considerable detail (cf. [4] and the bibliography therein). Let us show that there exist many radicals in Banach algebras that do not possess complements of a closed subspace.

<u>THEOREM 3.</u> Let Y be a closed subspace of a separable Banach space X. Then on X we may introduce a continuous commutative multiplicative operation so that, relative to this operation, Y becomes a radical.

<u>Proof.</u> It is known [5] that for any $\varepsilon > 0$ there exists a sequence $\hat{x}_n$, f_n, $n = 1, \infty$, $\hat{x}_n \in X/Y$, $f_n \in Y^\perp = \{f \in X^* : \forall y \in Y\ f(y) = 0\}$, such that $[\hat{x}_n]_1^\infty = X/Y$; $f_n(\hat{x}_m) = \delta_{nm}$; $\forall \hat{x} \in X/Y\ \exists n : f_n(\hat{x}) \neq 0$; $\|\hat{x}_n\| = 1$, $\|f_n\| < 1 + \varepsilon$. Let us consider arbitrary representatives $x_n \in \hat{x}_n$ with $\|x_n\| < 1 + \varepsilon$. For arbitrary $y, y' \in Y$ and an arbitrary finite number sequence $(a_n, b_n)_1^N$ we set

$$\left(\sum_1^N a_n x_n + y\right)\left(\sum_1^N b_n x_n + y'\right) = \sum_1^N \frac{1}{2^n (1+\varepsilon)^3} a_n b_n x_n. \tag{1}$$

The norm of the right side does not exceed $\max_n |a_n b_n|/(1+\varepsilon)^2$, and for some m, $1 \le m \le N$, $\left\| \sum_1^N a_n x_n + y \right\| \|f_m\| \geqslant \left| f_m \left(\sum_1^N a_n x_n + y \right) \right| = |a_m|$; therefore, $\left\| \sum_1^N a_n x_n + y \right\| \geqslant \max_n |a_n|/(1+\varepsilon)$. A similar inequality holds for $\sum_1^N b_n x_n + y'$. Therefore, the inequality $\|uv\| \le \|u\| \|v\|$ is satisfied on the linear hull $\mathrm{lin}(Y, (x_n)_1^\infty)$, which is dense in the space X. Obviously, the operation (1) is a commutative multiplication operation, and therefore it may be extended continuously to all X. As usual, Y belongs to the radical of the resulting algebra. If $x \notin Y$, then for some n, $f_n(x) \neq 0$, whence the multiplicative functional $(2^n(1+\varepsilon)^3)^{-1/2} f_n$ is not zero on x; therefore, x does not belong to the radical.

Remark. Numerous examples of noncomplementable subspaces are given in [6]. By Theorem 3, they are radicals of certain commutative Banach algebras. Theorem 3 may be easily carried over to separable Frechét spaces. It would be of considerable interest to determine if this theorem may be extended to nonseparable Banach spaces.

The following results are related to the next two questions.

1. Let X be a Banach algebra and suppose that the subspace $X^2 = \left\{ \sum_1^N x_i y_i : x_i, y_i \in X, n = \overline{1, \infty} \right\}$ has finite defect in X. Is it closed [7, p. 76]?

2. Let X be a Banach involution algebra and suppose that the subspace X^2 is closed and has finite defect in X. Is every positive functional on X continuous [8]?

In [8] it is shown that if an involution algebra is commutative and separable, and X^2 has finite defect, any positive functional X is continuous.

THEOREM 4. Let X be a semi-simple commutative Banach involution algebra and suppose that the unit sphere B(X) is compact in the weak topology $w(X, \Gamma)$, where Γ is the set of linear multiplicative functionals continuous on X. If X^2 has finite defect in X, the subspace X^2 is closed and every positive linear functional on X is continuous.

Proof. Let us show that the closure of the set $Z_m^n = \left\{ z = \sum_1^n x_i y_j : \|x_i\| = \|y_i\|, \sum_1^n \|x_i\| \|y_i\| \leqslant m\|z\| \right\}$ is in X^2. In fact, let $z^k \to z_0$, $z^k = \sum_{i=1}^n x_i^k y_i^k \in Z_m^n$. Since the sequence (z^k) is bounded, for some i the two sequences $(x_i^k)_{k=1}^\infty$ and $(y_i^k)_{k=1}^\infty$ are bounded. Therefore, by virtue of compactness there exist a sequence $k(s)$, $s = 1, \ldots, \infty$ and points x_i, y_i, $i = 1, \ldots, n$, that are limit points of the corresponding sets $\{x_i^{k(s)}, s = 1, \ldots, \infty\}$ and $\{y_i^{k(s)}, s = 1, \ldots, \infty\}$ in the topology $w(X, \Gamma)$. Since X is a semi-simple algebra, $\sum_{i=1}^n x_i y_i = z_0$.

Thus, the subspace X^2 will be a countable union of closed sets, that is, a Borel set. Since a Borel subspace of a separable Banach space of finite codimension is closed [9], it is easily deduced that this result also holds without assuming separability. Since X^2 has finite defect, X^3 also has finite defect. Thus [7, p. 77], any positive linear functional on X is continuous.

Remark. In a certain sense Theorem 3 strengthens a result found in [10] that, in turn, strengthens an unpublished result of the present author (cf. remark in [10]).

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LIMITS OF ANALYTIC VECTOR MEASURES

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The article attempts to determine when a vector measure is the limit of a sequence of analytic vector measures in the sense of convergence in semivariation and when it is the limit of a sequence of such measures in variation.

Suppose H is the linear hull of an orthonormal basis in an infinite-dimensional separable Hilbert space X and let μ be a vector measure defined on the sigma algebra B(X) of Borel subsets of X taking values in the Banach space Y. In the present article we investigate when the vector measure μ may be represented as the limit of H-analytic vector measures in the sense of convergence in semivariation, and when as the limit of such measures in variation.

The concept of analyticity of scalar measures was introduced in [1]. In [2] the relationship between analyticity and other differential properties of scalar measures was studied, and in [3] the limits of H-differentiable scalar measures were investigated. In view of the development of the theory of general vector measures (the basic concepts of this theory may be found in [4]), it is of interest to consider H-analyticity and the corresponding limits for vector measures.

1. By a shift of the vector measure μ by an element $h \in X$ we will understand the vector measure μ specified by means of the formula $\mu_h(E) = \mu(E + h)$. By [1], μ is said to be H-analytic if for any $h \in H$ and $E \in \mathscr{B}(X)$ the function $t \to \mu(E + th)$ is extended analytically in some neighborhood of zero to $\mathbb{C}$, independent of E. From the analyticity of μ along h it follows not only that it is infinitely differentiable, but also that it is quasi-invariant along h (the proof presented in [2], Proposition 3, part 2, is easily extended to the case of vector measures).

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