KHINCHIN'S INEQUALITY AND THE ASYMPTOTIC BEHAVIOR OF THE SUMS $\Sigma\varepsilon_n x_n$ IN BANACH LATTICES

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A series of generalizations of the classical Khinchin inequality to Banach lattices are given. The asymptotic behavior of $\|\Sigma_1^n \varepsilon_i x_i\|$ is investigated.

A substantial number of publications [1] have been devoted to the investigation of independent random variables (i.r.v.) in Banach spaces. At the same time, relatively little attention has been given to an important special case of Banach spaces, namely to Banach lattices [2-5]. This paper has been stimulated by the following generalization of the classical Khinchin inequality to q-concave ($q < \infty$) Banach lattices, obtained by Maurey [3, 6, pp. 49-50]:

$$C^{-1}\left\|\left(\sum_1^n |x_i|^2\right)^{1/2}\right\| \leqslant M\left\|\sum_1^n \varepsilon_i x_i\right\| \leqslant C\left\|\left(\sum_1^n |x_i|^2\right)^{1/2}\right\|, \tag{1}$$

where (ε_i) are independent copies of ε, ε is a symmetric Bernoulli r.v., $P(\varepsilon = \pm 1) = 1/2$, x_i are elements of a Banach lattice X.

We obtain some generalizations of the inequalities (1) and we apply them to the investigation of the asymptotic behavior of the variable $\|\Sigma_1^n \varepsilon_i x_i\|$.

We recall some definitions [6, 7]. A Banach lattice X is a vector lattice which at the same time is a Banach space with the following consistency of the order and the norm: $|x| \leqslant |y| \Rightarrow \|x\| \leqslant \|y\|$, where $|x|$ is the absolute value of the element $x \in X$. There exists a standard method of introducing into a Banch lattice functional operations, in particular, the operation $(\Sigma_1^n |x_i|^p)^{1/p}$ [6, 1.d]. Therefore, the following definition makes sense.

Definition. A Banach lattice X is said to be q-concave if there exists a constant $D_q(X)$ such that

$$\left(\sum_{i=1}^n \|x_i\|^q\right)^{1/q} \leqslant D_q(X)\left\|\left(\sum_1^n |x_i|^q\right)^{1/q}\right\|, \quad 1 \leqslant q < \infty,$$

$$\sup_{1\leqslant i\leqslant n} \|x_i\| \leqslant D_\infty(X)\left\|\bigvee_{i=1}^n |x_i|\right\|, \quad q = \infty$$

for any elements $x_1, \ldots, x_n$ from X and any n.

As examples of Banach lattices we mention the spaces C[0, 1], $L_p(\mu)$, and V[0, 1]. The first two are Banach lattices with the usual pointwise order. In the space V[0, 1] the order is generated by the duality between it and C[0, 1]. The lattice $L_p(\mu)$ is q-concave for $q \geqslant p$; V[0, 1] is q-concave for any $1 \leqslant q \leqslant \infty$; C [0, 1] is not q-concave for any $q < \infty$.

In the sequel by $(\xi_i)_1^\infty$ we shall denote a sequence of i.r.v. in a Banach lattice X, $M\xi_i = 0$, $\|\cdot\|$ is the norm in X, $\|\xi\|_p = (M\|\xi\|^p)^{1/p}$. If the Banach lattice X is nonseparable, then we shall assume the separability of ξ_i. The constant $0 < C_p(X) < \infty$ depends only on p and X and in various places it does not necessarily denote the same quantity.

Proposition 1. Assume that for some $q < \infty$ X is a q-concave Banach lattice. Then for $1 \leqslant p < \infty$ there exists a constant $C_p(X) < \infty$ such that

$$C_p^{-1}(X)\left\|\left(\sum_1^n |\xi_i|^2\right)^{1/2}\right\|_p \leqslant \left\|\sum_1^n \xi_i\right\|_p \leqslant C_p(X)\left\|\left(\sum_1^n |\xi_i|^2\right)^{1/2}\right\|_p. \tag{2}$$

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Proof. The inequalities (2) follow from (1) for p = 1 and $\xi_i = \varepsilon_i x_i$, $i \geqslant 1$. On the other hand, for $0 < p, r < \infty$ there exists a number $K_{p,r}$ such that

$$\Big\| \sum_1^n \varepsilon_i x_i \Big\|_p \leqslant K_{p,r} \Big\| \sum_1^n \varepsilon_i x_i \Big\|_r$$

(see [6, p. 74]). Consequently, inequalities (2) hold for $1 \leqslant p < \infty$ and $\xi_i = \varepsilon_i x_i$. Since, without loss of generality [1, p. 209], any independent symmetric r.v. $(\xi_i)_1^\infty$ in X can be considered represented in the form $\xi_i = \varepsilon_i \bar{\xi}_i$, where $(\varepsilon_i, \bar{\xi}_i)$ are independent in their totality, for any i ξ_i and $\bar{\xi}_i$ of the same distribution, from Fubini's theorem there follows that inequalities (2) hold for the symmetric r.v. $(\xi_i)_1^\infty$. The general case is obtained from the unconditionality of the sequence $(\xi_i)_1^\infty$ [1, p. 239] in $L_p(X)$, from Burkholder's known arguments (see Proposition 2), and from the estimates in the symmetric case.

A Banach space X is said to possess the unconditionality property for martingale differences ($X \in UMD$) [8] if for any X-valued martingale difference sequence $(d_i)_1^\infty$, $\| d_i \|_p < \infty$, any numbers θ_i ($\theta_i = \pm 1$), all $n \geqslant 1$ and $1 < p < \infty$ we have

$$\Big\| \sum_1^n \theta_i d_i \Big\|_p \leqslant C_p(X) \Big\| \sum_1^n d_i \Big\|_p . \tag{3}$$

Proposition 2. If a Banach lattice X has the UMD property, then for any $1 < p < \infty$ and any martingale difference sequence $(d_i)_1^\infty$, $\| d_i \|_p < \infty$ we have the inequalities

$$C_p^{-1}(X) \Big\| \Big(\sum_1^n |d_i|^2 \Big)^{1/2} \Big\|_p \leqslant \Big\| \sum_1^n d_i \Big\|_p \leqslant C_p(X) \Big\| \Big(\sum_1^n |d_i|^2 \Big)^{1/2} \Big\|_p .$$

Proof. It is known [8] that if a Banach space $X \in UMD$, then it is superreflexive. A superreflexive space has type greater than 1 [9, p. 91]. From here it follows [6, p. 92] that the Banach lattice X is q-concave for some $q < \infty$. Assume that the sequence $(\varepsilon_i)_1^\infty$ is such that the collections $(\varepsilon_i)_1^\infty$ and $(d_i)_1^\infty$ are independent among themselves. Then from (1) and Kahane's inequality [6, p. 74] we have

$$C_p^{-1}(X) \Big\| \Big(\sum_1^n |d_i|^2 \Big)^{1/2} \Big\|_p \leqslant \Big\| \sum_1^n d_i \varepsilon_i \Big\|_p \leqslant C_p(X) \Big\| \Big(\sum_1^n |d_i|^2 \Big)^{1/2} \Big\|_p .$$

From here and from the estimate (3) we obtain the inequalities of Proposition 2.

We mention that the spaces $L_p(\mu) \in UMD$ for $1 < p < \infty$, the Orlicz spaces $L_M \in UMD$, when M and M* satisfy the Δ_2-condition [10].

Remark. Inequalities (2) for $1 < p < \infty$ are valid for any martingale difference in R^1 (Burkholder's inequality). In the infinite-dimensional case this is not always so. For example, in the Banach lattice $L_1[0, 1]$, which is q-concave, for each $1 \leqslant q < \infty$ there exists a martingale difference sequence for which the inequalities (2) do not hold.

Indeed, if the inequalities of Proposition 2 would hold in $L_1[0, 1]$, then also inequality (3) would hold, i.e., $L_1[0, 1] \in UMD$, leading to a contradiction since $L_1[0, 1]$ is not reflexive.

We give some auxiliary statements.

LEMMA 1. Let X be a Banach space and let $x_i \in X$, $i = \overline{1, n}$. Then for any $k \geqslant 1$ we have

$$M \Big\| \sum_1^n \varepsilon_i x_i \Big\|^{2k} \leqslant (2k-1)^k \Big(M \Big\| \sum_1^n \varepsilon_i x_i \Big\|^2 \Big)^k .$$

The proof of Lemma 1 is contained in the proof of Theorem 1.e.13 [6, p. 76].

LEMMA 2. Assume that for some $q < \infty$ is a q-concave Banach lattice, $x_i \in X$, $i = \overline{1, n}$. Then

$$\Big(M \Big\| \sum_1^n \varepsilon_i x_i \Big\|^2 \Big)^{1/2} \leqslant K_q \Big\| \Big(\sum_1^n |x_i|^2 \Big)^{1/2} \Big\| ,$$

where $K_q = D_r(X) B_r$, $r = \max(2, q)$, $B_r = \sqrt{2}(\Gamma((r+1)/2)/\sqrt{\pi})^{1/r}$ is the Khinchin constant, $B_2 = 1$, $D_r(X)$ is the constant from the definition of q-concavity.

Proof. For $q \geqslant 2$ we have

$$\left(M \left\| \sum_1^n \varepsilon_i x_i \right\|^2\right)^{1/2} \leqslant \left(M \left\| \sum_1^n \varepsilon_i x_i \right\|^q\right)^{1/q} \leqslant \text{(taking into account q-concavity and the fact that here the mathematical expectation is a simple finite sum)} \leqslant$$

$$\leqslant D_q(X) \left\| \left(M \left| \sum_1^n \varepsilon_i x_i \right|^q\right)^{1/q} \right\| \leqslant D_q(X) B_q \left\| \left(\sum_1^n |x_i|^2\right)^{1/2} \right\|.$$

The proof of the last inequality is contained in the proof of Theorem 1.d.6 [6, p. 50]. For $1 \leqslant q < 2$ we make use of the fact that a q-concave lattice is 2-concave.

LEMMA 3. Let $(\zeta_n)_1^\infty$ be a sequence of r.v. in R^1, let $(A_n)_1^\infty$ be a real sequence, $A_n \uparrow \infty$ for $n \to \infty$, and let

$$P(\max_{1 \leqslant k \leqslant n} \zeta_k \geqslant t\sqrt{A_n}) \leqslant C \exp(-t^2).$$

Then almost surely (a.s.) we have

$$\overline{\lim_{n\to\infty}} \zeta_n / \sqrt{A_n \ln\ln A_n} \leqslant 1.$$

Proof. Assume that the numbers $\beta > \lambda > 1$ are fixed, let $V(i) = \sqrt{A_i \ln\ln A_i}$, let $I_n = \{i : \lambda^{n-1} \leqslant V(i) < \lambda^n\}$, let $(I_{n_k})_1^\infty$ be the sequence of index sets, obtained from $(I_n)_1^\infty$ if we discard the empty sets, and let $\bar{\zeta}_n = \max_{1 \leqslant k \leqslant n} \zeta_k$, $M_k = \max(i : i \in \mathcal{J}_{n_k})$, $m_k = \min(i : i \in \mathcal{J}_{n_k})$. Under the conditions of the lemma we have

$$\sum_{k=1}^\infty P(\bar{\zeta}_{M_k} \geqslant \beta V(M_k)) \leqslant C \sum_{k=1}^\infty (\ln A_{M_k})^{-\beta^2}. \tag{4}$$

Since for sufficiently large k we have $\ln A_{M_k} + \ln\ln\ln A_{M_k} > 2(n_k - 1)\ln\lambda$ and, consequently, $\ln A_{M_k} > (n_k - 1)\ln\lambda$, it follows that the series (4) converges and, by the Borel–Cantelli lemma, we have

$$P(\bar{\zeta}_{M_k} > \beta V(M_k) \text{ i.o.}) = 0 \tag{5}$$

(i.o. denotes infinitely often). Then from the conditions $\beta > \lambda > 1$ and the equality (5) we obtain

$$P(\zeta_n > \beta V(n) \text{ i.o.}) \leqslant P(\bar{\zeta}_{M_k} \geqslant \beta V(m_k) \text{ i.o.}) \leqslant P(\bar{\zeta}_{M_k} \geqslant (\beta/\lambda) V(M_k) \text{ i.o.}) = 0.$$

Consequently, a.s. $\overline{\lim_{n\to\infty}} \zeta_n / V(n) \leqslant \beta$. Since $\beta > 1$ is an arbitrary number, from here we obtain the assertion of the lemma.

A statement closely related to Lemma 3 is contained in [11].

Proposition 3. Let X be a Banach space and let

$$x_i \in X, \ i = \overline{1, n}, \ N_\delta = \max\left(1, \frac{1}{2}(\ln(1+\delta))^{-1/2}(1+\delta)^{-(2\ln(1+\delta))^{-1}}\right),$$

$$M_2(n) = M \left\| \sum_1^n \varepsilon_i x_i \right\|^2.$$

Then for all $\delta > 0$ and $t > 0$ we have

$$P\left(\left\| \sum_1^n \varepsilon_i x_i \right\| > t\left(e(1+\delta) M_2(n)\right)^{1/2}\right) \leqslant 2N_\delta \exp(-t^2/2). \tag{6}$$

Proof. It is sufficient to justify the estimate

$$M \exp\left(t \left\| \sum_1^n \varepsilon_i x_i \right\|\right) \leqslant 2N_\delta \exp\left(\frac{t^2}{2} e(1+\delta) M_2(n)\right) \tag{7}$$

for $\delta > 0$ and any finite t. The passage (7) $\Rightarrow$ (6) is known [12, p. 70 of the Russian edition]. Applying the known estimate $\exp(u) \leqslant 2\cosh(u)$ and Lemma 1, we obtain

$$M\exp\left(t\left\|\sum_1^n \varepsilon_i x_i\right\|\right)\leqslant 2\left(1+\frac{t^2M_2(n)}{2!}+\sum_{k=2}^{\infty}\frac{t^{2k}M\left\|\sum_1^n\varepsilon_i x_i\right\|^{2k}}{(2k)!}\right)\leqslant$$

$$\leqslant 2\left(1+\frac{t^2M_2(n)}{2!}+\sum_{k=2}^{\infty}\frac{(t^2M_2(n))^k}{k!}\,\frac{(2k-1)^k}{2^k(2k-1)!!}\right)\leqslant$$

$$\leqslant 2\left(1+\frac{t^2M_2(n)}{2!}+\sum_{k=2}^{\infty}\frac{(t^2M_2(n))^k}{k!}\left(\frac{\sqrt{\pi}\,(2k-1)^k}{\Gamma(k+1/2)\,2^{2k}}\right)\right). \quad (8)$$

We introduce into (8) the elementary estimate

$$\frac{\sqrt{\pi}\,(2k-1)^k}{\Gamma(k+1/2)\,2^{2k}}\leqslant\frac{e^k\sqrt{k/2}}{2^k}.$$

Then

$$M\exp\left(t\left\|\sum_1^n \varepsilon_i x_i\right\|\right)\leqslant 2\left(1+\frac{t^2M_2(n)}{2!}+\sum_{k=2}^{\infty}\frac{(t^2M_2(n)\,e\,(1+\delta)/2)^k}{k!}\sqrt{k/2}\,(1+\delta)^{-k}\right)\leqslant$$

$$\leqslant 2\max\left(1,\max_{k\geqslant 2}\sqrt{k/2}\,(1+\delta)^{-k}\right)\exp\left((1+\delta)\,eM_2(n)\,t^2/2\right)\leqslant 2N_\delta\exp\left((1+\delta)\,eM_2(n)\,t^2/2\right),$$

i.e., inequality (7) is established.

From Proposition 3 and Lemmas 2, 3 we obtain the following consequences.

COROLLARY 1. Let X be a Banach space, let $x_i\in X$, $i\geqslant 1$ and $M_2(n)\uparrow\infty$ for $n\to\infty$. Then we have a.s.

$$\overline{\lim_{n\to\infty}}\frac{\left\|\sum_1^n\varepsilon_i x_i\right\|}{\sqrt{2eM_2(n)\ln\ln M_2(n)}}\leqslant 1.$$

COROLLARY 2. Let X be a q-concave Banach lattice for some $q<\infty$ and let $x_i\in X$, $i\geqslant 1$, $\delta>0$. Then

$$P\left(\left\|\sum_1^n\varepsilon_i x_i\right\|>t\,(e(1+\delta))^{1/2}K_q\left\|\left(\sum_1^n|x_i|^2\right)^{1/2}\right\|\right)\leqslant 2N_\delta\exp(-t^2/2). \quad (9)$$

If $\left\|\left(\sum_1^n|x_i|^2\right)^{1/2}\right\|\uparrow\infty$ for $n\to\infty$, then we have a.s.

$$\overline{\lim_{n\to\infty}}\frac{\left\|\sum_1^n\varepsilon_i x_i\right\|}{\sqrt{2eK_q^2\left\|\left(\sum_1^n|x_i|^2\right)^{1/2}\right\|^2\ln\ln\left\|\left(\sum_1^n|x_i|^2\right)^{1/2}\right\|^2}}\leqslant 1, \quad (10)$$

where the constant K_q is defined in Lemma 2, while N_δ in Proposition 3.

We recall that X is called a space of type p if there exists a constant $C_p(X)$ such that for any finite collections $x_i\in X$ we have the inequality $\left\|\sum_1^n\varepsilon_i x_i\right\|_p\leqslant C_p(X)\left(\sum_1^n\|x_i\|^p\right)^{1/p}$.

From the estimates of [13], by a method close to Lemma 3 we derive the following corollary.

COROLLARY 3. Let X be a Banach space of type 2 and let $B_n^2=\sum_1^n\|x_i\|^2\uparrow\infty$, $\|x_n\|=o(B_n^2/\ln\ln B_n^2)^{1/2}$. Then we have a.s.

$$\overline{\lim_{n\to\infty}}\frac{\left\|\sum_1^n\varepsilon_i x_i\right\|}{\sqrt{2B_n^2\ln\ln B_n^2}}\leqslant 1.$$

Examples. 1. $X=L_p[0,1]$, $1\leqslant p\leqslant 2$. Then in the inequalities (9), (10) we have $K_q=1$. For $p>2$ $K_q=B_p$ is Khinchin's constant, while in the inequalities (9), (10) the quantity

$\left\|\left(\sum_1^n |x_i|^2\right)^{1/2}\right\|$ can be replaced by $\left(\sum_1^n \|x_i\|^2\right)^{1/2}$.

2. X = V[0, 1] is the Banach lattice of functions of bounded variation. V[0, 1] is a 2-concave Banach lattice in $D_2(V[0, 1]) = 1$ since V[0, 1] = C*[0, 1], while C[0, 1] is a p-concave Banach lattice for any $1 \leqslant p < \infty$, $D^{(2)}(C[0, 1]) = 1$ (see Proposition 1.d.4 from [6]).

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SEMISCALAR EQUIVALENCE AND THE FACTORIZATION OF POLYNOMIAL MATRICES

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One considers the problem of the factorization of polynomial matrices over an arbitrary field in connection with their reducibility by semiscalar equivalent transformations to triangular form with the invariant factors along the principal diagonal. In particular, one establishes a criterion for the representability of a polynomial matrix in the form of a product of factors (the first of which is unital), the product of the canonical diagonal forms of which is equal to the canonical diagonal form of the given matrix. There is given also a method for the construction of such factorizations.

Let P be an arbitrary field, let P[x] be the ring of polynomials over P, and let P_n and $P_n[x]$ be the rings of n × n matrices over P and P[x], respectively. By $\mu_k^A(x)$ we shall denote the k-th invariant factor of the matrix $A(x) \in P_n[x]$, by $D^A(x)$ the canonical diagonal form of A(x), i.e., $D^A(x) = U(x)A(x)V(x) = \operatorname{diag}(\mu_1^A(x), \ldots, \mu_n^A(x))$, $\mu_i^A(x) \mid \mu_{i+1}^A(x)$, $i = 1, \ldots, n-1$, for some

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