THEORY OF REGULARIZABILITY IN TOPOLOGICAL VECTOR SPACES

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Two known definitions of regularizability for topological vector spaces are found to be equivalent. Regularizability in the sense of Tikhonov is considered in reflexive linear metric spaces. In particular, an example is presented of a linear continuous injective operator on a reflexive Fréchet space whose inverse cannot be regularized. The latter indicates the sharp difference between regularizability in Fréchet spaces and in Banach spaces, respectively.

Whereas the regularizability of linear inverse problems in Banach spaces has been extensively studied (see [1, 2] and the bibliography), the conditions of regularizability in topological vector spaces (TVS), or even in Fréchet spaces, have been hardly studied. In this paper we show that the two definitions of regularizability given in [3, 4] are equivalent, and we consider regularizability in reflexive linear metric spaces. In particular, we present an example of linear compactification (of a continuous injective operator) on a reflexive Fréchet space whose inverse cannot be regularized. The latter indicates the sharp difference between regularizability in Fréchet spaces and in Banach spaces, respectively.

Let X and Y be TVS, let $\mathscr{T}(X)$ and $\mathscr{T}(Y)$ be their topologies, let $\mathcal{U}$ be the base of a filter of neighborhoods of the origin in X, let f: X → Y be a mapping with a domain of definition D(f), and let S be a system of subsets of X.

<u>Definition 1 [3].</u> A mapping R: S → 2^Y is said to be an A-regularizer of f with respect to the system S if the following conditions hold:

1) $\forall A\in S,\ A\cap D(f)\neq 0: R(A)\neq\varnothing$;

2) $\forall x\in D(f),\ \forall G\subset\mathscr{T}(Y),\ G\ni f(x),\ \exists V\in\mathscr{T}(X),\ V\ni x,\ \forall A\in S,\ V\supset A\ni x: R(A)\subset G.$

A mapping f is said to be A-regularizable with respect to the system S if there exists an A-regularizer of f with respect to S.

<u>Definition 2 [4].</u> A family of mappings $R_U: X\to Y,\ U\in\mathcal{U}$, is said to be a T-regularizer of a mapping f in a base $\mathcal{U}$, if $\forall x\in D(f),\ \forall G\in\mathscr{T}(Y),\ f(x)\in G,\ \exists U_0\in\mathcal{U},\ \forall U\in\mathcal{U},\ U\subset U_0: R_U(x+U)\subset G.$

A mapping f is said to be T-regularizable in the base $\mathcal{U}$, if there exists a T-regularizer of f in the base $\mathcal{U}$.

We shall assume that the base $\mathcal{U}$ consists of balanced sets that satisfy the condition $U\in\mathcal{U}\Rightarrow U+U\in\mathcal{U}$.

<u>THEOREM 1.</u> A-regularizability of a mapping f with respect to a system $S=\{x+U,\ x\in X,\ U\in\mathcal{U}\}$ is equivalent to T-regularizability of f in a base $\mathcal{U}$.

<u>Proof.</u> Let R be an A-regularizer of the mapping f, and let R_U: X → Y be a mapping in $R_U(x)\in R(x+U)$ for any $x\in D(f)$ and $U\in\mathcal{U}$. Since R is an A-regularizer, it follows that $\forall x\in D(f),\ \forall G\in\mathscr{T}(Y),\ G\ni f(x),\ \exists U_1\in\mathcal{U},\ \forall A\in S,\ x+U_1\supset A\ni x: R(A)\subset G.$ Let us take a $U_0\in\mathcal{U}$, such that $U_0+U_0\subset U_1$. Then $\forall U\subset U_0,\ U\in\mathcal{U}$, and by taking A = v + U in the definition of an A-regularizer, we obtain

$$R_U(x+U)=\bigcup_{v\in x+U}R_U(v)\subset\bigcup_{v\in x+U}R(v+U)\subset G,$$

because $v+U\subset x+U+U\subset x+U_1$, and since U is balanced, we have $v+U\ni x$. Hence $\{R_U\}$ is a T-regularizer.

Let $\{R_U\}$ be a T-regularizer for f. Let us define the mapping R: S → 2^Y by the relation $R(x+U)=R_{U+U}(x+U)$. Since $\{R_U\}$ is a T-regularizer, it follows that $\forall x\in D(f),\ \forall G\in\mathscr{T}(Y),\ G\ni f(x),\ \exists U_0\in\mathcal{U},\ \forall U\in\mathcal{U},\ U\subset U_0: R_U(x+U)\subset G.$ Let us take a $W\in\mathcal{U}$ such that

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$$W+W+W+W\subset U_0, \tag{1}$$

and let V = x + W. If $A=v+U\ni x$, $A\subset V$, then it follows from the fact that U is balanced that

$$v\in x+U. \tag{2}$$

From v + U ⊂ x + W it follows that $v-x\in W$, and since every element $u\in U$ has the form $u=x-v+w$, $w\in W$, we obtain U ⊂ W + W, and by virtue of (1) we have

$$U+U\subset U_0. \tag{3}$$

Thus,

$$R(A)=R(v+U)=R_{U+U}(v+U)\subset(2)\subset R_{U+U}(x+U+U)\subset(3)\subset G.$$

Hence R is an A-regularizer.

We shall assume that a mapping f is linearly (finite-dimensional linearly) regularizable if there exists a T-regularizer $\{R_U\}$, where all the R_U's are linear (finite-dimensional) continuous mappings.

Definition 3. A mapping f with a domain of definition D(f) ⊂ X and a set of values in Y has the property FD for a base $\mathcal{U}$ of neighborhoods of the origin in Y if for any $U\in\mathcal{U}$ there exists a finite-dimensional linear continuous operator B_U: X → Y such that the net B_Ux converges to f(x) for any $x\in D(f)$. The set of net indexes $\{B_Ux\}$ is ordered by the relation $U_1\geqslant U_2\Leftrightarrow U_1\subset U_2$.

Definition 4. A TVS X is said to be a space with the bounded approximation property in the base $\mathcal{U}$, if there exists a family of linear continuous finite-dimensional operators $\{B_U:U\in\mathcal{U}\}$, $B_U:X\to X$ such that the net B_Ux converges to x for any $x\in X$.

THEOREM 2. Let X be a semi-reflexive space with the property of bounded approximation in the base $\mathcal{U}$, and let Y be a locally convex space (LCS). For any linear compactification T: X → Y, the mapping $T^{-1}|_{T(P)}$ will then have the property FD in the base $\mathcal{U}$ for any bounded set P ⊂ X.

Proof. It is easy to see that the image T^*Y^* is compact in X^* in a strong topology of X^*. Indeed, since T is an injective operator, it follows that $T^*Y^*\subset X^*$ is a total subspace, i.e., for any x ≠ 0 there exists an $f\in T^*Y^*$ in f(x) ≠ 0. It hence follows by virtue of the semireflexivity of X that any functional in X^{**} is determined by an $x\in X$; for any $\varphi\in X^{**}$ there then exists an $f\in T^*Y^*$, such that $\varphi(f)\neq 0$. But precisely this implies that T^*Y^* is compact in X^*, since otherwise T^*Y^* would be contained in a closed hyperplane, and hence there would exist a functional $\varphi\in X^{**}$, such that $\varphi(f)=0$ for any $f\in T^*Y^*$.

Since X has the property of bounded approximation, it then follows that for any $U\in\mathcal{U}$ there exist elements $x_U^1, x_U^2,\dots,x_U^{n_U}$ in X and $f_U^1, f_U^2,\dots,f_U^{n_U}$ in X^* such that for any $x\in X$ the net $\left\{\sum_{i=1}^{n_U} f_U^i(x)\,x_U^i\right\}$ converges to x.

Next, since T^*Y^* is compact in X^*, it is possible to select for any bounded set P ⊂ X elements $g_U^1, g_U^2,\dots,g_U^{n_U}$ in T^*Y^* such that for any $x\in P$ we have

$$\sum_{i=1}^{n_U}(f_U^i(x)-g_U^i(x))\,x_U^i\in U.$$

Indeed, for any $U\in\mathcal{U}$ there exists a $V\in\mathcal{U}$, such that $\underbrace{V+V+\dots+V}_{n_U}\subset U$; then there exists a δ > 0 such that for any ε, |ε| < δ we have $\varepsilon x_U^i\in V$. Next let us select $g_U^i\in T^*Y^*$ such that $|f_U^i(x)-g_U^i(x)|<\varepsilon$ for any $x\in P$. Hence

$$\sum_{i=1}^{n_U}(f_U^i(x)-g_U^i(x))\,x_U^i\in U.$$

Let us write $B_Uy=\sum_{i=1}^{n_U}h_U^i(y)\,x_U^i$, where $y\in Y$, $h_U^i\in(T^*)^{-1}g_U^i$. Hence $B_Uy\to T^{-1}y$ for any $y\in T(P)$. Indeed, if $y=Tx$, $x\in P$, then

$$B_U y = \sum_{i=1}^{n_U} h_U^i (Tx) x_U^i = \sum_{i=1}^{n_U} g_U^i (x) x_U^i = \sum_{i=1}^{n_U} f_U^i (x) x_U^i + \sum_{i=1}^{n_U} (g_U^i (x) - f_U^i (x)) x_U^i \to x,$$

where the first term converges to x, and the second term to zero.

Definition 5. A mapping T^{-1} is said to be boundedly (linearly or finite-dimensionally) regularizable if the image T(P) of each bounded set $P \subset X$ has its regularizer, i.e., there exists a family $\{R_U\}$ such that $\forall x \in P,\ \forall V \in \mathcal{T}(X),\ x \in V,\ \exists U_0 \in \mathcal{U},\ \forall U \in \mathcal{U},\ U \subset U_0 : R_U (Ax + U) \subset V$.

From Theorem 2 and the results of [4] we obtain the following theorem.

THEOREM 3. Let X be a metrizable semireflexive space with the property of bounded approximation in a base $\mathcal{U}$, and let Y be a metrizable LCS in a base $\overline{\mathcal{U}}$. Then for any linear compactification t: $X \to Y$ the mapping T^{-1} is boundedly finite-dimensionally linearly regularizable in the base $\overline{\mathcal{U}}$.

Remark. In the case of metrizable spaces it is possible not to refer to a base $\mathcal{U}$, since according to [4] regularizability does not depend in this case on a base. Hence it also follows that for linear metric spaces the above definition of regularizability is equivalent to the conventional definition of regularizability in metric spaces [1, p. 178].

COROLLARY. Any linear compactification $T : \mathcal{E} \to Y$ $(T : \mathcal{L} \to Y)$, where $\mathcal{E}$ is the space of all infinitely differentiable functions on the real axis, $\mathcal{L}$ a space of infinitely differentiable fast decreasing functions, and Y a metrizable LCS, has a boundedly finite-dimensionally linearly regularizable inverse T^{-1}.

This corollary follows from the fact that the spaces $\mathcal{E}$ and $\mathcal{L}$ have bases and they are reflexive [5, 6].

Prior to presenting an example of a reflexive Fréchet space on which there exists a linear compactification with a nonregularizable inverse, let us formulate an extension of a well-known result for Banach spaces.

Assertion 1. Let X be a Fréchet space whose topology is determined by a countable set of norms $\| \ \|_k$, $k = \overline{1, \infty}$, and $B_k = \{x \in X : \| x \|_k < 1\}$. Let T: $X \to Y$ be a linear compactification, and Y a normed space. On the space X let us introduce a norm $\|x\|_0 = \|Tx\|_Y$. If for any $k > 0$ the closure $\bar{B}_k$ of the sphere B_k in the norm $\| \ \|_0$ is a $\| \ \|_1$-unbounded set, then the operator T^{-1} is not regularizable.

Proof. Indeed, if the operator T^{-1} would be regularizable, then it would be a mapping of first Borel class [1, p. 184], i.e., the sphere B_1 would have to be a union of a countable number of $\| \ \|_0$-closed sets V_i, $i = \overline{1, \infty}$. But according to our condition none of the sets V_i can contain shifts of homothetic images of B_k. According to Baire's theorem on categories, the sets B_1 are not neighborhoods of the origin. We have arrived at a contradition.

Let X be a "Slovikovskii space" (a Montel space, i.e., a separable and reflexive space which is not a Schwarz space) [7, p. 258]. It consists of double sequences $x = (x_{mn})_{m,n=1}^{\infty}$, for which $\| x \|_{kl} = \sup_{m,n} a_{klmn} | x_{mn} | < \infty$, $k, l = \overline{1, \infty}$, where $a_{klmn} = m^k \max (1, n^{l-m})$ with a set of norms $\| \ \|_{k\ell}$. The unit vectors constitute a base of this space. It can be naturally embedded in a continuous manner in the Banach spaces $Z = c_0 (\mathbb{N} \times \mathbb{N})$. For a fixed m_0 let us denote by Z_{m_0} a subspace of Z that consists of the sequences $x = (x_{mn} : x_{mn} = 0 \text{ for } m \neq m_0)$. The space Z_{m_0} (which is isometric to c_0) is nonquasireflexive. Therefore we can define on it a weaker norm $\| \ \|_{m_0}$ such that the closure of the unit sphere of the space Z_{m_0} in the norm $\| \ \|_{m_0}$ would be unbounded in the original norm [1, p. 80]. On the space Z let us define a norm $\| x \|_0 = \sum_{m=1}^{\infty} 2^{-m} \| x_m \|_m$, where x_m is a natural projection of the element x on the subspace Z_m. Let us fix k and ℓ. Then for $m = \ell$ the norm $\| \ \|_{k\ell}$ will be equivalent (even proportional) to the norm of the space Z, and $X \cap Z_m$ is compact in Z_m. Therefore the $\| \ \|_0$-closure of the sphere $B_{kl} = \{x \in X : \ \|x\|_{kl} \leqslant 1\}$ is unbounded in the norm of Z.

For completing the construction of the example, let us take as Y a space Z with a norm $\| \ \|_0$, and as T the identity embedding of X in Y. It hence follows from Assertion 1 (where the role of the norms $\| \ \|_k$, $k > 1$ is played by the norms $\| \ \|_{ki}$, numbered by natural numbers in any order, whereas the role of $\| \ \|_1$ is played by the restriction of the norm of the space Z to X) that the inverse operator T^{-1} cannot be regularized.

Let us formulate a simple sufficient condition of regularizability in Frechet spaces.

Assertion 2. Let $T: X \to Y$ be a linear continuous injective operator, let X and Y be separable Fréchet spaces, with the topology in X being defined by a sequence of norms $\| \ \|_n$, $n = 1, \infty$, and for any n the mapping T^{-1} is regularizable with respect to Y in $(X, \| \ \|_n)$. Then T^{-1} is regularizable with respect to Y in X.

Proof. It is sufficient to prove that for any n the sphere $B_n = \{x \mid \|x\|_n < 1\}$ is a union of a countable number of sets which are closed in the topology of preimages of open sets Y under a mapping T [1, p. 182]. But since $T^{-1}: Y \to (X, \| \ \|_n)$ is regularizable, it follows that this is indeed so [1, p. 184].

COROLLARY. Let D be an open circle of finite or infinite radius and which is centered at the origin in the complex plane, and let A(D) be a space of functions which are analytic in D and which have a topology defined by a set of norms $\|x\|_n = \max_{z \in D_n} |x(z)|$, where $D_n \subset D$ is a closed circle centered at the origin, $\bigcup_n D_n = D$. Let Γ be a compact subset of D that consists of infinitely many points. Then the operator $T: A(D) \to C(\Gamma)$ [$C(\Gamma)$ being a space of functions which are continuous on Γ and have maximum norm] which assigns to a function $x(z) \in A(D)$ its restriction to Γ will be linear, continuous, and injective, whereas the operator T^{-1} is regularizable.

Proof. Beginning with some n_0 we have $\Gamma \subset D_n$. Hence for $n > n_0$ the operator T^{-1} will be regularizable with respect to $C(\Gamma)$ in $(A(D), \| \ \|_n)$ [1, p. 200]. According to Assertion 2 it is regularizable with respect to $C(\Gamma)$ in A(D).

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