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KHINCHIN'S INEQUALITY FOR k-FOLD PRODUCTS OF INDEPENDENT RANDOM VARIABLES

I. K. Matsak and A. N. Plichko

1. A. Ya. Khinchin [1] proved the following inequality (see also [2; 3. p. 66]). Let $1 \le p < \infty$. Then there exist constants k_p, K_p such that for any finite collection of real numbers (a_i) we have

$$k_p \left(\sum a_1^2\right)^{1/2} \leqslant \left(\mathsf{M}\left|\sum a_i r_i\right|^p\right)^{1/p} \leqslant K_p \left(\sum a_i^2\right)^{1/2}, \tag{1}$$

where r_i is a sequence of independent Rademacher variables, $P(r_i = \pm 1) = 1/2$. As it is known, $k_p \ge 1/\sqrt{2}$ for $p < 2$, $k_p = 1$ for $p \ge 2$, $K_p = 1$ for $p \le 2$, $K_p = O(\sqrt{p})$ for $p \to \infty$. Inequality (1) has found wide application in probability theory and analysis. Various generalizations of Khinchin's inequality are known [4-8].

In this paper a relation of the form (1) is established for the case when the terms are k-fold products of independent random variables. We consider its application to the determination of estimates of the moments of random determinants and permanents. We give some results of the type of the law of iterated logarithms.

2. Let ξ_n be a sequence of independent, identically distributed random variables in $\mathbf{R}$, $\mathsf{M}\xi_n = 0$, $\xi_n \in L_p = L_p(\Omega, \Sigma, \mathsf{P})$, i.e., $\|\xi_n\|_p = (\mathsf{M}|\xi_n|^p)^{1/p} < \infty$, $1 \leqslant p < \infty$. For a fixed natural number k we consider the collection of all k-fold products $\eta_{n_1\ldots n_k} = \xi_{n_1}\cdot\xi_{n_2}\cdot\xi_{n_k}$, where the numbers $n_1, \ldots, n_k$ are mutually distinct. We enumerate the collection $(\eta_{n_1\ldots n_k})$ in an arbitrary manner: $(\eta_{n_1\ldots n_k}) = (\eta_i)_1^\infty$.

<u>THEOREM 1.</u> Let $1 < p < \infty$, $r = \max(p, 2)$, $q = \min(p, 2)$, $p^* = \max(p, p/(p-1)) - 1$, let k be a natural number, and let $\xi_n \in L_2$. Then for any finite collection of real numbers (a_i) we have

$$\left(\|\xi_1\|_q \cdot k_p/p^*\right)^k \left(\sum a_i^2\right)^{1/2} \leqslant \left\|\sum a_i \eta_i\right\|_p \leqslant \left(\|\xi_1\|_r K_p p^*\right)^k \left(\sum a_i^2\right)^{1/2}. \tag{2}$$

Clearly, for $p \le 2$ the right-hand inequality and for $p \ge 2$ the left-hand inequality are obvious. Before proceeding to the proof, we recall that by the unconditional constant of a sequence of nonzero elements x_n of a Banach space X we mean the smallest of the numbers K for which for any finite collection of real numbers (a_n) and any collection of signs $\varepsilon_n = \pm 1$ we have

$$\left\|\sum \varepsilon_n a_n x_n\right\| \leqslant K \left\|\sum a_n x_n\right\|.$$

<u>Proof of Theorem 1.</u> For each n let x_n be an arbitrary nonzero element of the linear hull $\operatorname{lin}(\eta_{n_1\ldots n_k}: n_1 < n_2 < \ldots < n_{k-1} < n_k = n)$. It is easy to verify that $(\Sigma_1^n x_i)_{n=1}^\infty$ is a Martingale with respect to the sequence of σ-algebras F_n, generated by the random variables

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$\xi_1, \xi_2, \ldots, \xi_n$ [9, Chap. 2, Sec. 8] and, consequently, (x_n) is a sequence of Martingale-difference sequence in the space L_p does not exceed the number p^* [10]. By the known Orlicz and Kadets estimates (see, for example, [8]), for each N we have

$$(k_p/p^*)\left(\sum_{n=1}^{N}\|x_n\|_q^2\right)^{1/2} \leqslant \left\|\sum_1^N x_n\right\|_p \leqslant K_p p^*\left(\sum_{n=1}^{N}\|x_n\|_r^2\right)^{1/2}. \tag{3}$$

The proof of the theorem will be carried out by induction on k. For $k = 1$ we have $\eta_i = \xi_i$ and (2) follows from the inequalities (3) if we set there $x_n = a_n\xi_n$.

Assume that the theorem has been proved for $k - 1$. We represent the finite sum $k - 1$ in the form

$$\sum_i a_i\eta_i = \sum_m x_m, \tag{4}$$

where

$$x_m = \xi_m \sum_j a_j^{(m)} v_j^{(m)}, \quad v_j^{(m)} = \eta_{n_1 n_2 \ldots n_{k-1}}, \quad n_i = n_i(j, m) < m.$$

By the induction hypothesis, for each m we have

$$(\|\xi_1\|_q k_p/p^*)^{k-1}\left(\sum_j |a_j^{(m)}|^2\right)^{1/2} \leqslant \left\|\sum_j a_j^{(m)} v_j^{(m)}\right\|_p \leqslant (\|\xi_1\|_r K_p p^*)^{k-1}\left(\sum_j |a_j^{(m)}|^2\right)^{1/2}. \tag{5}$$

From the independence of the sequence $\{\xi_m\}_1^\infty$ there follows [9, Chap. 2]

$$\left\|\xi_m \sum_j a_j^{(m)} v_j^{(m)}\right\|_p = \|\xi_m\|_p \left\|\sum_j a_j^{(m)} v_j^{(m)}\right\|_p. \tag{6}$$

The application of the relations (3), (5), and (6) to the equality (4) yields

$$\|\xi_1\|_q k_p/p^*)^k \left(\sum_{m,j} |a_j^{(m)}|^2\right)^{1/2} \leqslant \left\|\sum_i a_i\eta_i\right\|_p \leqslant (\|\xi_1\|_r K_p p^*)^k \left(\sum_{m,j} |a_j^{(m)}|^2\right)^{1/2},$$

i.e., inequality (2) is proved.

<u>Remark 1.</u> The left-hand estimate in (2) for p close to 1 is very coarse. We refine it under the assumption of the existence of the fourth moment.

<u>LEMMA 1 [3, p. 66].</u> Let $x \in L_4$ and assume that $\|x\|_4 \leq B\|x\|_2$. Then $\|x\|_1 \geqslant B^{-2}\|x\|_2$.

<u>COROLLARY 1.</u> Let x_n be an orthonormal system in the space L_2, $x_n \in L_4$ and assume that there exists a constant B such that $\|\Sigma_i a_i x_i\|_4 \leqslant B(\Sigma_i a_i^2)^{1/2}$ for any finite collection of scalars (a_i). Then

$$\left\|\sum_i a_i x_i\right\|_1 \geqslant B^{-2}\left(\sum_i a_i^2\right)^{1/2}.$$

As one can easily see, the system $(\eta_i)_{i=1}^\infty$ is orthogonal. Therefore, combining this corollary and the right-hand inequality in (2) for $r = 4$, we obtain: if $1 \leq p < 2$, then

$$\left\|\sum a_i\eta_i\right\|_p \geqslant \|\eta_i\|_2 \left\|\sum a_i \frac{\eta_i}{\|\eta_i\|_2}\right\|_1 \geqslant \left(\frac{\|\xi_1\|_2}{\|\xi_1\|_4 K_4 4^*}\right)^{2k} \|\xi_1\|_2^k \left(\sum a_i^2\right)^{1/2}.$$

Taking into account that $K_4 \leq 3^{\frac{1}{4}}$ [2], we obtain

$$\left\|\sum a_i\eta_i\right\|_p \geqslant \frac{\|\xi_1\|_2^{3k}}{\|\xi_1\|_4^{2k}\, 3^{5k/2}}\left(\sum a_i^2\right)^{1/2}. \tag{7}$$

Remark 2. An inequality, similar to the estimates (2), is valid also for nonidentically distributed random variables. For the right-hand inequality it is sufficient to require that $\sup_n \|\xi_n\|_r < \infty$, while for the left-hand inequality $\inf_n \|\xi_n\|_q > 0$. Moreover, in (2) instead of $\|\xi_1\|_q$ and $\|\xi_1\|_r$ one has to set the infimum and the supremum, respectively.

3. We apply the inequalities (2) and (7) to the determination of estimates of the moments of the permanent and the determinant of a random matrix [11, 12]. Let $A = (a_{i,j})_{i,j=1}^n$ be a real matrix, let per A, det A be the permanent and the determinant of the matrix A, let A, $A(\xi_{ij}) = (a_{ij}\cdot\xi_{ij})_{i,j=1}^n$, $A^{(2)} = (a_{ij}^2)_{i,j=1}^n$. We recall that by the permanent of a matrix A we mean the sum $\Sigma a_{1i_1}a_{2i_2}\ldots a_{ni_n}$, taken over all permutations $\langle i_1, \ldots, i_n\rangle$ of the numbers 1, 2, ..., n.

COROLLARY 2 (of Theorem 1 and Remark 1). Let $(\xi_{ij})_{i,j=1}^n$ be independent, identically distributed random variables, $M\xi_{ij} = 0$. If $1 < p < \infty$ and $\xi_{ij} \in L_r$, then

$$\left(\frac{\|\xi_{11}\|_q k_p}{p^*}\right)^n (\operatorname{per} A^{(2)})^{1/2} \leqslant (M\,|\det A(\xi_{ij})|^p)^{1/p} \leqslant (\|\xi_{11}\|_r K_p p^*)^n (\operatorname{per} A^{(2)})^{1/2},$$

$$\left(\frac{\|\xi_{11}\|_q k_p}{p^*}\right)^n (\operatorname{per} A^{(2)})^{1/2} \leqslant (M\,|\operatorname{per} A(\xi_{ij})|^p)^{1/p} \leqslant (\|\xi_{11}\|_r K_p p^*)^n (\operatorname{per} A^{(2)})^{1/2},$$

If $\xi_{ij} \in L_4$, then for $1 \le p < 2$ we have

$$(M\,|\det A(\xi_{ij})|^p)^{1/p} \geqslant \frac{\|\xi_{11}\|_2^{3n}}{\|\xi_{11}\|_4^{2n}\, 3^{5n/2}} (\operatorname{per} A^{(2)})^{1/2}.$$

A similar inequality holds also for per $A(\xi_{ij})$.

4. We consider the question on the limiting behavior of the variables $S_n = \Sigma_{i=1}^n a_i\eta_i$. It is known (see, for example, [13]), that from (2) for $p > 2$ there follows the convergence of S_n almost surely (a.s.) for $\Sigma_{i=1}^\infty a_i^2 < \infty$, i.e., (η_i) is a convergence system. Below we shall assume that

$$A_n^2 = \Sigma_1^n a_i^2 \uparrow \infty,\ A_n/A_{n+1} \to 1 \text{ for } n \to \infty,\ M\xi_n^2 = 1.$$

THEOREM 2. A. Let $|\xi_n| \leqslant L$, $L < \infty$. Then almost surely we have

$$\overline{\lim_{n\to\infty}}\, |S_n|/(A_n^2(\ln\ln A_n)^{3^k})^{1/2} \leqslant M_k < \infty,$$

B. There exists an enumeration of the sequence $(\eta_i) = (\eta_{n_1 n_2 \ldots n_k})$ such that a.s.

$$\overline{\lim_{n\to\infty}}\, \left|\sum_{i=1}^n \eta_i\right|/(n(\ln\ln n)^k)^{1/2} \geqslant m_k > 0.$$

The numbers L, M_k, m_k are nonrandom.

For the proof of this theorem we need the following auxiliary statements. We set $S_n^* = \sup_{m\leqslant n} \left|\sum_{i=1}^m a_i\eta_i\right|$.

LEMMA 2. If $\xi_n \in L_p$, $p > 2$, then

$$\|S_n^*\|_p \leqslant C_\varepsilon (K_p p^* \|\xi_1\|_p)^k A_n, \tag{8}$$

where C_ε does not depend on p; $p \ge 2 + \varepsilon$, $\varepsilon > 0$.

The estimate (8) follows from the right-hand inequality of (2) and Lemma 2 of [14].

LEMMA 3. If $|\xi_n| \le L$, then

$$M \exp(\lambda\, |S_n^*/A_n|^{1/3^k}) \leqslant 2\exp(\lambda^2 B_k L^{2/3}), \tag{9}$$

where B_k is a constant depending only on k.

Proof of Lemma 3. The known estimate $\exp t \leqslant 2\sum_{p=0}^{\infty} t^{2p}/(2p)!$ gives the inequality

$$\mathsf{M}\exp(\lambda\,|S_n^*/A_n|)^{1/3k} \leqslant 2\sum_{p=0}^{\infty}\frac{\lambda^{2p}}{(2p)!}\,\mathsf{M}\,|S_n^*/A_n|^{(2p)/(3k)} = 2R_0.$$

We estimate the quantity $R_1 = \sum_{p=[9k/2]+1}^{\infty}\frac{\lambda^{2p}}{(2p)!}\,\mathsf{M}\,|S_n^*/A_n|^{(2p)/(3k)}$. We apply the inequality (8) and the estimate $K_p \leq Cp^{\frac{1}{2}}$:

$$R_1 \leqslant \sum_{p=[9k/2]+1}^{\infty}\frac{\lambda^{2p}}{(2p)!}\left(K_{2p/3k}\left(\frac{2p}{3k}\right)^{*}L\right)^{2p/3}C_1^{2p/3k} \leqslant$$

$$\leqslant \sum_{p=[9k/2]+1}^{\infty}\frac{\lambda^{2p}}{(2p)!}C_1^{2p/3k}\left(CL\left(\frac{2p}{3k}\right)^{3/2}\right)^{2p/3} \leqslant \sum_{p=[9k/2]+1}^{\infty}\frac{\lambda^{2p}}{(2p)!}\,p^p\,(2C_1^{2/3k}(CL)^{2/3}/3k)^p. \tag{10}$$

Let $R_2 = R_0 - R_1$. Then

$$R_2 \leqslant \sum_{p=0}^{[9k/2]}\frac{\lambda^{2p}}{(2p)!}\,\|S_n^*/A_n\|_3^{2p/3k} \leqslant$$

$$\leqslant 1 + \sum_{p=1}^{[9k/2]}\frac{\lambda^{2p}}{(2p)!}(C_1(K_3 3^* L)^k)^{2p/3k} \leqslant 1 + \sum_{p=1}^{[9k/2]}\frac{\lambda^{2p}}{(2p)!}(C_1^{2/3k}(4L)^{2/3})^p. \tag{11}$$

We set $B_k = \max(4^{2/3}, 2C^{2/3}/3k)c_1^{2/3k}$. Then the inequality (9) follows directly from the estimates (10), (11).

Proof of Theorem 2. A. For $\lambda > 0$ and $d > 0$ we have

$$\mathsf{P}\{|S_n^*| > d^{3k}(A_n^2(\ln\ln A_n)^{3k})^{1/2}\} =$$

$$= \mathsf{P}\{\lambda\,|S_n^*/A_n|^{1/3k} \geqslant d\lambda(\ln\ln A_n)^{1/2}\} \leqslant \mathsf{M}\exp(\lambda\,|S_n^*/A_n|^{1/3k})\exp(-d\lambda(\ln\ln A_n)^{1/2}).$$

We apply the estimate (9) to the last inequality for $\lambda = (\ln\ln A_n)^{1/2}$, $d = \gamma + B_k L^{2/3}$, $\gamma > 1$; B_k and L are defined in Lemma 3. We obtain

$$\mathsf{P}\{|S_n^*| > d^{3k}(A_n^2(\ln\ln A_n)^{3k})^{1/2}\} \leqslant 2\ln^{-\gamma} A_n. \tag{12}$$

In order to conclude the proof of Part A of Theorem 2 we need

LEMMA 4 [15]. Let Y_n be a random sequence, $a_n > 0$, $a_n \uparrow \infty$, $a_{n+1}/a_n \to 1$ for $n \to \infty$, $x \in (1, 1+\beta)$, $\beta > 0$, $\psi(x)$ is a positive nondecreasing function, $\sum_{n=1}^{\infty} 1/n\psi(n) < \infty$ and $\mathsf{P}(\max_{1\leqslant k\leqslant n} Y_k > xa_n) \leqslant B/\psi(a_n)$ for all sufficiently large n. Then a.s. $\overline{\lim_{n\to\infty}}\, Y_n/a_n \leqslant 1$.

We apply Lemma 4, setting $a_n = d^{3k}(A_n^2 \ln\ln^{3k} A_n)^{1/2}$. From the estimate (12) we obtain that a.s.

$$\overline{\lim_{n\to\infty}}\,|S_n|/(A_n^2\ln\ln^{3k}A_n)^{1/2} \leqslant M_k,$$

where $M_k \leqslant (1 + B_k L^{2/3})^{3k}$.

Part B of Theorem 2 follows from the equality obtained in [16]:

$$\overline{\lim_{n\to\infty}}\,U_{k,n}/(\mathsf{M}U_{k,n}^2(\ln\ln \mathsf{M}U_{k,n}^2)^k)^{1/2} = (2^k/k!)^{1/2} \text{ a.s.}$$

where

$$U_{k,n} = \sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\eta_{i_1\ldots i_k}.$$

This concludes the proof of Theorem 2.

We note that there remains open the problem of the liquidation of the gap between the exponents 3k and k in parts A and B of Theorem 2. There are reasons to believe that the exponent 3k in part A can be reduced to k.

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COMPONENTWISE SPLITTING MIXED ABELIAN GROUPS

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We consider mixed abelian groups and some of their p-mixed quotient groups. These quotient groups are assumed to be splitting. We study the connection of the groups of the class under consideration with the homomorphisms of their quotient groups modulo the periodic parts into the direct product of the primary components of these periodic parts.

We use the following notation:

$\bar{G}$ is the quotient group of the mixed abelian group G modulo its maximal periodic subgroup $T = T(G)$;

$\bar{g}$ is the coset $g + T$ ($g \in G$);

π_α is the epic projection $\Pi_\alpha A_\alpha \to A_\alpha$;

P is the set of all prime numbers.

J is an index set that contains the symbol 0. Other notation and terminology are used in the sense of [1].

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