and it then follows from Theorem 3 that not every function (4.3) of finite R-order has the property that ln M(σ, F) ~ ln u(σ, F) as σ → +∞, outside of some set of zero density. But if function (4.3) has zero R-order, i.e., if |a_n| ≤ exp {−λ_n ln (ln n)}, n ≥ 0, where ψ ∈ L and ϕ(x)/ln x → +∞ (x → +∞), then relation (4.4) implies the condition (1.4) and, according to Theorem 1, ln M(σ, F) ~ ln u(σ, F) as σ → +∞, outside of some set of zero density. Thus we arrive at the following theorem.

**THEOREM 4.** In order that every function (4.3) of finite R-order ρ_R have the property that ln M(σ, F) ~ ln u(σ, F) as σ → +∞, outside of some set of zero density, it is necessary and sufficient that ρ_R = 0.

**LITERATURE CITED.**

In this definition and further on an equality or inequality between elements of an SFS is understood in the "almost everywhere" sense.

A sequence of elements \((y_n)\) of an SFS will be called almost disjoint, if there exists a sequence \((x_n) \subset E\), such that \(x_n(t)x_m(t) = 0\) for arbitrary \(n \neq m\) and \(\|y_n - x_n\|_E = 0\) for \(n \to \infty\).

Following [5], we shall call a system of elements \((x_n)\) of a Banach space an \([\ell_p]\)-system \((1 \leq p \leq \infty)\), if each of its subsequences contains a subsequence equivalent to the standard basis of the space \(\ell_p(\ell_{m'})\).

The main result of the present note is the following.

**Theorem 1.** There exists an orthonormal function system \((g_n(t))\), \(t \in [0,1]\), \(n = 1, \infty\), with the following properties:

1. \(g_n \in L^\infty[0,1]\) \(\sup_n \|g_n\|_{L^\infty} < \infty\);
2. the system \((g_n)\) is a system of any separable SFS \(E\), for which the continuous embeddings \(G \subseteq E \subseteq G^*\) hold, where \(G\) is the closure of the class \(L^\infty\) in the Orlicz space \(L_\infty^\Phi\) with \(M(u) = e^{u^2} - 1\), and \(G^*\) is its adjoint;
3. \((g_n)\) is an \([\ell_1]\)-system in the space \(E\) and an \([\ell_1]\)-system in \(L_\infty[0,1]\).

In [5] it has been shown that if a separable Banach space \(X\) contains a subspace isomorphic to \(\ell_p\), then \(X\) has a complete minimal system, whose every subsystem contains a subsequence spanning a space isomorphic to \(\ell_p\). The following theorem is a variant of the above statement for bases.

**Theorem 2.** Let a Banach space \(X\) contain a complementable subspace \(Y\), isomorphic to \(\ell_p\) \((1 \leq p \leq \infty; \ell_{m'}^* \text{ def } c_0)\), such that its complement \(Z = X \ominus Y\) has a basis. Then the space \(X\) has a basis, which is an \([\ell_p]\)-system.

We shall remark, that the space \(L_1[0,1]\) contains a subspace isomorphic to \(\ell_2\), but no basis of \(L_1\) is an \([\ell_2]\)-system, as it follows from the cited above result of Szarek [2]. The presence of a basis in the complement \(Z = X \ominus Y\) is also necessary. As a counterexample we can take the direct sum \(\ell_p \oplus Z\), where \(Z\) does not have the approximation property.

**Theorem 3.** The space \(L_1\) has a basis generated by a function system which does not contain any almost disjoint subsystems.

Now we shall turn to the proofs of the formulated results. The following statement, obviously, is well known.

**Lemma 1.** Let \(X, Y, Z\) be Banach spaces; \(X = Y \oplus Z\); \((y_n)\) is a basis of \(Y\); \((z_m)\) is a basis of \(Z\); then the set \((y_n) \cup (z_m)\) is a basis of the space \(X\), independently of its indexing, provided it preserves the original ordering of the elements of \((y_n)\) and the elements of \((z_m)\).

**Proof.** According to the projective basis criterion it is enough to verify the boundedness in the family of projections \(P_{nm}\) onto the subspace \([y_i]_n \oplus [z_j]_m\) parallel to \([y_i]_{n-1} \oplus [z_j]_{m-1}\) (the notation \([x]_n\) means the closure of the linear hull of the vectors \((x_r)_{r \leq n}\)). Let \(P\) be a projection from \(X\) onto \(Y\) parallel to \(Z\); \(x = y + z \subseteq X\); \(y \subseteq Y\); \(z \subseteq Z\). Then \(\| P_{nm} x \| = \| P_{nm} (y + z) \| \leq \| P_{nm} y \| + \| P_{nm} z \|\), where \(P_n\) is the projection in \(Y\) onto \([y_i]_n\) parallel to \([y_i]_{n+1}\), and \(P_m\) is the projection in \(Z\) onto \([z_j]_m\) parallel to \([z_j]_{m+1}\).

Since \(\| y \| \leq \| P\|\), \(\| z \| \leq \| I - P\|\), and the projections \(P_n\) and \(P_m\) are jointly bounded, then the numbers \(\| P_{nm}\|\) are also jointly bounded. The lemma has been proved.

A sequence \((X_n)\) of finite-dimensional subspaces of a Banach space \(X\) is called its finite-dimensional decomposition, if for every \(x \in X\) there exists a unique sequence of elements \(x_n \equiv X_n\) \((n = 1, 2, \ldots)\) for which \(x = \sum X_n x_n\). For every basis \((x_n)\) of the space \(X\) and an arbitrary increasing sequence of indices \(n_m\) the subspaces \([x_n]_{n_m}, \ldots, [x_n]_{n_m-1}\) constitute a finite-dimensional decomposition of the space \(X\). The following lemma is partially converse to this fact.

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LEMMA 2 [6]. Let $X_n$ be a finite-dimensional decomposition of a space $X$, and let each $X_n$ have a basis $(x^*_n)_{n=1}^k$, whose basis constant is less than a number $K$. Then the sequence $(x^*_1, x^*_2, x^*_3, \ldots)$ is a basis of the space $X$.

Proof of Theorem 1. Let $(r_n)_{n=1}^\infty$ be a Haar system: $r_1 = 0, r_{n+1} = r_n + 2^{n-1}, n = 1, \ldots, 2^n - 1$. Consider the orthonormal function system $(f_n)_{n=1}^\infty$, introduced by Olevskii in [7]: $f_n = r_n; f_{n+1} = r_{n+1}/2^n; f_{n+2} = f_n$ ($l = 1, \ldots, 2^n; n = 0, 1, \ldots$). Obviously, $f_{2^n} = r_n$ ($n = 1, 2, \ldots$).

Analogously [8, Proposition 1.2], one can show that the system $(f_k)$ is a basis of every separable symmetric space. If an SFS $E$ satisfies condition (1), then the norms of the spaces $E$ and $L_2$ are equivalent on the space $[r_n]_n$, because $(r_n)$ generates in $E$ a space isomorphic to $l_2$. Moreover, this subspace is complementable in $E$, and its complement is the closed linear hull of the remaining functions $f_k$ (see [9, p. 134]). From this it follows, that if an SFS $E$ satisfies condition (1), then there exists a constant $K_E$, such that for all $k = 1, 2, \ldots$ and any $x \in [r_n]_n$, $a \in R$

$$K_E^{-1} \| x + af_k \|_{L_2} \leq \| x \|_{E} \leq K_E \| x + af_k \|_{L_2}.$$  

(2)

We shall decompose the set $(f_k)_{k=1}^\infty (k = 2^n, n = 0, \infty)$ with an increasing sequence of indices $(r_n)_{n=1}^\infty: y_1, y_2, y_3, \ldots, y_n$, $\ldots$, preserving the former order, i.e., if $k = y_m$, then $k < k'$, then $m < m'$. The sequence $(r_m)$ will be defined so that $r_m - r_{m-1} = 2^m$, where the integer number $s_m$ is chosen from the condition

$$2^{s_m} \| y_{r_m} \|_{L_2} < 2^{-m}.$$  

(3)

We shall decompose the functions $(f_k)_{k=1}^\infty$ with the remaining natural numbers preserving the previous order: $y_1, y_2, \ldots, y_{r_2}, y_{r_3}, \ldots$. By Lemma 1 the sequence is a basis in every SFS $E$, which satisfies condition (1).

According to Lemma 1 in [10] for every $m$ there exists an orthogonal matrix $(a^m_{n,i}; i_{m-1} < k < i_m)$, such that for

$$g_k = \sum_{i_{m-1} < k < i_m} a^m_{n,i} y_i; i_{m-1} < k < i_m; a^m_{n,i} = 2^{-s_m/2}$$

and

$$\| g_k \|_{L_2} \leq (1 + \sqrt{2}) \max_{i_{m-1} < k < i_m} \| y_i \|_{L_\infty} + 2^{-s_m/2} \| y_{i_m} \|_{L_\infty} \leq (1 + \sqrt{2}) + 2^{-m}.$$  

(4)

The last inequality in (5) follows from the fact that $\| y_i \|_{L_\infty} = 1$ for $i = i_m$, $m = 1, \infty$, and from (3).

For the sequence $(g_k)_{k=1}^\infty$ condition (a) follows from (5). If an SFS $E$ satisfies condition (1), i.e., if (2) is satisfied, then by virtue of the orthonormality of the matrices $(a^m_{n,i})$ the basic constants of the system $(g_k; i_{m-1} < k < i_m)$ are jointly bounded. Therefore, by Lemma 2, the sequence $(g_k)_{k=1}^\infty$ is a basis of the space $E$, and condition b) is satisfied.

At last, we shall verify condition c). Let $(g_m)_{m=1}^\infty$ be a subsequence of the system $(g_n)$. We shall choose from it a subsequence $(g_n(k))_{k=1}^\infty$, such that $g_n(k) \in \{g_n: m_{j-1} < n < m_j\}$. Then by (4) and (3)

$$\| g_n(k) - z_j \|_{E} \leq \| g_n(k) - z_j \|_{L_\infty} < 2^{-m_j}.$$  

(6)

By the orthogonality of the matrix $(a^m_{n,i})$ and by inequality (2), for sufficiently big j's

$$\| z_j \|_{E} > K_E^{1/2} \| z_j \|_{L_2} \geq K_E^{1/2} \| g_n(k) - \sum_{m_{j-1} < n < m_j} [a^m_{n,i}(1/2)]^{1/2} \|_2 \geq \frac{1}{2}.$$  

(7)

The system $(z_j)$ is a block-basis of the Rademacher system, which in the space $E$ is equivalent to the standard basis of the space $l_2$. Since the inequalities (6) and (7) are satisfied, then by the theorem on the stability of bases (see, e.g., [11, p. 5]) the system $(g_n(k))_{k=1}^\infty$ is also equivalent in $E$ to the standard basis of the space $l_2$. The fact, that the functions $(g_n)$ create an $(l_1)$-system in the space $L_\infty$, follows, for instance, from Corollary 4 in [10]. Theorem 1 has been proved.
Proof of Theorem 2. Let \((y_n)_{n=1}^{\infty}\) be a normalized system, being a basis of the space \(Y\), equivalent to the standard basis of the space \(\ell_p\); let \((z_m)_{m=1}^{\infty}\) be a normalized basis of the space \(Z\), and \(i_k\) an increasing sequence of natural numbers \(k = 1, 2, \ldots; i_1 = 1; i_k - i_{k-1} = 2^k\) for \(k > 1\). We shall enumerate the elements of \((z_m)\) with the indices \(i_k\) preserving the previous order, and the elements \(y_n\) with the remaining (except for \(i_k\)) natural numbers, also preserving the previous order. According to Lemma 2 we obtain a basis \((e_i)_{i=1}^{\infty}\) of the space \(X\). For every fixed \(k\) we put

\[
u_k = \sum_{i < i_k} e_i; \quad \nu_k = \tfrac{e_i}{c_i} + \nu_k; \quad x_k = \frac{x_{i_k}}{c_k}; \quad x_i = \frac{x_{i_k}}{c_i}; \quad y_k = \nu_k \quad (i = 1)\]
LITERATURE CITED


PROPERTIES OF SOLUTIONS OF CONJUGACY PROBLEMS FOR CERTAIN IRREGULAR EQUATIONS

N. Kh. Agakhanov

We study the differential properties of solutions of a certain class of weakly irregular equations (see [1]). We establish the dependence of the smoothness of the solutions on the spectral properties of the differential operators that occur in the equation.

In the cylinder V = [-T_1, T_2] \times Q, where T_s > 0 for s = 1, 2 and Q = \{x \in \mathbb{R}^n : 0 \leq x_r \leq 2\pi, r = 1, n\}, we consider

\[ Lu(t, x) \equiv D_x^2 u(t, x) - A(t, -iD_x) u(t, x) = f(t, x), \]  

(1)

\[ A(t, -iD_x) = \begin{cases} A_1(-iD_x), & t \equiv (-T_1, t), \\ A_2(-iD_x), & t \equiv (0, T_2), \end{cases} \]

where A_s(-iD_x), s = 1, 2, are differential operations of arbitrary order in the variable x with constant complex coefficients. Let u = (u_1, u_2), f = (f_1, f_2), where u_1, u_2 and f_1, f_2 are the restrictions of the functions u(t, x) and f(t, x) to V_1 = [-T_1, 0] \times Q and V_2 = [0, T_2] \times Q, respectively. To Eq. (1) let us adjoint the periodicity condition with respect to each variable x_r, r = 1, n,

\[ u(t, x_1, \ldots, x_r + 2\pi, \ldots, x_n) = u(t, x_1, \ldots, x_r, \ldots, x_n), \]  

(2)

boundary conditions with respect to t of the form

\[ \sum_{p=0}^{2} [P_0^{(m)} D_p^0 u_1(-T_1, x) + P_0^{(m)} D_p^0 u_2(T_2, x)] = 0 \]  

(3)

(m = 1, 2, 3, x \equiv Q,