CENTRAL LIMIT THEOREM IN A BANACH SPACE

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We consider a sequence $(\xi_n)_1^\infty$ of independent identically distributed random variables (i.i.d.r.v.) with values in a separable Banach space B with norm $\|\cdot\|$ satisfying $M\xi_1 = 0$. We will say that the central limit theorem (CLT) holds for the sequence $(\xi_n)_1^\infty$ if the distribution $S_n/\sqrt{n}$ converges weakly to the Gaussian distribution in B, where $S_n=\sum_1^n \xi_k$. In [1] there is an extensive bibliography on this topic.

In the present article we consider the CLT in Banach spaces with an unconditional basis. Using a known result [2], we construct an example of a sequence of i.i.d.r.v.'s which are bounded and have a Gaussian correlation operator, for which the variables $\|S_n/\sqrt{n}\|$ are uniformly stochastically bounded, but the CLT does not hold.

1. CLT in a Banach Space with Unconditional Basis. Let c_0 be the Banach space of sequences which converge to 0: $c_0=\{x=(x_i)_1^\infty : \lim\limits_{i\to\infty} x_i=0\}$, $\|x\|_0=\sup\limits_{i\geqslant 1}|x_i|$, $\xi\in c_0$, $M\xi=0$, $\xi=(\eta_i x_i \ln^{-1/2}(i+1))_1^\infty$, $x=(x_i)_1^\infty\in c_0$, where η_i are r.v.'s satisfying

$$M\|(\eta_i)_1^\infty\|_0^2=K<\infty. \tag{1}$$

It was proved in [3-5] that if $(\xi_n)_1^\infty$ are independent copies of ξ and (1) is satisfied, then the CLT holds in c_0 for the sequence $(\xi_n)_1^\infty$. We prove below that an analogous result holds in any Banach space with an unconditional basis. All the definitions and results that we use from the theory of Banach spaces can be found in [6].

THEOREM 1. Let B be a Banach space with an unconditional basis $(e_i)_1^\infty$, $\xi=\Sigma_1^\infty \eta_i x_i \ln^{-1/2}(i+1)e_i \in B$, $M\eta_i=0$, $i\geqslant 1$, $(x_i)_1^\infty$ where $(x_i)_1^\infty$ is a given sequence in R^1, such that $\Sigma_1^\infty x_i e_i \in B$. Let $(\xi_n)_1^\infty$ be independent copies of ξ. If the condition (1) holds, then the CLT is valid for the sequence $(\xi_n)_1^\infty$.

Remark. The condition (1) holds, in particular, if $|\eta_i|\leqslant C$ almost surely (a.s.), for some $C < \infty$.

For the proof of the theorem we need the following results.

LEMMA 1. Let $(e_i)_1^\infty$ be an unconditional basis for the space B, $\Sigma_1^\infty x_i e_i \in B$ and $0 < L < \infty$. Then the set $Z=\{z=\Sigma_1^\infty c_i x_i e_i : |c_i|\leqslant L\}$ is compact.

Proof. There exists a constant K_e depending on the unconditional basis, such that if $x=\Sigma_1^\infty a_i e_i\in B$ and $|c_i| \leqslant |a_i|$ for all $i\geqslant 1$, then $\Sigma_1^\infty c_i e_i\in B$ and $\|\Sigma_1^\infty c_i e_i\|<K_e\|x\|$. Let $\varepsilon > 0$ be arbitrary, and choose j large enough in order that $\|\Sigma_j^\infty x_i e_i\|<\varepsilon/2K_e L$. Cover the compact set $\{z = \Sigma_1^{j-1} c_i x_i e_i : |c_i|\leqslant L\}$ with a finite set $\varepsilon/2$-net $z_1, \ldots, z_n$. Let $z = \Sigma_1^{j-1} c_i x_i e_i + \Sigma_j^\infty c_i x_i e_i \in Z$. Then $\|\Sigma_j^\infty c_i x_i e_i\| < K_e L \|\Sigma_j^\infty x_i e_i\| < \varepsilon/2$ and we can find z_k in the $\varepsilon/2$-net $z_1, \ldots, z_n$, such that $\|\Sigma_1^{j-1} c_i x_i e_i - z_k\| < \varepsilon/2$. It follows that $\|z-z_k\|<\varepsilon$. Therefore, for any $\varepsilon > 0$, the set Z can be covered by a finite ε-net. Clearly Z is closed and hence compact.

We denote by R the covariance operator of the variable ξ [7]. The covariance operator R is called Gaussian if there is a Gaussian r.v. in B whose covariance operator is R.

LEMMA 2. Let ξ be a r.v. in a Banach space B with a basis $(e_i)_1^\infty$. Suppose that $M\|\xi\|^2 < \infty$, $M\xi = 0$ and that ξ has a Gaussian covariance operator. Let $(\xi_n)_1^\infty$ be independent copies of ξ. Denote by $V_m: B \to B$ the operator $V_m(\Sigma_1^\infty x_i e_i) = \Sigma_{m+1}^\infty x_i e_i$. The following statements are equivalent:

a) $\forall \varepsilon>0 \ \lim\limits_{m\to\infty} \sup\limits_{n\geqslant 1} P(\|V_m(S_n/\sqrt{n})\|>\varepsilon)=0$;

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b) $\lim_{m\to\infty} \sup_{n\geqslant 1} M \| V_m(S_n) \| / \sqrt{n} = 0$;

c) $\lim_{m\to\infty} \sup_{n\geqslant 1} M \| V_m(S_n) \|^2 / n = 0$;

d) The CLT holds for the sequence $(\xi_n)_1^\infty$.

Proof. The implications c) ⇒ b) ⇒ a) are obvious. It is also clear that c) implies that $\sup_{n\geqslant 1} M \| S_n/\sqrt{n} \| < \infty$. This inequality, the fact that R is Gaussian, and condition a) ensure that the CLT holds ([8, 7, p. 54]). If condition d) is satisfied, then, for all $\varepsilon > 0$, $P(\| V_m(S_n)/\sqrt{n} \| > \varepsilon) \to P(\| V_m(\gamma) \| > \varepsilon)$ as $n \to \infty$, since $\| V_m(x) \|$ is a continuous functional on B, where γ is a Gaussian variable in B. Consequently $\lim_{m\to\infty} \overline{\lim_{n\to\infty}} P \| V_m(S_n)/\sqrt{n} \| > \varepsilon) = \lim_{m\to\infty} P(\| V_m(\gamma) \| > \varepsilon) = 0$, which is equivalent to condition a). For the proof of the lemma it now suffices to show that a) ⇒ c). We will use the inequalities

$$P(\eta \geqslant \lambda M\eta) \geqslant (1-\lambda)^2 (M\eta)^2 / M\eta^2, \tag{2}$$

where η is a real r.v. with finite variance, $0 < \lambda < 1$ [9] and

$$M \| S_n \|^2 - (M \| S_n \|)^2 \leqslant \Sigma_1^n M \| \xi_k \|^2 \tag{3}$$

[10]. Suppose that

$$\overline{\lim_{m\to\infty}} \sup_{n\geqslant 1} M \| V_m(S_n) \|^2 / n = d > 0. \tag{4}$$

It is well known [6] that $\sup_{m\geqslant 1} \| V_m \| < \infty$ and $\| V_m(x) \| \to 0$ as $m \to \infty$ for any $x \in B$. Applying Lebesgue's bounded convergence theorem, under the hypotheses of Lemma 2, we have

$$\lim_{m\to\infty} M \| V_m(\xi) \|^2 = 0. \tag{5}$$

In (2), put $\eta = \| V_m(S_n)/\sqrt{n} \|$. Then

$$P(\| V_m(S_n/\sqrt{n}) \| \geqslant \lambda M \| V_m(S_n/\sqrt{n}) \|) \geqslant (1-\lambda)^2 \frac{(M \| V_m(S_n) \|)^2}{M \| V_m(S_n) \|^2}. \tag{6}$$

In inequality (3), replace ξ_k by $V_m(\xi_k)$ and apply the Eqs. (4) and (5). We deduce that there exist subsequences m_k, $n_k \uparrow \infty$ as $k \uparrow \infty$, such that

$$\lim_{k\to\infty} (M \| V_{m_k}(S_{n_k}) \|)^2 / n_k = \lim_{k\to\infty} M \| V_{m_k}(S_{n_k}) \|^2 / n_k = d > 0.$$

Hence, on account of (6), we arrive at a contradiction to condition a), i.e., Lemma 2 is proved.

LEMMA 3. If (1) is satisfied, then $M \| \xi \|^2 < \infty$.

Proof. In fact, under the hypotheses of Theorem 1, we have [6, Lemma 1.4]

$$\| \xi \| \leqslant \| (\eta_i)_1^\infty \|_0 \sup_{\theta_i = \pm 1} \left\| \sum_1^\infty \theta_i x_i \ln^{-1/2}(i+1) e_i \right\|,$$

and the variable $\sup_{\theta_i=\pm 1} \| \Sigma_1^\infty \theta_i x_i \ln^{-1/2}(i+1) e_i \|$ is bounded [6, p. 120], since the series $\Sigma_1^\infty x_i \ln^{-1/2}(i+1) e_i$ is unconditionally convergent.

LEMMA 4. The r.v. ξ has a Gaussian covariance operator.

Proof. Consider a Gaussian sequence $\gamma_i \in R^1$ such that

$$M\gamma_i = 0, \quad M\gamma_i\gamma_j = M\eta_i\eta_j = r_{ij}, \quad i, j \geqslant 1. \tag{7}$$

The fact that (10) is satisfied ensures that $M\gamma_i^2$ is bounded in i. It is well known [7, p. 255] that this implies $P(\sup_{i\geqslant 1} |\gamma_i| \ln^{-1/2}(i+1) < \infty) = 1$. Consequently $\gamma = \Sigma_1^\infty \gamma_i x_i \ln^{-1/2}(i+1) e_i \in B$, since $\Sigma_1^\infty x_i e_i$ is unconditionally convergent in B.

From this, together with (7), we have, for all x^*, $y^* \in B$,

$$Mx^*(\gamma)\,y^*(\gamma)=\sum_{i,j=1}^{\infty} r_{ij}x_ix_j\,[\ln(i+1)\ln(j+1)]^{-1/2}x^*(e_i)\,y^*(e_j)=Mx^*(\xi)\,y^*(\xi).$$

The latter inequalities ensure [7, p. 139] that the covariance operators of ξ and γ coincide.

Proof of Theorem 1. It follows from Lemmas 3 and 4 that the finite-dimensional distributions $S_n/\sqrt{n}$ converge to finite-dimensional distributions γ [1]. It remains to prove the density of the measures P_n corresponding to the r.v.'s $S_n/\sqrt{n}$. First of all we consider the case when the ξ_k are symmetrically distributed. Let $\xi_k=\Sigma_{i=1}^{\infty}\eta_{ki}x_i\ln^{-1/2}(i+1)e_i,\ k\geqslant 1$; then $S_n/\sqrt{n}=\Sigma_{i=1}^{\infty}(n\ln(i+1))^{-1/2}(\Sigma_{k=1}^{n}\eta_{ki})x_ie_i$. We shall show that

$$\lim_{L\to\infty}\sup_{n\geqslant 1}P\Big(\sup_{i\geqslant 1}(n\ln(i+1))^{-1/2}\Big|\sum_{k=1}^{n}\eta_{ki}\Big|>L\Big)=0. \tag{8}$$

The density of the family of measures P_n will then follow from Lemma 1 and (8).

We will need the following inequality [11, p. 70]:

$$P\Big(\Big|\sum_{1}^{n}\varepsilon_k a_k\Big|\geqslant z\Big(\sum_{1}^{n}a_k^2\Big)^{1/2}\Big)\leqslant 2\exp(-z^2/2), \tag{9}$$

where $a_k\in R^1$, $P(\varepsilon_k=\pm 1)=1/2$, and ε_k are independent, $k=1,\ldots,n$. A sequence $(\eta_k)_1^n$ of independent symmetric variables is equivalent to a sequence $(\varepsilon_k\eta_k)_1^n$, where $(\varepsilon_k)_1^n$ does not depend on $(\eta_k)_1^n$. Therefore we obtain from (9) that $P(|\Sigma_1^n\eta_k|\geqslant z(\Sigma_1^n\eta_k^2)^{1/2})\leqslant 2\exp(-z^2/2)$. From this it follows that, for all $i\geqslant 1$,

$$P\Big((n\ln(i+1))^{-1/2}\Big|\sum_{k=1}^{n}\eta_{ki}\Big|\geqslant z\Big(\sum_{k=1}^{n}\eta_{ki}^2/n\Big)^{1/2}\Big)\leqslant 2/(i+1)^{z^2/2}. \tag{10}$$

On the other hand, applying the strong law of large numbers in R^1 together with condition (1), we have

$$\sup_{i\geqslant 1}\frac{1}{n}\sum_{k=1}^{n}\eta_{ki}^2\leqslant\frac{1}{n}\sum_{k=1}^{n}\sup_{i\geqslant 1}\eta_{ki}^2=\frac{1}{n}\sum_{k=1}^{n}\|(\eta_{ki})_{i=1}^{\infty}\|_0^2\xrightarrow{\text{a.s.}}K.$$

Consequently,

$$\sup_{n\geqslant 1}\sup_{i\geqslant 1}\sum_{k=1}^{n}\eta_{ki}^2/n=\eta<\infty\ \text{a.s.} \tag{11}$$

Now the bounds (10), (11) give

$$P\Big(\sup_{i\geqslant 1}(n\ln(i+1))^{-1/2}\Big|\sum_{k=1}^{n}\eta_{ki}\Big|>z\eta^{1/2}\Big)\leqslant\sum_{1}^{\infty}2/(i+1)^{z^2/2}=f(z)\to 0,\quad z\to\infty. \tag{12}$$

Since the function $f(z)$ in (12) does not depend on n, we deduce from (11) the equality (8). We have thus proved the theorem in the symmetric case.

Let ξ be any r.v. in B satisfying the conditions of Theorem 1. Put $\xi_k^s=\xi_k-\xi_k'$, where (ξ_k,ξ_k') are independent copies of ξ for $k\geqslant 1$, and let $S_n^s=\Sigma_{k=1}^{n}\xi_k^s$. Clearly the ξ_k^s are symmetrically distributed and satisfy the conditions of Theorem 1. It follows that the CLT holds for the sequences $(\xi_n^s)_{n=1}^{\infty}$ and so does condition c) of Lemma 2, i.e., $\lim_{m\to\infty}\sup_{n\geqslant 1}M\|V_m(S_n^s)\|^2/n=0$.

It is well known [1] that $M\|X+Y\|^2\geqslant M\|Y\|^2$ if X and Y are independent and $MX=0$. Therefore $M\|V_m(S_n^s)\|^2\geqslant M\|V_m(S_n)\|^2$ and condition c) of Lemma 2 holds for the sequence $(\xi_n)_1^{\infty}$. Now the CLT holds for $(\xi_n)_1^{\infty}$. Theorem 1 has been proved.

2. Counterexample to the CLT in the Space c_0. In [2] there was constructed, for any separable Banach space uniformly containing l_∞^n, a r.v. ξ in B such that

$$\|\xi\|\leqslant 1\ \text{a.s.}, \tag{13}$$

$$\xi \text{ has Gaussian covariance operator} \tag{14}$$

but the CLT does not hold for $(\xi_n)_1^\infty$ (where ξ_n, $n \geqslant 1$, are independent copies of ξ). We construct an example of a r.v. ξ in the space c_0 such that the conditions (13), (14) are satisfied, as well as

$$\sup_{n\geqslant 1} M\|S_n\|^2/n < \infty, \tag{15}$$

but there exist $\varepsilon > 0$, $\beta > 0$, for which

$$\overline{\lim_{m\to\infty}} \sup_{n\geqslant 1} P(\|V_m(S_n/\sqrt{n})\| > \varepsilon) > \beta. \tag{16}$$

It follows from Lemma 2 that the CLT does not hold for $(\xi_n)_1^\infty$.

Let $(e_i)_1^\infty$ be the natural basis for the space c_0, $(\eta_{ki})_{k,i=1}^\infty$ independent r.v.'s, $P(\eta_{ki} = 1) = P(\eta_{ki} = -1) = \ln^{-1}(i+7)$, $P(\eta_{ki} = 0) = 1 - 2\ln^{-1}(i+7)$, $x_i = (\ln\ln(i+7)/\ln(i+7))^{1/2}$, $\xi_n = \Sigma_{i=1}^\infty \eta_{ni} x_i e_i$, $n, i \geqslant 1$.

THEOREM 2. Conditions (13)-(16) are satisfied for the sequence $(\xi_n)_1^\infty$.

Proof. It is not hard to see that the given variables ξ_n satisfy conditions (13), (14) [2]. We prove the inequality (16). Put $A_{ni} = \bigcap_{k=1}^n (\eta_{ki} = 1)$, $N_n = [\exp(n\ln n + \ln\ln n)]$ and $A_n = \bigcup_{i=n}^{N_n} A_{ni}$. Then $P(A_n) = 1 - P\left(\bigcap_{i=n}^{N_n} \bar{A}_{ni}\right) = 1 - \prod_{i=n}^{N_n}(1 - \ln^{-n}(i+7))$. We will write $a_n \sim b_n$ if $a_n/b_n \to 1$ as $n \to \infty$. For large values of n, we have

$$\prod_{i=1}^{N_n-7}(1-\ln^{-n}(i+7)) \leqslant (1-\ln^{-n}N_n)^{N_n-7} \sim (1-(n\ln n + \ln\ln n)^{-n})^{N_n}$$

$$= (1-\exp(-n\ln(n\ln n + \ln\ln n)))^{N_n} \sim \left(1 - \frac{1}{N_n}\exp\left(-n\ln\left(1+\frac{\ln\ln n}{n\ln n}\right)\right)\right)^{N_n} \to e^{-1}$$

as $n \to \infty$, since $\exp\left(n\ln\left(1+\frac{\ln\ln n}{n\ln n}\right)\right) \sim \exp\left(\frac{\ln\ln n}{\ln n}\right) \xrightarrow[n\to\infty]{} 1$. In addition $\Pi_{i=1}^n(1-\ln^{-n}(i+7)) \geqslant (1-\ln^{-n}8)^n \to 1$ as $n \to \infty$. Consequently, $\underline{\lim}_{n\to\infty} P(A_n) \geqslant 1 - e^{-1}$. If $\omega \in A_n$, then

$$\|V_n(S_n(\omega)/\sqrt{n})\| \geqslant \max_{n\leqslant i\leqslant N_n}\left|\frac{x_i}{\sqrt{n}}\sum_{k=1}^n \eta_{ki}(\omega)\right| \geqslant \frac{n\ln\ln^{1/2}N_n}{n^{1/2}\ln^{1/2}N_n} \sim \frac{\ln^{1/2}(n\ln n + \ln\ln n)}{(\ln n + (\ln\ln n)/n)^{1/2}} \xrightarrow[n\to\infty]{} 1.$$

Therefore $\overline{\lim_{m\to\infty}}\sup_{n\geqslant 1} P(\|V_m(S_n/\sqrt{n})\| > 1/2) \geqslant \underline{\lim}_{n\to\infty} P(A_n) \geqslant 1 - e^{-1}$, i.e., inequality (16) is proved. We will show that

$$\lim_{C\to\infty}\sup_{n\geqslant 1} P(\|S_n/\sqrt{n}\| > C) = 0. \tag{17}$$

We obtain the bound (15) from this by using arguments analogous to those in the proof of Lemma 2. We have $P(\|S_n/\sqrt{n}\| > C) = P(\sup_{i\geqslant 1}|\Sigma_{k=1}^n \eta_{ki}|x_i/\sqrt{n} > C) \leqslant \Sigma_{\sqrt{n}x_i \geqslant C} P(|\Sigma_{k=1}^n \eta_{ki}| > C\sqrt{n}/x_i) \stackrel{\text{def}}{=} I(C,n)$. In order to evaluate I(C, n), we need the following inequalities [11, p. 76, 12]:

$$P\left(\left|\sum_{k=1}^n Y_k\right| > x\sqrt{n}\right) \leqslant 2\exp(-2^{-1}x\sqrt{n}\,\mathrm{arsh}(x/2\sigma^2\sqrt{n}), \quad x > 0, \tag{18}$$

and

$$P\left(\left|\sum_{k=1}^n Y_k\right| > x\sqrt{n}\right) \leqslant 2\exp(-x\sqrt{n}/4), \tag{19}$$

for $x > \sigma^2\sqrt{n}$ [11, p. 73], where $(Y_k)_1^n$ are symmetric i.i.d.r.v.'s, $MY_k^2 = \sigma^2$, and $|Y_k| \leqslant 1$. The function arcsinh (z) satisfies the following inequalities [13, p. 52]:

$$\operatorname{arsh}(z)\geqslant z-z^3/6,\ |z|<1;\ \operatorname{arsh}(z)\geqslant \ln 2z,\ |z|\geqslant 1. \tag{20}$$

Put

$$A_1=\{i\geqslant 1: C\leqslant \sqrt{n}x_i,\ C<2\sigma_i^2\sqrt{n}x_i\},\ A_2=\{i\geqslant 1: C\leqslant \sqrt{n}x_i,$$
$$C\geqslant 2\sigma_i^2\sqrt{n}x_i,\ \sigma_i^2\sqrt{n}\leqslant 1\},\ A_3=\{i\geqslant 1: C\leqslant \sqrt{n}x_i,\ C\geqslant 2\sigma_i^2\sqrt{n}x_i,$$
$$\sigma_i^2\sqrt{n}>1\},\ \sigma_i^2=M\eta_{ki}^2=2/\ln(i+7),\ P(i,C,n)=P(|\Sigma_{k=1}^n\eta_{ki})>C\sqrt{n}/x_i),$$
$$I_j(C,n)=\Sigma_{i\in A_j}P(i,C,n),\ j=1,2,3.$$

Then $I(C, n) = \Sigma_{j=1}^{3} I_j(C, n)$. For large C we have from (18) and (20),

$$I_1(C,n)\leqslant 2\sum_{i\in A_1}\exp\left(-\frac{C\sqrt{n}}{2x_i}\,\frac{5C}{6\cdot 2\sigma_i^2\sqrt{n}x_i}\right)=2\sum_{i\in A_1}\exp\left(-\frac{5C^2}{24\sigma_i^2x_i}\right)$$
$$=2\sum_{i\in A_2}\exp\left(-\frac{5C^2\ln^2(i+7)}{48\ln\ln(i+7)}\right)\leqslant 2\sum_{i\in A_1}(i+7)^{-K_1C^2}, \tag{21}$$

$$I_2(C,n)\leqslant 2\sum_{i\in A_1}\exp\left(-\frac{C\sqrt{n}}{2x_i}\ln\left(\frac{C}{\sigma_i^2\sqrt{n}x_i}\right)\right)\leqslant 2\sum_{i\in A_2}\exp\left(-\frac{C\sqrt{n}}{2x_i}\ln(C/x_i)\right)$$

$$\leqslant 2\sum_{i\in A_2}\exp\left(-\frac{C^2}{2x_i^2}\ln\left(\frac{\ln^{1/2}(i+7)}{\ln\ln^{1/2}(i+7)}\right)\right)=2\sum_{i\in A_2}\exp\left(-\frac{C^2}{4}\ln(i+7)\left(1-\frac{\ln\ln\ln(i+7)}{\ln\ln(i+7)}\right)\right)\leqslant 2\sum_{i\in A_2}(i+7)^{-K_2C^2}. \tag{22}$$

In the range A_3 we apply (19):

$$I_3(C,n)\leqslant 2\sum_{i\in A_3}\exp\left(-\frac{C\sqrt{n}}{4x_i}\right)\leqslant 2e^{2\sqrt{n}}e^{-CK_3\sqrt{n}}=2e^{-\sqrt{n}(CK_3-2)}. \tag{23}$$

In the estimates (21)-(23), $K_1 > 0$, $K_2 > 0$, and $K_3 > 0$ are absolute constants. Therefore $I(C, n) = \Sigma_{j=1}^{3} I_j(C, n) \to 0$ as $C \to \infty$, uniformly in n, i.e., (17) is proved.

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