NONNORMALIZING SUBSPACES AND INTEGRAL OPERATORS WITH UNREGULARIZABLE INVERSE

A. N. Plichko

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Let X be a Banach space and X^* its adjoint space. We say that a subspace $F \subset X^*$ is normalizing if the seminorm $|||x||| = \sup\{|f(x)|: f \in F, \|f\| \leq 1\}$ is a norm equivalent to the original norm $\|x\|$ of the space X. Otherwise F is said to be nonnormalizing. According to a theorem of Davis and Lindenstrauss (see [1], p. 78), under the condition $\dim X^{**}/X = \infty$ (a Banach space is identified with its image in the second adjoint space) X^* contains a nonnormalizing total subspace F. The fact that F is total means that for any $x \in X$, $x \neq 0$, there is a functional $f \in F$ with $f(x) \neq 0$. Let A be a linear continuous injective operator from the Banach space X to a Banach space Y. The inverse operator A^{-1} is said to be regularizable if there is a family of mappings R_δ: $Y \to X$, $\delta \in (0, \delta_0)$, such that for any $x \in X$

$$\sup\{\|x - R_\delta y\| : y \in Y, \|y - Ax\| \leqslant \delta\} \to 0 \text{ as } \delta \to 0.$$

In [2, 3] examples have been published of integral operators in separable spaces with nonregularizable inverse.

We shall prove a generalization of the theorem of Davis and Lindenstrauss for separable spaces and on the basis of it we shall obtain whole classes of function spaces on which there are integral operators with nonregularizable inverses. From this as special cases there will follow the examples from [2, 3]. We can regard Theorem 1 below as a new shorter proof of the theorem of Davis and Lindenstrauss, and from Theorem 2 there will follow to some extent an answer to the question posed in [3] of an explicit form of an integral operator from L_1 to L_2 with a nonregularizable inverse.

We denote by $M^\perp$ the annihilator in X^* of a set $M \subset X$, and by $M^\top$ the annihilator in X of a set $M \subset X^*$; $[x_n]_1^\infty$ is the closed linear hull of the sequence (x_n). The angle between the subspaces Y and Z of a Banach space X is the quantity $\inf\{\|y - z\|: y \in Y, z \in Z, \|y\| = \|z\| = 1\}$. If two closed subspaces intersect trivially, then the angle between them is greater than zero if and only if their sum is closed. As we know, a total subspace $F \subset X^*$ is normalizing if and only if the angle between the subspaces X and $F^\perp$ of X^{**} is nonzero ([1], p. 32). The following two assertions are well known.

LEMMA 1. Let X be a closed subspace of a Banach space Y, and Z an infinite-dimensional subspace of Y that intersects X trivially. Then Z contains an infinite-dimensional subspace Z_0 whose closure intersects X trivially.

LEMMA 2. Let X be a separable locally convex topological vector space, Z a finite-dimensional subspace of it, and V a closed convex bounded subset of X with $V \cap Z = \emptyset$. Then there is a functional $f \in X^*$ that strictly separates V and Z.

THEOREM 1. Let X be a separable Banach space, let F be a total subspace of X^*, and suppose that $\dim X^{**}/(F^\perp + X) = \infty$. Then F contains a total nonnormalizing subspace.

Proof. By the remark before the statement of Lemma 1 the sum $F^\perp + X$ can be assumed to be closed. By Lemma 1 there is a closed infinite-dimensional subspace $Y \subset X^{**}$ that intersects $F^\perp$ and X trivially. We choose normalized basic sequences (x_n) in X and (y_n) in Y ([4], p. 4). According to the theorem on stability of the basis ([4], p. 5) there is a sequence $a_n > 0$, $\Sigma_n a_n < \infty$ such that $z_n = x_n + a_n y_n$ is a basic sequence, and the sum $F^\perp + Z$, where $Z = [z_n]_1^\infty$, is closed. We show that $F^\perp \cap (Z + X) = 0$. For suppose that $\varphi = \Sigma_n b_n z_n + x$, $\varphi \in F^\perp$, $x \in X$. Then $\varphi - \sum_n b_n x_n - x = \sum_n b_n a_n y_n$, so all the b_n are zero and $\varphi = x = 0$. We also note that by construction every infinite-dimensional subspace Z makes a zero angle with X.

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From what we have said above, we can regard $X + Z$ as a subspace of F^*, endowed with the weak topology $w(X + Z, F)$. The space X will be a closed subspace of F^* in the norm of F^*, and according to Lemma 1 Z contains an infinite-dimensional subspace Z_0 whose closure in F^* intersects X trivially. We cover the set $X\backslash 0$ by a sequence of balls V_n of the space $X + Z_0$, closed in the norm of F^*, in such a way that $V_n \cap Z_0 = \emptyset$ for any n. Obviously all the balls V_n are bounded and closed in the topology $w(X + Z, F)$.

We construct sequences $z_n \in Z_0$ and $f_n \in F$ for which for any n

1) $z_n \notin [z_i]_1^{n-1}$, $z_n \in [(f_i)_1^{n-1}]^{\perp}$,

2) f_n strictly separates V_n and $[z_i]_1^n$.

We complete the construction as follows. We choose $z_1 \in Z_0$, $z_1 \neq 0$. By Lemma 2 we separate the subspace $[z_1]$ and the set V_1 by a functional $f_1 \in F$. Suppose that for $n - 1$ the elements have been constructed. The subspace $Z_0 \cap ([f_i]_1^{n-1})^{\perp}$ is infinite-dimensional, so we can choose an element z_n with the condition 1. By Lemma 2 we separate the set V_n and the subspace $[z_i]_1^n$ by a functional f_n.

We put $G = \text{lin}\,(f_n)_1^{\infty}$. Since $G^{\perp}\cap\left(\bigcup_n V_n\right)=\varnothing$ we have $G^{\perp} \cap X = 0$, so the subspace G is total. But since $G^{\perp} \cap Z \supset (z_n)_1^{\infty}$ is infinite-dimensional, G is nonnormalizing.

Let us recall some other definitions. A Markushevich basis of a Banach space X is a sequence (x_n, f_n), $x_n \in X$, $f_n \in X^*$ such that $f_n(x_m) = \delta_{nm}$ (δ is the Kronecker symbol), $[x_n]_1^{\infty} = X$ and $[f_n]_1^{\infty}$ is a total subspace. Let X be a Banach space of functions, say on the interval $[0, 1]$, that are algebraically, topologically and densely embedded in $L_1[0, 1]$. We denote by X' the space dual to X, that is, the collection of all functions $y(t)$, measurable on $[0, 1]$, for which $\sup\left\{\int_0^1 x(t)\,y(t)\,dt<\infty\colon \|x\|\leqslant 1\right\}<\infty$. The space X' is naturally identified with the subspace X^*. Since $L_\infty[0, 1] = L_1[0, 1]'$ is total on L_1, it is a fortiori total on X. Also we assume that $L_\infty[0, 1]$ is a linear subspace of X^*. A Banach space X of measurable functions with the condition $|x(t)| \leq |y(t)|$ almost everywhere, $y \in X \Rightarrow x \in X$, $\|x\| \leq \|y\|$, is called an ideal Banach space. We say that the norm of an ideal space is absolutely continuous if for any $x \in X$ and any decreasing sequence of measurable sets E_n with empty intersection $\|\chi_{E_n} x\|_X \to 0$ as $n \to \infty$.

THEOREM 2. Let X and Y be separable Banach spaces of functions on $[0, 1]$ that are algebraically, topologically and densely embedded in $L_1[0, 1]$ and suppose that $\dim X^{**}/(L_\infty[0, 1]^{\perp} + X) = \infty$. Then there is a compact injective integral operator from X to Y with nonregularizable inverse.

Proof. By Theorem 1 we find a total nonnormalizing subspace $F \subset L_\infty[0, 1]$. Because of [4], p. 43, there are Markushevich bases (x_n, f_n), $x_n \in X$, $\|x_n\| = 1$, $f_n \in F$, and (y_n, g_n), $y_n \in Y$, $\|y_n\| = 1$, $g_n \in Y^*$. For any $x \in X$ we put

$$Ax=\sum_{n=1}^{\infty}\frac{f_n(x)}{2^n\|f_n\|_{L_\infty}}y_n=\sum_{n=1}^{\infty}\frac{\int_0^1 f_n(s)\,x(s)\,ds}{2^n\|f_n\|_{L_\infty}}y_n(t). \tag{1}$$

The operator A is injective and compact, and its inverse is nonregularizable ([1], p. 190). Since $\|y_n\|_{L_1} \leq \|y_n\|_Y = 1$ and $\int_0^1 |f_n(s)\,x(s)|\,ds\leqslant\|f_n\|_{L_\infty}\|x\|_{L_1}$ the symbols of summation and integration in (1) can change place:

$$Ax=\int_0^1\left[\sum_{n=1}^{\infty}\frac{f_n(s)\,y_n(t)}{2^n\|f_n\|_{L_\infty}}\right]x(s)\,ds.$$

Consequently, A is an integral operator.

COROLLARY 1. If the separable spaces of functions X and Y are algebraically, topologically and densely embedded in L_1, and X is not isomorphic to the adjoint space, then there is an integral injective operator from X to Y with nonregularizable inverse.

Proof. It is sufficient to verify that under the conditions of Corollary 1 $\dim X^{**}/(L_\infty[0, 1]^\perp + X) = \infty$. But otherwise there is a weakly* closed subspace $L_\infty^\perp \subset Z \subset X^{**}$ with the condition $Z \oplus X = X^{**}$. Then $(Z^\top)^*$ is isomorphic to X.

In particular, for X we can take the spaces $C^{(n)}$, $n = 0, \infty$, the Lipschitz spaces lip_α, $0 < \alpha < 1$, or the space $L_1[0, 1]$.

Remark. Menikhes [2] constructed an example of an integral operator from C to L_2 with an infinitely differentiable symmetric kernel. It is not difficult to see that this can be achieved in our construction [in the proof of Theorem 2 instead of the Markushevich basis (x_n, f_n) we can take a total biorthogonal system]. In fact, the idea of using biorthogonal systems to construct integral operators is taken from [2].

COROLLARY 2. Suppose that the function spaces X and Y are algebraically and topologically embedded in L_1, and that X is a nonreflexive ideal space with absolutely continuous norm. Then there is an integral injective operator from X to Y with nonregularizable inverse.

Proof. As we know, under the conditions of Corollary 2, $X' = X^*$ ([5], p. 65) and $\dim X^{**}/X = \infty$ ([6], p. 35). We construct an operator A: $X \to Y$ by formula (1), taking for F any nonnormalizing subspace of X^*, and taking $\|f_n\|_{X^*}$ instead of $\|f_n\|_{L_\infty}$. Since $\int_0^1 |f_n(s)\,x(s)|\,ds \leqslant \| |f_n| \|_{X^*} \| |x| \|_X = \|f_n\|_{X^*} \|x\|_X$ the symbols of summation and integration in (1) can again change places, and we obtain an integral operator.

Under unimportant additional restrictions the integral operator is completely continuous ([7], p. 156). If, in addition, it is defined on a nonseparable space of functions on [0, 1], then its inverse is not regularizable [8].

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