$$\sum_{0<\lambda_j\leqslant\lambda_n}\frac{1}{\lambda_j}\sim\ln\lambda_n \qquad (n\to\infty) \tag{4.4}$$

and it then follows from Theorem 3 that not every function (4.3) of finite R-order has the property that $\ln M(\sigma, F) \sim \ln\mu(\sigma, F)$ as $\sigma\to+\infty$, outside of some set of zero density. But if function (4.3) has zero R-order, i.e., if $|a_n|\leqslant\exp\{-\lambda_n\psi(\lambda_n)\}$, $n\geqslant 0$, where $\psi\in L$ and $\psi(x)/\ln x\to+\infty$ $(x\to+\infty)$, then relation (4.4) implies the condition (1.4) and, according to Theorem 1, $\ln M(\sigma, F)\sim\ln\mu(\sigma, F)$ as $\sigma\to+\infty$, outside of some set of zero density. Thus we arrive at the following theorem.

THEOREM 4. In order that every function (4.3) of finite R-order ρ_R have the property that $\ln M(\sigma, F)\sim\ln\mu(\sigma, F)$ as $\sigma\to+\infty$, outside of some set of zero density, it is necessary and sufficient that $\rho_R = 0$.

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BASES OF SYMMETRIC FUNCTION SPACES

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This paper is devoted to the investigation of the following problem: which symmetric Banach spaces of functions have a basis generated by a system of functions, with properties close to those of trigonometric functions?

Besides the orthonormality property a trigonometric system has also the property of the uniform integrability, and even the property of the uniform boundedness, but it constitutes a basis only in symmetric spaces which have unconditional bases (see [1, pp. 242, 247]). This property belongs to only some symmetric spaces among those for which the continuous embeddings $L_p\subseteq E\subseteq L_q$ hold for some $1 < p, q < \infty$.

A Haar system, which is a basis in every separable symmetric space (see [1]), in distribution from a trigonometric system, has the following property: its every subsystem contains an almost disjoint sequence. Therefore, a Haar system is not uniformly integrable.

The well-known result of Szarek [2] destroyed all the hopes to find at least one basis in the space $L_1[0, 1]$ with the uniform integrability property. This result, in particular, implies several other former negative results on bases in the space L_1; for example, L_1 does not have any bounded [3], or even order bounded basis [4]. However, as we shall show in this note, in the space L_1 there exists a basis generated by a function system without any almost disjoint subsystems.

Before we formulate and prove the results of this paper, we shall recall the definitions of several terms.

A symmetric function space (in short SFS) is (see [1]) a Banach space E (of the classes of) measurable functions over the interval [0, 1], such that

a) $x(t)\in E$ and $|y(t)|\leqslant|x(t)|$ implies $y(t)\in E$ and $\|y\|_E\leqslant\|x\|_E$;

b) $x(t)\in E$, and $|y(t)|$ is equimeasurable with $|x(t)|$, then $y(t)\in E$ and $\|y\|_E = \|x\|_E$.

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In this definition and further on an equality or inequality between elements of an SFS is understood in the "almost everywhere" sense.

A sequence of elements (y_n) of an SFS will be called almost disjoint, if there exists a sequence $(x_n) \subset E$, such that $x_n(t)x_m(t) = 0$ for arbitrary $n \neq m$ and $\| y_n - x_n \|_E / \| y_n \|_E \to 0$ for $n \to \infty$.

Following [5], we shall call a system of elements (x_n) of a Banach space an $[\ell_p]$-system $(1 \leqslant p \leqslant \infty)$, if each of its subsequences contains an subsequence equivalent to the standard basis of the space l_p $(l_\infty \overset{\text{def}}{=} c_0)$.

The main result of the present note is the following.

THEOREM 1. There exists an orthonormal function system $(g_n(t))$, $t \in [0, 1]$, $n = 1, \infty$, with the following properties:

a) $g_n \in L_\infty[0, 1]$; $\sup_n \| g_n \|_{L_\infty} < \infty$;

b) the system (g_n) is a basis of any separable SFS E, for which the continuous embedings

$$G \subseteq E \subseteq G^* \tag{1}$$

hold, where G is the closure of the class L_∞ in the Orlicz space L_M^* with $M(u) = e^{u^2} - 1$, and G^* is its adjoint;

c) (g_n) is an $[\ell_2]$-system in the space E and an $[\ell_1]$-system in $L_\infty[0, 1]$.

In [5] it has been shown that if a separable Banach space X contains a subspace isomorphic to ℓ_p, then X has a complete minimal system, whose every subsystem contains a subsequence spanning a space isomorphic to ℓ_p. The following theorem is a variant of the above statement for bases.

THEOREM 2. Let a Banach space X contain a complementable subspace Y, isomorphic to ℓ_p $(1 \leqslant p \leqslant \infty;\ l_\infty \overset{\text{def}}{=} c_0)$, such that its complement Z $(X = Y \oplus Z)$ has a basis. Then the space X has a basis, which is an $[\ell_p]$-system.

We shall remark, that the space $L_1[0, 1]$ contains a subspace isomorphic to ℓ_2, but no basis of L_1 is an $[\ell_2]$-system, as it follows from the cited above result of Szarek [2]. The presence of a basis in the complement $Z = X \ominus Y$ is also necessary. As a counterexample we can take the direct sum $\ell_p \oplus Z$, where Z does not have the approximation property.

THEOREM 3. The space L_1 has a basis generated by a function system which does not contain any almost disjoint subsystems.

Now we shall turn to the proofs of the formulated results. The following statement, obviously, is well known.

LEMMA 1. Let X, Y, Z be Banach spaces; $X = Y \oplus Z$; (y_n) is a basis of Y; (z_m) is a basis of Z; then the set $(y_n) \cup (z_m)$ is a basis of the space X, independently of its indexing, provided it preserves the original ordering of the elements of (y_n) and the elements of (z_m).

Proof. According to the projective basis criterion it is enough to verify the boundedness in the family of projections P_{nm} onto the subspace $[y_i]_1^n \oplus [z_j]_1^m$ parallel to $[y_i]_{n+1}^\infty \oplus [z_j]_{m+1}^\infty$ (the notation $[x_r]_m^n$ means the closure of the linear hull of the vectors $(x_r)_{r=m}^n$). Let P be a projection from X onto the subspace Y parallel to Z; $x = y + z \in X$; $y \in Y$, $z \in Z$, $\| x \| = 1$. Then $\| P_{nm}x \| = \| P_{nm}(y + z) \| \leqslant \| P_{nm}y \| + \| P_{nm}z \| = \| P_n y \| + \| P_m z \|$, where P_n is the projection in Y onto $[y_i]_1^n$ parallel to $[y_i]_{n+1}^\infty$, and P_m is the projection in Z onto $[z_j]_1^m$ parallel to $[z_j]_{m+1}^\infty$.

Since $\| y \| \leqslant \| P \|$, $\| z \| \leqslant \| I - P \|$, and the projections P_n and P_m are jointly bounded, then the numbers $\| P_{nm} \|$ are also jointly bounded. The lemma has been proved.

A sequence (X_n) of finite-dimensional subspaces of a Banach space X is called its finite-dimensional decomposition, if for every $x \in X$ there exists a unique sequence of elements $x_n \in X_n$ $(n = 1, 2, \ldots)$ for which $x = \sum_{n=1}^\infty x_n$. For every basis (x_n) of the space X and an arbitrary increasing sequence of indices n_m the subspaces $[x_{n_m+1}, \ldots, x_{n_{m+1}}]$ constitute a finite-dimensional decomposition of the space X. The following lemma is partially converse to this fact.

LEMMA 2 [6]. Let X_n be a finite-dimensional decomposition of a space X, and let each X_n have a basis $(x_i^n)_{i=1}^{k_n}$, whose basic constant is less than a number K. Then the sequence $(x_1^1, \ldots, x_{k_1}^1, x_1^2, \ldots, x_{k_2}^2, x_1^3, \ldots)$ is a basis of the space X.

Proof of Theorem 1. Let $(r_n)_{n=0}^{\infty}$ be a Rademacher system $r_n(t) = \operatorname{sign} \sin 2^n \pi t$, $t \in [0, 1]$; $(h_k)_{k=1}^{\infty}$ is a Haar system: $h_1 = r_0$, $h_{2^n+l} = r_{n+1}\chi_{[(l-1)/2^n, l/2^n]}$ $(l = 1, \ldots, 2^n;\ n = 0, 1, \ldots)$. Consider the orthonormal function system $(f_k)_{k=1}^{\infty}$, introduced by Olevskii in [7]: $f_1 = h_1$; $f_{2^n+l} = r_{n+1}h_l/\| r_{n+1} \cdot h_l \|_{L_2}$ $(l = 1, \ldots, 2^n;\ n = 0, 1, \ldots)$. Obviously, $f_{2^n} = r_n$ $(n = 1, 2, \ldots)$.

Analogously [8, Proposition 1.2], one can show that the system (f_k) is a basis of every separable symmetric space. If an SFS E satisfies condition (1), then the norms of the spaces E and L_2 are equivalent on the space $[r_n]_0^{\infty}$, because (r_n) generates in E a space isomorphic to ℓ_2. Moreover, this subspace is complementable in E, and its complement is the closed linear hull of the remaining functions f_k (see [9, p. 134]). From this it follows, that if an SFS E satisfies condition (1), then there exists a constant K_E, such that for all k = 1, 2,... and any $x \in [r_n]_{n=0}^{\infty}$, $a \in R$

$$K_E^{-1} \| x + af_k/\| f_k \|_E \|_E \leqslant \| x + af_k \|_{L_2} \leqslant K_E \| x + af_k/\| f_k \|_E \|_E. \tag{2}$$

We shall denumerate the set $(f_k)_{k=1}^{\infty}$ $(k \neq 2^n,\ n = 0, \infty)$ with an increasing sequence of indices $(i_m)_{m=1}^{\infty}$: $y_{i_1}, y_{i_2}, \ldots, y_{i_m}, \ldots$ preserving the former order, i.e., if $f_k = y_{i_m}$, $f_{k'} = y_{i_{m'}}$ and $k < k'$, then $m < m'$. The sequence i_m will be defined so that $i_m - i_{m-1} = 2^{s_m}$, where the integer number s_m is chosen from the condition

$$2^{-s_m/2} \| y_{i_m} \|_{L_\infty[0,1]} < 2^{-m}. \tag{3}$$

We shall denumerate the functions $(f_{2^n})_{n=0}^{\infty}$ with the remaining natural numbers preserving the previous order: $y_1, y_2, \ldots, y_{i_1-1}, y_{i_1+1}, \ldots$. By Lemma 1 the sequence is a basis in every SFS E, which satisfies condition (1).

According to Lemma 1 in [10] for every m there exists an orthogonal matrix $(a_{k,i}^m : i_{m-1} < k,\ i \leqslant i_m)$, such that for

$$g_k = \textstyle\sum_{i_{m-1} < i \leqslant i_m} a_{k,i}^m y_i, \qquad i_{m-1} < k \leqslant i_m, \tag{4}$$
$$a_{k,i_m}^m = 2^{-s_m/2}$$

and

$$\| g_k \|_{L_\infty} \leqslant (1 + \sqrt{2}) \max_{i_{m-1} < i < i_m} \| y_i \|_{L_\infty} + 2^{-s_m/2} \| y_{i_m} \|_{L_\infty} \leqslant (1 + \sqrt{2}) + 2^{-m}. \tag{5}$$

The last inequality in (5) follows from the fact that $\| y_i \|_{L_\infty} = 1$ for $i \neq i_m$, $m = 1, \infty$, and from (3).

For the sequence $(g_k)_{k=1}^{\infty}$ condition a) follows from (5). If an SFS E satisfies condition (1), i.e., if (2) is satisfied, then by virtue of the orthogonality of the matrices $(a_{k,i}^m)$ the basic constants of the system $(g_k : i_{m-1} < k \leqslant i_m)$ are jointly bounded. Therefore, by Lemma 2, the sequence $(g_k)_{k=1}^{\infty}$ is a basis of the space E, and condition b) is satisfied.

At last, we shall verify condition c). Let $(g_{n_m})_{m=1}^{\infty}$ be a subsequence of the system (g_n). We shall choose from it a subsequence $(g_{n(j)})_{j=1}^{\infty}$, such that $g_{n(j)} \in \{g_n : i_{m_j-1} < n \leqslant i_{m_j}\}$ $(m_1 < m_2 < \ldots < m_j < \ldots)$. Put $z_j = \sum_{i_{m_j-1} < i < i_{m_j}} a_{n(j),i}^{m_j} y_i$. Then by (4) and (3)

$$\| g_{n(j)} - z_j \|_E \leqslant \| g_{n(j)} - z_j \|_{L_\infty} < 2^{-m_j}. \tag{6}$$

By the orthogonality of the matrix $(a_{k,i}^{m_j})$ and by inequality (2), for sufficiently big j's

$$\| z_j \|_E > K_E^{-1} \| z_j \|_{L_2} \geqslant K_E^{-1} \left(\textstyle\sum_{i_{m_j-1} < i < i_{m_j}} [a_{n(j),i}^{m_j}]^2\right)^{1/2} > \frac{1}{2}. \tag{7}$$

The system (z_j) is a block-basis of the Rademacher system, which in the space E is equivalent to the standard basis of the space ℓ_2. Since the inequalities (6) and (7) are satisfied, then by the theorem on the stability of bases (see, e.g., [11, p. 5]) the system $(g_{n(j)})_{j=1}^{\infty}$ is also equivalent in E to the standard basis of the space ℓ_2. The fact, that the functions (g_n) create an $[\ell_1]$-system in the space L_∞, follows, for instance, from Corollary 4 in [10]. Theorem 1 has been proved.

Proof of Theorem 2. Let $(y_n)_{n=1}^{\infty}$ be a normalized system, being a basis of the space Y, equivalent to the standard basis of the space ℓ_p; let $(z_m)_{m=1}^{\infty}$ be a normalized basis of the space Z, and i_k an increasing sequence of natural numbers $k = 1, 2, \ldots$; $i_1 = 1$; $i_k - i_{k-1} = 2^k$ for $k > 1$. We shall denumerate the elements of (z_m) with the indices i_k preserving the previous order, and the elements y_n with the remaining (except for i_k) natural numbers, also preserving the previous order. According to Lemma 2 we obtain a basis $(e_i)_{i=1}^{\infty}$ of the space X. For every fixed k we put $u_k = \sum_{i_k < i < i_{k+1}} e_i$; $\bar{x}_{i_k} = \frac{e_{i_k}}{c_k} + u_k$; $x_{i_k} = \frac{\bar{x}_{i_k}}{\|\bar{x}_{i_k}\|}$; $x_i = \frac{e_{i_k}}{c_k} + e_i$; $i_k < i < i_{k+1}$, where the constants c_k have the form $c_k = \|\sum_{i_k<i<i_{k+1}} e_i^*\| \geqslant K_q 2^{k/q}$; e_i^* are functionals biorthogonal to e_i; $p^{-1} + q^{-1} = 1$; K_q is some positive number. By Proposition 2.1 from [12] and Lemma 2 the system $(x_i)_{i=1}^{\infty}$ is a basis of the space X. We will show that it is an $[\ell_p]$.

Every subsequence (x_i) contains either an infinite subsequence $(x_{i(s)})_{s=1}^{\infty}$, $i_{k_s} < i(s) < i_{k_s+1}$, $k_1 < k_2 < \ldots < k_s < \ldots$, or an infinite subsequence $(x_{i_{k_s}})_{s=1}^{\infty} \subset (x_{i_k})$.

In the first case for $p > 1$ the inequality

$$\| x_{i(s)} - e_{i(s)} \| = \| e_{i_{k_s}} \| / c_{k_s} < 1/K_q \cdot 2^{k_s/q}$$

is satisfied, so that by the theorem on stability of bases $(x_{i(s)})_1^{\infty}$ is equivalent to the standard basis of the space ℓ_p. If, however, $p = 1$, then for any finite choice of real numbers (a_s) we have

$$\|\textstyle\sum_s a_s x_{i(s)}\| \leqslant \sum_s |a_s| (\| e_{i_{k_s}} \|/c_{k_s} + \| e_{i(s)} \|) \leqslant (\sum_s |a_s|)(1/K_\infty + 1)$$

and

$$\|\textstyle\sum_s a_s x_{i(s)}\| \geqslant (1/\| P \|) \|\sum_s a_s e_{i(s)}\| \geqslant K_1 \sum_s |a_s|,$$

where P is the projection from X onto Y parallel to Z, and K_1 is a constant depending on the norm $\|P\|$, and on the equivalence constant of the basis (y_n) with respect to the standard basis of the space ℓ_1. Therefore $(x_{i(s)})$ is equivalent to the basis of ℓ_1.

In the second case

$$\left\| x_{i_{k_s}} - \frac{u_{k_s}}{\|\bar{x}_{i_{k_s}}\|} \right\| = \frac{\| e_{i_{k_s}} \|}{c_{k_s} \|\bar{x}_{i_{k_s}}\|} = \frac{1}{\| e_{i_{k_s}} + c_{k_s} u_{k_s} \|} \leqslant (c_{k_s} \| u_{k_s} \| - 1)^{-1} \leqslant 1/(K_q 2^{k_s/q} K_p 2^{k_s/p} - 1),$$

where K_p is the constant of the equivalence between the basis (y_n) and the standard basis of ℓ_p. By the theorem on the bases stability the system $(x_{i_{k_s}})$ and (u_{k_s}) are equivalent. The sequence $u_{k_s}/\|\bar{x}_{i_{k_s}}\|$ is a block-basis with respect to a system equivalent to the standard basis of ℓ_p. Moreover, $\| u_{k_s}/\| x_{i_{k_s}} \| \| \to 1$ for $s \to \infty$. Therefore it is equivalent to the standard basis of the space ℓ_p [11, p. 53]. Consequently, the same is true also for $(x_{i_{k_s}})$. Theorem 2 has been proved.

Proof of Theorem 3. Let $\bar{h}_{n(i)}$ be a subsystem of the normalized in L_1 Haar system $(\bar{h}_n)$ such that $\bar{h}_{n(i)} \bar{h}_{n(j)} = 0$ and $\{t: \bar{h}_{n(i)} \neq 0\} \subset [0, 1/2]$. We shall consider a function system $(x_n)_1^{\infty}$, where $x_i = \bar{h}_{n(i)} + r_i \chi_{[1/2, 1]}$. It is easy to verify that $[x_i]_1^{\infty}$ is isomorphic to the space ℓ_1, and it has a complement in L_1, generated by the closed linear hull of those Haar functions, which do not belong to the system $\bar{h}_{n(i)}$ [and which will be denoted further by (z_n) with preservation of the initial order of the Haar system].

For each $k = 1, 2, \ldots$ we put

$$y_{3k-2} = \frac{z_k}{c_k} + x_{2k-1} + x_{2k};$$

$$y_{3k-i} = \frac{z_k}{c_k} + x_{2k-i} \qquad (i = 0, 1).$$

where $c_k = \| x_{2k-1}^* + x_{2k}^* \|_k$; x_{2k-i}^* are functionals biorthogonal to x_{2k-i} $(i = 0, 1)$, and the norm $\|\cdot\|_k$ is taken in the space $([x_{2k-i}]_0^1)^*$. Analogous to the proof of Theorem 2 and applying Proposition 2.1 from [12] we can easily show that the system $(y_k)_1^{\infty}$ is a basis of the space L_1. Obviously, $(y_k)_1^{\infty}$ does not contain any almost disjoint subsystems.

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PROPERTIES OF SOLUTIONS OF CONJUGACY PROBLEMS FOR CERTAIN IRREGULAR EQUATIONS

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We study the differential properties of solutions of a certain class of weakly irregular equations (see [1]). We establish the dependence of the smoothness of the solutions on the spectral properties of the differential operators that occur in the equation.

In the cylinder $V = [-T_1, T_2] \times Q$, where $T_s > 0$ for $s = 1, 2$ and $Q = \{x \in \mathbf{R}^n : 0 \leqslant x_r \leqslant 2\pi, r = \overline{1, n}\}$, we consider

$$Lu(t, x) \equiv D_t^3 u(t, x) - A(t, -iD_x)\, u(t, x) = f(t, x), \tag{1}$$

$$A(t, -iD_x) = \begin{cases} A_1(-iD_x), & t \in (-T_1, 0), \\ A_2(-iD_x), & t \in (0, T_2), \end{cases}$$

where $A_s(-iD_x)$, $s = 1, 2$, are differential operations of arbitrary order in the variable x with constant complex coefficients. Let $u = (u_1, u_2)$, $f = (f_1, f_2)$, where u_1, u_2 and f_1, f_2 are the restrictions of the functions u(t, x) and f(t, x) to $V_1 = [-T_1, 0] \times Q$ and $V_2 = [0, T_2] \times Q$, respectively. To Eq. (1) let us adjoint the periodicity condition with respect to each variable x_r, $r = \overline{1, n}$,

$$u(t, x_1, \ldots, x_r + 2\pi, \ldots, x_n) = u(t, x_1, \ldots, x_r, \ldots, x_n), \tag{2}$$

boundary conditions with respect to t of the form

$$\sum_{p=0}^{2} [\beta_p^{(m)} D_t^p u_1(-T_1, x) + \gamma_p^{(m)} D_t^p u_2(T_2, x)] = 0 \tag{3}$$
$$(m = 1, 2, 3) \quad x \in Q,$$

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