SOME REMARKS ON OPERATOR RANGES

A. N. Plichko UDC 517.982

Let X be a Banach space and let X^* be its conjugate space. A system (x_γ, f_γ), $\gamma \in \Gamma$, $x_\gamma \in X$, $f_\gamma \in X^*$, Γ being some set, is said to be *biorthogonal* if $f_\gamma(x_\beta) = 0$ for $\gamma \neq \beta$ and $= 1$ for $\gamma = \beta$, and *fundamental* if the closed linear hull $[x_\gamma: \gamma \in \Gamma] = X$. A fundamental biorthogonal system in which for each $x \in X$, $x \neq 0$, there exists an index $\gamma \in \Gamma$ such that $f_\gamma(x) \neq 0$ is called a Markushevich basis (abbreviated: an M-basis). We say that a linear subspace $Y \subset X$ is an *operator range* if there exist a Banach space Z and a bounded linear operator $T: Z \to X$ such that $TZ = Y$. By $\ell_1(\Gamma)$ we shall denote the space of absolutely summable sequences $x(\gamma)$, $\gamma \in \Gamma$. The space $\ell_1(\Gamma_0)$, $\Gamma_0 \subset \Gamma$, will be identified with a subspace of $\ell_1(\Gamma)$. All the spaces are assumed to be real.

THEOREM 1. Assume that there exists a continuous linear injective operator $T: \ell_1(\Gamma) \to X$ with a dense range. Then X has two dense operator ranges, intersecting only at zero.

Proof. Let Γ_i: $i \in J$, be a partition of Γ into countable subsets: $\Gamma = \cup_{i \in J}\Gamma_i$. In each space $\ell_1(\Gamma_i)$ there exist dense operator ranges Y_i, Z_i, $Y_i \cap Z_i = 0$ [1]. We denote by Y the subspace consisting of the absolutely convergent series $\Sigma_{i\in J}y_i$, $y_i \in Y_i$, and by Z the subspace consisting of the absolutely convergent series $\Sigma_{i \in J}z_i$, $z_i \in Z_i$. Then TY and TZ will be the required operator ranges.

COROLLARY 1. A Banach space X with a fundamental biorthogonal system $(x_\gamma, f_\gamma, \gamma \in \Gamma)$ has two dense operator ranges, intersecting only at zero.

Proof. Obviously, we can assume that $\|x_\gamma\| = 1$. Let e_γ be the unit vectors of the space $\ell_1(\Gamma)$. We set $Te_\gamma = x_\gamma$. The mapping T extends to a bounded linear operator from $\ell_1(\Gamma)$ into X. It is easy to see that it is injective.

COROLLARY 2. In the space $\ell_\infty(\Gamma)$ there exist two dense operator ranges, intersecting only at zero.

Indeed, $\ell_\infty(\Gamma)$ has a fundamental biorthogonal system [2].

Theorem 1 and the corollaries are connected with the question from [3] regarding the existence in an arbitrary Banach space, in particular in ℓ_∞, of two operator ranges, intersecting only at zero. This question is related to the investigation of supportless convex sets. We recall the definitions. A subset A of a normed space is said to be *supportless* if no nonzero continuous linear functional attains its supremum and infimum on A. A normed space is said to be *supportless* if it contains a closed, bounded, convex, supportless set. We say that a Banach space is weakly compactly generated if it is the closed linear hull of a weakly compact subset. By the symbol $c_0(\Gamma)$ we shall denote the space of those sequences $x(\gamma)$ such that for any $\varepsilon > 0$ there exist only a finite number of coordinates exceeding ε in absolute value, with the supremum norm.

LEMMA 1. In the space $\ell_1(\Gamma)$ there exists a dense operator range which intersects the linear hull $\mathrm{lin}(e_\gamma: \gamma \in \Gamma)$ of the unit vectors of $\ell_1(\Gamma)$ at zero.

Proof. Let Γ_i: $i \in J$, be a partition of Γ into countable subsets. In each $\ell_1(\Gamma_i)$ there exists a dense operator range U_i, $U_i \cap \mathrm{lin}(e_\gamma: \gamma \in \Gamma_i) = 0$; this follows from the results of [1]. Then the subspace U, consisting of the absolutely convergent series $\Sigma_{i\in J}u_i$, $u_i \in U_i$, is the required one.

THEOREM 2. In each weakly compactly generated space X there exists a dense, supportless operator range.

Proof. Let Y be a reflexive space and let R be a dense, injective operator from Y into X [4]. Let $(y_\gamma, g_\gamma)_{\gamma\in\Gamma}$ be an M-basis in Y, $\|y_\gamma\| = 1$ [5, p. 693] and let e_γ be the standard unit vectors of the space $\ell_1(\Gamma)$. We set $S(e_\gamma) = y_\gamma$. The mapping S can be extended to a continuous linear injective operator from $\ell_1(\Gamma)$ into Y. According to Lemma 1, we select a dense operator range $U \subset \ell_1(\Gamma)$, $U \cap \mathrm{lin}_{\gamma\in\Gamma}e_\gamma = 0$. Then the operator $T = RS$ is weakly compact, injective and, since S is a conjugate operator, $A = T(B_U)$ (B_U is the ℓ_1-ball of the subspace U) is a closed subset of the normed space $Z = (TU, \|\ \|_X)$. We mention that the operator S^* maps Y^* into the space $c_0(\Gamma) \subset \ell_\infty(\Gamma) = \ell_1(\Gamma)^*$. Therefore, if a is a support point of the set A with support

Translated from Teoriya Funktsii, Funktsional'nyi Analiz i Ikh Prilozheniya, No. 53, pp. 69-70, 1990. Original article submitted September 21, 1987.

functional f, then $T^{-1}a$ is a support point of the ball B_U with support functional $T^*f \in c_0(\Gamma)$. It is easy to see that if b is a support point of the ball of the space $\ell_1(\Gamma)$ and the corresponding support functional is $g \in c_0(\Gamma)$, then b necessarily has a finite number of nonzero coordinates, i.e. it does not belong to U. Thus, the set A does not have support points.

Remark. For separable Banach spaces this theorem has been proved in [3]; also there one has formulated the conjecture of its validity for weakly compactly generated spaces.

Question. In an arbitrary Banach space X, is it possible to imbed densely and injectively some $\ell_1(\Gamma)$? In particular, in $X = m_0(\Delta)$, the space of bounded functions on a set Δ, taking nonzero values in at most a countable number of points.

LITERATURE CITED

1. V. V. Shevchik, "On subspaces of a Banach space that coincide with the ranges of continuous linear operators," Rev. Roumaine Math. Pures Appl., **31**, No. 1, 65-71 (1986).
2. W. J. Davis and W. B. Johnson, "On the existence of fundamental and total bounded biorthogonal systems in Banach spaces," Stud. Math., **45**, No. 2, 173-179 (1973).
3. J. M. Borwein and D. W. Tingley, "On supportless convex sets," Proc. Am. Math. Soc., **94**, No. 3, 471-476 (1985).
4. J. Diestel, Geometry of Banach Spaces — Selected Topics, Lecture Notes in Math., No. 485, Springer, Berlin (1975).
5. I. Singer, Bases in Banach Spaces. II, Springer, Berlin (1981).

THE NEUMANN BOUNDARY VALUE PROBLEM IN AN INFINITE LAYER

I. I. Antypko and V. M. Borok

UDC 517.956

The purpose of this paper is the investigation of the correct solvability in classes of smooth functions, having various growth or decrease character (for $\|x\| \to \infty$), of the Neumann problem in the layer $\Pi_T = \mathbf{R}^m \times [0, T]$ for the second-order equation (with respect to t)

$$\frac{\partial^2 u(x, t)}{\partial t^2} + P\left(\frac{\partial}{\partial x}\right)\frac{\partial u(x, t)}{\partial t} + Q\left(\frac{\partial}{\partial x}\right)u(x, t) = 0, \tag{1}$$

$(x, t) \in \Pi_T$, P(s), Q(s) are arbitrary complex-valued polynomials; the Neumann boundary conditions have the form

$$\frac{\partial u(x, 0)}{\partial t} = u_0(x), \quad \frac{\partial u(x, T)}{\partial t} = u_T(x), \quad x \in \boldsymbol{R}^m. \tag{2}$$

Let $\gamma \in \mathbf{R}$; we denote $C_\gamma^k = \{\varphi \in C^k(\boldsymbol{R}^m) : \|\varphi\|_{\gamma,k} = \sup\limits_{R^m} \max\limits_{|\alpha|<k} |D^\alpha\varphi| \times (1 + \|x\|)^{-\gamma} < \infty\}$.

Definition 1. The problem (1)—(2) is said to be correctly solvable from the space C_γ^{k1} into the space C_γ^{k2} if for any functions $u_0(x) \in C_\gamma^{k1}$, $u_T(x) \in C_\gamma^{k1}$ the problem (1)—(2) has a unique solution u(x, t), belonging for each $t \in]0, T[$ to the space C_γ^{k2} and satisfying the estimate

$$\sup_{[0, T]} \| u(x, t)\|_{\gamma,k_2} < C\{\| u_0(x)\|_{\gamma,k_1} + \| u_T(x)\|_{\gamma,k_1}\}. \tag{3}$$

Translated from Teoriya Funktsii, Funktsional'nyi Analiz i Ikh Prilozheniya, No. 53, pp. 71-78, 1990. Original article submitted October 21, 1987.