ON THE STRUCTURE OF A σ-ALGEBRA OF BOREL SETS AND THE CONVERGENCE OF CERTAIN STOCHASTIC SERIES IN BANACH SPACES

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1. Every Banach space E can be allotted a measurable space structure by defining in it a σ-algebra $\mathfrak{A}$ (E), generated by open sets. The elements of $\mathfrak{A}$ (E) are called Borel sets.

Study of the structure of the set $\mathfrak{A}$ (E) enables certain propositions of functional analysis and probability theory to be proved. For example, we shall prove the following.

Proposition 1. The set of all distinct (i.e., nonisometric or nonisomorphic) separable Banach spaces has the power of a continuum.

Proposition 2. Let E be a separable Banach space, and $\mathfrak{T}$ a separable topology in E, matched with the structure of the vector space E. Then $\mathfrak{A}$ (E) is identical with the σ-algebra $\mathfrak{A}$ ($\mathfrak{T}$), generated by open sets of the topology $\mathfrak{T}$.

Note 1. Proposition 2 is false if the topology $\mathfrak{T}$ is not matched with the structure of the vector space E. The proposition is likewise false if E is an incomplete linear normed space [1].

Let E' be the space adjoint to the separable Banach space E. It is well known [2] that the σ-algebra $\mathfrak{A}$ (E) is the same as the σ-algebra induced by all the half-spaces

$$\mathfrak{U}_{x'} = \{x \in E : x'(x) \leqslant c\},\ x' \in E',\ c \in R,$$

when x' runs over E'; and c is the set of real numbers. In Sec. 2 of the present article we shall prove a stronger assertion.

THEOREM 1. Let E be a separable Banach space; and M a subspace of E', everywhere dense in the weak topology σ(E', E). Then the σ-algebra $\mathfrak{A}$ (E) of space E is the same as the σ-algebra generated by the half-spaces $\mathfrak{U}_{x'}$, when x' runs over the entire subspace M; and c is the number axis.

We recall that a set T $\subseteq$ E' is said to be total in E if, for all x' $\in$ T, the fact that x'(x) = 0 implies x = 0. If we recall that the linear hull L(T) of a total set T is dense in E' in the σ(E', E) topology (see [3]) and that $\mathfrak{A}(L(T)) = \mathfrak{A}(T)$ [$\mathfrak{A}(L(T))$, $\mathfrak{A}(T)$ are the σ-algebras generated by the sets $\mathfrak{U}_{x'}$, where x' runs over L(T) and T, respectively], then the following is easily proved.

COROLLARY 1. Theorem 1 remains in force if M is a set, total in E.

This proposition enables conditions to be established, under which a series of independent random variables, taking values in Banach space E, is convergent with probability unity. Let $\{\Omega, \mathfrak{F}, P\}$ be a probability space; by the E-valued random variable (E-v. r. v.) ξ we shall understand the $\{\mathfrak{F}, \mathfrak{A}\}$ measurable mapping $\xi : \Omega \to E$.

The E-v. r. v. ξ induces into $\{E, \mathfrak{A}\}$ the probability measure

$$\mu_\xi(A) = P\{\omega : \xi(\omega) \in A\},\ A \in \mathfrak{A},$$

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which we call the distribution of ξ. We shall say that ξ is a symmetric E-v. r. v. if $\mu_\xi = \mu_{-\xi}$. We define in the standard way (see, e.g., [4]) the independence of a collection of E-v. r. v.'s.

Let $M \subseteq E$; we shall say that the sequence of E-v. r. v.'s $\{S_n, n \geq 1\}$ is M-weakly convergent to the E-v. r. v. S with probability unity if, for any $x' \in M$, we have $P\{x'(S_n) \to x'(S)\} = 1$.

It is well known that, in the case of a nonrandom sequence $\{x_n, n \geq 1\} \subset E$, its M-weak convergence (even if $M = E'$) to $x \in M$ does not imply its strong convergence. However, in the case of a series of independent symmetric E-v. r. v.'s, we have the following theorem.

THEOREM 2. Let $\{\xi_k, k \geq 1\}$ be a sequence of independent symmetric E-v. r. v.'s, and M a set, total in E. The necessary and sufficient condition for the series $\sum_{k=1}^{\infty} \xi_k$ to be strongly (i.e., in the norm of space E) convergent with probability unity is that it be M-weakly convergent with probability unity to an E-v. r. v. S.

The example of nonrandom sequences shows that the condition of Theorem 2 that the E-v. r. v.'s be symmetric is essential and cannot be dropped. We give the proof of Theorem 2 in Sec. 3, where an example will be found, showing that the condition that M be total likewise cannot in general be weakened. We do not know if Theorem 2 can be extended to the nonseparable case. Notice that Theorem 2 was proved by Ito and Nisio in [5] for the case $M = E'$.

2. The proof of Proposition 1 is based on the following considerations. We know ([6], Chap. 2, Sec. 30, ¶ III) that the set $\mathfrak{A}(E)$ has the power of a continuum. The family of all closed subspaces of the space C[0, 1] therefore has a power $\leq c$, since every closed subspace is a Borel set. Since space C[0, 1] is universal, the power of the set $\mathfrak{M}$ of all distinct separable Banach spaces does not exceed that of the continuum. On the other hand, the spaces l_p, l_q $(1 < p \neq q > 1)$ have different linear dimensionalities, so that they are not isomorphic (see [7]). Hence there exists a continual family of nonisomorphic Banach spaces l_p $(1 < p < \infty)$ so that the power of $\mathfrak{M}$ is equal to c.

To prove Proposition 2, we consider the closure $\overline{S_1(E)}$ of the unit sphere $S_1(E)$ of space E in the topology $\mathfrak{T}$. By Proposition 14 of ([3], Chap. II, Sec. 2), it is convex and balanced. Let us show that $\overline{S_1(E)}$ does not contain a straight line. In fact, since $\mathfrak{T}$ is separable, there exist, for every $x \neq 0$, open sets $V_1, V_2 \in \mathfrak{T}$ such that $0 \in V_1$, $x \in V_2$, and $V_1 \cap V_2 = \emptyset$. Since $\mathfrak{T}$ is weaker than the initial topology of space E, we have $V_1 \supset S_\alpha(E)$ for some α. By hypothesis, the topology $\mathfrak{T}$ is matched with the structure of the vector space E, so that

$$0 \in S_1(E) \subset \frac{1}{\alpha} V_1, \quad \frac{1}{\alpha} x \in \frac{1}{\alpha} V_2, \quad \frac{1}{\alpha} V_1 \cap \frac{1}{\alpha} V_2 = \varnothing,$$

so that the closure of $S_1(E)$ in the topology $\mathfrak{T}$ will not contain x/α and hence will not contain the entire line $\{\lambda x\}$. The gauge function of the set $\overline{S_1(E)}$ is the norm $\|\cdot\|^*$ defined in the vector space E, where $\|x\|^* \leq \|x\|$, and in this norm $\|\cdot\|^*$ the set $\overline{S_1(E)}$ is the unit sphere. By Suslin's theorem ([6], Chap. 3, Sec. 39), $\mathfrak{A}(E)$ is the same as the σ-algebra $\mathfrak{A}(E^*)$ of space E, equipped with the norm $\|\cdot\|^*$. Let us show that $\mathfrak{A}(E^*) \subset \mathfrak{A}(\mathfrak{T})$. We denote by G_0 the collection of all open sets of space E^*. It is easily seen that $G_0 \subset \mathfrak{A}(\mathfrak{T})$. For, the open sphere $\{x : \|x\|^* < 1\}$, as the countable union of closed spheres $S_{\alpha_n} = \{x : \|x\|^* \leq \alpha_n\}$ as $\alpha_n \to 1$, $\alpha_n < 1$, belongs to $\mathfrak{A}(\mathfrak{T})$. From this and Theorem 2 of ([6], Chap. 2, Sec. 21, ¶ II), it follows that any set, open with respect to the norm $\|\cdot\|^*$, belongs to $\mathfrak{A}(\mathfrak{T})$. Hence the minimal σ-algebra $\mathfrak{A}(E^*)$, containing G_0, is the subset $\mathfrak{A}(\mathfrak{T})$.

In short,

$$\mathfrak{A}(E) = \mathfrak{A}(E^*) \subset \mathfrak{A}(\mathfrak{T}) \subset \mathfrak{A}(E),$$

and hence $\mathfrak{A}(E) = \mathfrak{A}(\mathfrak{T})$.

As an example of a topology $\mathfrak{T}_1$, not matched with the structure of the vector space in R^1, we can quote the collection of open sets symmetric with respect to zero. This is a separable topology, but it is clear that the class $\mathfrak{A}(\mathfrak{T}_1)$, consisting of open sets symmetric about zero, is in fact the class of all Borel sets.

Let us prove Theorem 1. We first introduce the following notation: $\overline{M}^s$ is the weak sequential closure of the set M (i.e., the limits of all sequences of M, convergent in a weak topology); $\overline{M}_f$ is the strong closure of M; and $\mathfrak{A}(M)$ is the σ-algebra generated by the half-spaces , $x' \in M$.

Obviously,

$$\mathfrak{A}(M) \subset \mathfrak{A}(\overline{M}_f) \subset \mathfrak{A}(\overline{M}^s), \tag{1}$$

since $M \subset \overline{M}_f \subset \overline{M}^s$. We shall show that $\mathfrak{A}(M) = \mathfrak{A}(\overline{M}^s)$. Let $x' \in \overline{M}^s$; then there exists a sequence $x'_n \in M$ which is weakly convergent to x'. We have

$$\mathfrak{U}_{x'} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \left\{x; x'_n(x) < c + \frac{1}{m}\right\},$$

and hence $\mathfrak{U}_{x'} \in \mathfrak{A}(M)$ and $\mathfrak{A}(M) = \mathfrak{A}(\overline{M}^s)$. It follows from this and the inclusions (1) that $\mathfrak{A}(M) = \mathfrak{A}(\overline{M}_f)$.

We consider the Mackey topology $\tau(E, \overline{M}_f)$, generated by the strongly closed subspace $\overline{M}_f$ (see [3]). By Lemma 2 of [8], this topology $\tau(E, \overline{M}_f)$ majorizes some normed topology $\mathfrak{T}$. It is easy to see that the σ-algebra $\mathfrak{A}(\mathfrak{T})$, generated by the topology $\mathfrak{T}$, is the same as $\mathfrak{A}$. For, let $\|x\|_{\mathfrak{T}}$ be the norm which induces the topology $\mathfrak{T}$; and $E_{\mathfrak{T}}$ the linear space E, equipped with the norm $\|x\|_{\mathfrak{T}}$. The operator j of natural imbedding of Banach space E into the linear normed space $E_{\mathfrak{T}}$ is continuous, since the topology $\mathfrak{T}$ is weaker than the initial topology of space E. By Suslin's theorem (see [6], Chap. 3, Sec. 39), any one-to-one continuous image of a Borel set $B \in \mathfrak{A}$ is a Borel set, so that $\mathfrak{A} \subset \mathfrak{A}(\mathfrak{T})$. Since $\mathfrak{A}(\mathfrak{T}) \subset \mathfrak{A}$, it follows from the previous inclusion that $\mathfrak{A} = \mathfrak{A}(\mathfrak{T})$. By hypothesis, the topology $\mathfrak{T}$ is majorized by the Mackey topology $\tau(E, \overline{M}_f)$, so that the space $E'_{\mathfrak{T}}$, adjoint to the linear normed space $E_{\mathfrak{T}}$, is imbedded in the space $\overline{M}_f$. Hence $\mathfrak{A}(\mathfrak{T}) \subset \mathfrak{A}(\overline{M}_f)$.

We thus have $\mathfrak{A} = \mathfrak{A}(\mathfrak{T}) \subset \mathfrak{A}(\overline{M}_f) = \mathfrak{A}(M) \subset \mathfrak{A}$, whence it follows that $\mathfrak{A} = \mathfrak{A}(M)$. QED.

3. Proof of Theorem 2. The necessity is obvious. Let us prove the sufficiency.

Let the series $\sum_{k=1}^{\infty} \xi_k$ be M-weakly convergent with probability unity to the E-v. r. v. S, and let $x'_1, \ldots, x'_q$ ($q \geq 1$) be any finite collection of functionals of M; then the sequence of random vectors $\vec{\eta}_n = (x'_1(S_n), \ldots, x'_q(S_n))$, $n \geq 1$, where $S_n = \sum_{k=1}^{n} \xi_k$ is convergent with probability unity to the random vector $\vec{\eta} = (x'_1(S), \ldots, x'_q(s))$. In view of this, and the fact that $\vec{\eta}_m - \vec{\eta}_n$ and $\vec{\eta}_n$ are independent for any $m > n \geq 1$, we have

$$P\{S_n \in C_1, S - S_n \in C_2\} = P\{S_n \in C_1\} P\{S - S_n \in C_2\}, \tag{2}$$

which holds for any $C_1, C_2 \in C_M$, where C_M is the collection of sets of the form $\{x \in E: (x'_1(x), \ldots, x'_p(x)) \in B^{(p)}\}$, where $x'_1, \ldots, x'_p \in M$, $p \geq 1$, and $B^{(p)}$ is a Borel set in R^p. Since the σ-algebra generated by C_M is the same as $\mathfrak{A}(M)$, and, by Corollary 1 to Theorem 1 (we recall that M is a total set), $\mathfrak{A}(M)$ is the same as $\mathfrak{A}$, we can easily show that Eq. (2) holds for any Borel sets $C_1, C_2 \in E$. Consequently, the E-v. r. v.'s $S - S_n$, and S_n, are independent for any $n \geq 1$. It is easily shown by similar arguments that the E-v. r. v.'s $S - S_n$ ($n \geq 1$) have a symmetric distribution. We write

$$S(\omega) = S_n(\omega) + [S(\omega) - S_n(\omega)], \quad \omega \in \Omega,$$

and put

$$\tilde{S}(\omega) = S_n(\omega) - [S(\omega) - S_n(\omega)], \quad \omega \in \Omega.$$

Since, given any $n \geq 1$, the E-v. r. v.'s S, $S - S_n$ are independent and the $S - S_n$ have a symmetric distribution, it can easily be seen that the E-v. r. v.'s S and $\tilde{S}$ are equidistributed. Notice that, given any convex set $B \subset E$, and any $n \geq 1$,

$$\{\omega : S_n(\omega) \bar{\in} B\} \subset \{\omega : S(\omega) \bar{\in} B\} \cup \{\omega : \tilde{S}(\omega) \bar{\in} B\}.$$

If, at the same time, B is a Borel set, then

$$\sup_{n \geq 1} P\{S_n \bar{\in} B\} \leqslant 2P\{S \bar{\in} B\}. \tag{3}$$

Since S is an E-v. r. v., there exists, for any $\varepsilon > 0$ (see, e.g., [2]), a compact set $K_\varepsilon \subset E$ such that $P\{S \bar{\in} K_\varepsilon\} < \varepsilon/2$. If we recall that the convex hull of a compactum is a compactum, and use the inequality (3), it is clear that, given any $\varepsilon > 0$, there exists a compact set $\tilde{K}_\varepsilon$ ($\tilde{K}_\varepsilon$ is the convex hull of K_ε) such that, for all $n \geq 1$, $P\{S_n \in \tilde{K}_\varepsilon\} \geq 1 - \varepsilon$. In short, the distributions of the partial sums of the series $\sum_{k=1}^{\infty} \xi_k$ are uniformly compact, and this fact is sufficient (see [5, 9]) for the series $\sum_{k=1}^{\infty} \xi_k$ to be strongly convergent with probability unity. The sufficiency, and hence Theorem 2, is proved.

The following example shows that the condition in Theorem 2 that the set M be total is essential and cannot be weakened.

Example. Let $M \subset E'$, and let the set M be not total in E. There then exists an element $x \in E$ such that, for all $x' \in M$, we have $x'(x) = 0$ but $x \neq 0$. Let $\{\xi_k, k \geq 1\}$ be a sequence of independent random variables, where ξ_k takes the values $\pm k$ with probability 1/2. We put $\zeta_k = \xi_k x$. It is easily seen that $\{\zeta_k, k \geq 1\}$ is a sequence of independent symmetric E-v. r. v.'s, and that, for every $x' \in M$, $\sum_{k=1}^{\infty} x'(\zeta_k) = 0$ with probability unity. Hence the series $\Sigma \zeta_k$ is M-weakly convergent to zero with probability unity. But

$$\|\zeta_k\| = k\|x\| \underset{k\to\infty}{\to} \infty, \tag{4}$$

i.e., the series $\sum_{k=1}^{\infty} \zeta_k$ is divergent with probability unity in the sense of the norm of space E.

Note 2. If M is a countable set, total in E (notice that, since E is separable, such a set always exists exists), then the fact that the series $\sum_{k=1}^{\infty} \xi_k$, where ξ_k are independent and symmetric E-v. r. v.'s, is M-weakly convergent to S with probability unity, implies that $\sum_{k=1}^{\infty} \xi_k \overset{\text{a. s.}}{=} S$.

For, let us number the elements of M, $M = \{x'_k, k \geq 1\}$. Then

$$1 \geqslant P\left\{\sum_{k=1}^{\infty} \xi_k = S\right\} = 1 - P\left\{\sum_{k=1}^{\infty} \xi_k \neq S\right\} = 1 - P\left\{\bigcup_{m=1}^{\infty}\left\{x'_m\left(\sum_{k=1}^{\infty} \xi_k\right) \neq x'_m(S)\right\}\right\} \geqslant 1 - \sum_{m=1}^{\infty} P\left\{\sum_{k=1}^{\infty} x'_m(\xi_k) \neq x'_m(S)\right\} = 1,$$

i.e., $P\left\{\sum_{k=1}^{\infty} \xi_k = \xi\right\} = 1$.

4. In this section we shall give some corollaries to Theorem 2.

THEOREM 3. Given in E a norm $\|\cdot\|_1$ which is weaker than the natural norm of E. Further, let $\{\xi_k, k \geq 1\}$ be a sequence of independent symmetric E-v. r. v.'s.

Then, if the series $\sum_{k=1}^{\infty} \xi_k$ is convergent with probability unity to the E-v. r. v. S in the norm $\|\cdot\|_1$, then it converges to S in the norm $\|\cdot\|_E$ also, with probability unity.

The proof follows readily from Theorem 2 and Note 2, if we also observe that $E_1' \subset E'$, where E_1' is the space adjoint to E_1 (E_1 is the space E equipped with the norm $\|\cdot\|_1$), that E_1 is a separable normed space, and that E_1' is a set total in E (this last follows from the Hahn – Banach theorem on the continuation of linear functionals in a normed space).

Let S be a compact topological space. We consider the space C(S) of all real-valued functions x(s), continuous in S. Notice that C(S) is a separable Banach space with uniform norm. We define the linear functional $\delta_s \in C'(S)$, $s \in S$, as follows: $\delta_s(x) = x(s)$. It is easily seen that the set $\{\delta_s, s \in Q\}$ is total in C(S), if the set Q is dense in S. We consider the stochastic field $\{\xi(s), s \in S\}$. If the field $\xi(s)$ is sample-wise continuous with probability unity, it can be regarded as a C(S)-v. r. v. We shall say that the field $\xi(s)$ is symmetric if its distribution in C(S) is a symmetric measure (see Sec. 1). For the field $\xi(s)$ to be symmetric, it is sufficient that every finite-dimensional distribution of it be symmetric in the relevant finite-dimensional space. In the present case, Theorem 2 can be stated as follows.

THEOREM 4. Let $\{(\xi_k(s), s \in S), k \geq 1\}$ be a sequence of independent symmetric random fields, sample-wise continuous with probability unity, and let the field $\xi(s)$ be also sample-wise continuous with probability unity. If, for every $s \in S$,

$$\xi(s) \overset{a.s.}{=} \sum_{k=1}^{\infty} \xi_k(s),$$

then the series $\sum_{k=1}^{\infty} \xi_k(s)$ must be uniformly convergent to $\xi(s)$ with probability unity.

Note 3. It can be required in Theorem 4 that the condition (4) be satisfied only for s belonging to a countable set, everywhere dense in S.

It can easily be seen that a trivial consequence of Theorem 4 is the uniform convergence, with probability unity, of the Karhunen – Loeve expansion of a Gaussian field $\xi(s)$, continuous with probability unity ($M\xi(s) = 0$). This fact has been proved [10, 11] by a variety of methods.

Let H be a Hilbert space with scalar product (,), and let L be a isonormal Gaussian linear functional in H (see, e.g., [12]), i.e., L is a linear mapping of H into a set of real Gaussian random variables with

$$ML(x) = 0, \quad ML(x)L(y) = (x, y)$$

for any x, $y \in H$. If $\{\varphi_n, n \geq 1\}$ is a basis orthonormalized in H, then, given any $x \in H$, we have $L(x) = \sum_{n=1}^{\infty} L(\varphi_n)(x, \varphi_n)$, and $\{L(\varphi_n), n \geq 1\}$ is a sequence of independent standard Gaussian random variables. It can easily be seen that, given any $x \in H$, the series $\sum_{n=1}^{\infty} L(\varphi_n)(x, \varphi_n)$ is convergent to L(x) with probability unity. If we regard φ_n as a continuous linear functional, $\varphi_n(x) \equiv (x, \varphi_n)$, then formally, $L = \sum_{n=1}^{\infty} L(\varphi_n)\varphi_n$ and $\{L(\varphi_n)\varphi_n, n \geq 1\}$ is a sequence of independent Gaussian H'-v. r. v.'s. But L is not an H-v. r. v., since $M\| L(\varphi_n)\varphi_n\|^2_{H'} = \sum_{n=1}^{\infty} M[L(\varphi_n)]^2 = \infty$, and hence the series $\sum_{n=1}^{\infty} L(\varphi_n)\varphi_n$ is divergent in the norm of space H' with probability unity. Now let K be a topological compactum in H. We shall say that K has the GC property if L, contracted into K, is continuous with probability unity.

COROLLARY 2. The compactum K has the GC property if and only if, given any orthonormalized basis $\{\varphi_n, n \geq 1\} \subset H$, the series $\sum_{n=1}^{\infty} L(\varphi_n)\varphi_n(x)$ is convergent uniformly with respect to $x \in K$ with probability unity.

This proposition is a trivial consequence of Theorem 4, if we note that $\{L(\varphi_n)\varphi_n(x), n \geq 1\}$ is a sequence of independent symmetric C(K)-v. r. v.'s, and use the previous arguments.

Notice that this corollary was obtained in [12]. The number of examples could be extended, illustrating how propositions which previously had to be strengthened in various ways, follow readily from Theorem 2. However, instead of doing this, we shall quote a new result concerning the nature of the convergence of the expansion of a Wiener process.

Let $\{w(t), t \in [0, 1]\}$ be a standard Wiener process. It is easily shown by the previous arguments that, given any orthonormalized basis $\{\psi_n(t), n \geq 1\}$ in $\mathfrak{L}^2[0,1]$

$$w(t) = \sum_{n=1}^{\infty} \xi_n \int_0^t \psi_n(\tau)\, d\tau, \tag{5}$$

where $\xi_n = \int_0^1 w(t)\psi_n(t)dt$, $n \geq 1$ are independent standard Gaussian random variables, and the series on the right side of (5) is uniformly convergent with respect to $t \in [0, 1]$ with probability unity. This assertion was first proved in [5]. Let us show that, under certain conditions, a stronger type of convergence holds. For this, we take the space $\Lambda_\alpha[0, 1]$ of continuous real functions x(t), $t \in [0, 1]$, satisfying a Lipschitz condition with exponent α $(0 < \alpha \leq 1)$, i.e., $|x(t) - x(t')| = O(|t - t'|^\alpha)$ uniformly with respect to all t, $t' \in [0, 1]$. $\Lambda_\alpha[0, 1]$ is a separable Banach space with respect to the norm

$$\|x\|_\alpha = \sup_{t\in[0,1]} |x(t)| + \sup_{t\neq t'} \frac{|x(t) - x(t')|}{|t - t'|^\alpha}.$$

COROLLARY 3. Let $\{\psi_n(t), n \geq 1\}$ be an orthonormalized basis in $\mathfrak{L}_2[0, 1]$ such that, for all $n \geq 1$, $\sup_{t\in[0,1]} |\psi_n(t)| < \infty$; then the series $\sum_{n=1}^{\infty} \xi_n \int_0^t \psi_n(\tau)d\tau$ is convergent to w(t) in the Lipschitz norm with exponent $\alpha < 1/2$ with probability unity.

Proof. It is well known that, with probability unity,

$$|w(t) - w(t')| = O(|t - t'|^{\alpha}), \quad 0 < \alpha < \frac{1}{2},$$

uniformly with respect to all t, $t' \in [0, 1]$. Using this fact, it can easily be shown that w(t), $t \in [0, 1]$, is an $\Lambda_\alpha[0, 1]$-v. r. v. $(0 < \alpha < 1/2)$. On the other hand, by hypothesis

$$\left|\int_0^{t_1} \psi_n(\tau)\, d\tau - \int_0^{t} \psi_n(\tau)\, d\tau\right| = O(|t_1 - t|),$$

i.e., $y_n(t) = \int_0^t \psi_n(\tau)d\tau$ belongs to $\Lambda_\alpha[0, 1]$ for any $0 < \alpha \leq 1$.

The series $\sum_{n=1}^{\infty} \xi_n y_n$ thus consists of independent symmetric $\Lambda_\alpha[0, 1]$-v. r. v.'s $(0 < \alpha \leq 1)$ and converges to w with probability unity in the norm $\|\cdot\|_{C[0,1]}$. The norm $\|\cdot\|_{C[0,1]}$ is weaker than the norm $\|\cdot\|_\alpha$, and hence, by Theorem 3, the series $\sum_{n=1}^{\infty} \xi_n y_n$ converges to w in the norm $\|\cdot\|_\alpha$ $(0 < \alpha < 1/2)$ also, with probability unity.

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