BASES AND COMPLEMENTS IN NONSEPARABLE BANACH SPACES. II

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This article is a continuation of the paper [1], and uses the same notation and terminology. We begin with the problem of extensions of the Markushevich basis. It is known (see [1, p. 231]) that in a separable Banach space, any M-basis of a subspace can be extended to an M-basis of the whole space. We show that there is no analog of this fact for nonseparable spaces. This gives a negative and conditional answer to a problem of Singer [2, p. 832], conditional in the sense that in the proof we use the nonseparable Shelach space [3], which has no uncountable biorthogonal systems. The existence of such a space was established in the hypothesis of the constructivity axiom V = L. Countable biorthogonal systems exist in any infinite-dimensional Banach space and are fairly good. We show that any Banach space has an orthonormal bibasic sequence. This answers one of Terenzi's problems [4]. Without assuming orthonormality, the existence of bounded bibasic systems was established in [4-6]. We also note that the natural projective basis of the space of continuous functions on the ordinals $C[\omega^2]$, can never be divided into two (or a finite number of) sequences, each of which is a basis with brackets in its closed linear hull. This is an answer to Singer's problem 13.4 of [2]. We note an application of M-bases to spaces, which is of interest from the point of view of vector measures. We prove that a space with an M-basis is an RNPG-space (Radon–Nikodym property generated). Hence, and from the previous results, it follows that there exists an operator from l_∞ into an RNPG-space which is not weakly compact; this answers a problem of Diestel [7]. We also show that the property of being an RNPG-space is not inherited by complemented subspaces. It is known (see [8, p. 115]) that for any separable subspace X of the WCG-space Y, there exists a separable complemented subspace $Z \subset Y$, containing X. In an arbitrary Banach space, this is not so in general. It is not known whether for any separable subspace X of the Banach space Y, there exists a subspace complemented in Y and of continual weight, containing X. We show that this fact holds for spaces with weakly* Angel conjugate.

Therefore, we shall assume that there exists a nonseparable Banach space Sh, which has only countable biorthogonal systems.

Proposition 1. There exists a Banach space with an M-basis and a complemented subspace $U \subset X$ also with an M-basis, such that no M-basis of the subspace U can ever be extended to an M-basis of the whole space X.

Proof. The space Sh, like any Banach space, has a total biorthogonal system [1]. It can be only countable, since Sh* is weakly* separable. By Theorem 2 of [1], the space Sh can be embedded isometrically and complementedly in a space X with an M-basis. Moreover, this complement U has an M-basis. Suppose that some M-basis u_i, $i \in I$, of the subspace U can be extended to an M-basis $(u_i)\cup(z_j)$ for the whole of X. It is easily seen that the factor-images $\hat{z}_j$ form a complete minimal system in the factor-space X/U, i.e., $[\hat{z}_j] = X/U$ and for any j, $\hat{z}_j \notin [\hat{z}_k : k \neq j]$. The functionals biorthogonal to z_j are defined in the natural way: $f_j(\hat{z}_k) = \delta_{jk}$. The factor-space X/U, which is isometric to Sh, is nonseparable, since the system $\hat{z}_j$, f_j is not countable, and this is impossible. ■

We recall that the biorthogonal sequence $(x_n, f_n)_1^\infty$ is called *bibasic*, if both (x_n) and (f_n) are bases in their norm-closed linear hulls, and *orthonormed*, if for any n $\|x_n\| = \|f_n\| = 1$. We introduce another definition. Let X be a Banach space, X* its conjugate and let $0 < \lambda \leqslant 1$. We say that the subset G of the sphere S(X*) [respectively, $Y \subset S(X)$] *λ-norms the subset* $z \subset X$ [respectively, $H \subset X^*$), if for any $z \in Z$

$$\|z\| \leqslant \lambda^{-1} \sup\{g(z):\ g \in G\}$$

[respectively, for any $h \in H$, $\|h\| \leqslant \lambda^{-1} \sup\{h(y):\ y \in Y\}$].

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LEMMA. A. Let Z and H be finite-dimensional subspaces of X and X*, and let $\lambda < 1$. There exist finite subsets $G \subset S(X^*)$ and $Y \subset S(X)$, which λ-norm Z and H, respectively. B. Let Z and H be subspaces of X and X*. There exist subsets $G \subset S(X^*)$ and $Y \subset S(X)$, of cardinality not greater than dens Z and dens Y, respectively, which 1-norm Z and H. C. Let Z be a closed subspace of X, and let the set $G \subset S(X^*)$ λ-norm Z. Then the norm of the projector $P: Z + G^{\perp} \to Z$ parallel to the annihilator $G^{\perp}$ is not greater than λ^{-1}. D. If, moreover, G is the unit sphere of some weakly* closed subspace, and S(Z) λ-norms G, then $Z \oplus G^{\perp} = X$.

The proofs are standard and are well known.

Proposition 2. Let $0 < \lambda < 1$. Any infinite-dimensional Banach space X has an orthonormal bibasic sequence x_n, f_n, and the norms of the natural projectors $[x_i]_1^{\infty} \to [x_i]_1^n$ and $[f_i]_1^{\infty} \to [f_i]_1^n$ are not greater than λ^{-1}.

Proof. We construct a sequence of elements $x_n \in X$, $f_n \in X^*$ and finite subsets $Y_n \subset S(X)$, $G_n \subset S(X^*)$ such that for any n

1) $\|x_n\| = \|f_n\| = f_n(x_n) = 1$;

2) $x_n \in Y_n$, $f_n \in G_n$, $Y_{n-1} \subset Y_n$, $G_{n-1} \subset G_n$;

3) Y_n λ-norms $[f_i]_1^n$, and G_n λ-norms $[x_i]_1^n$;

4) $x_{n+1} \in G_n^{\perp}$, $f_{n+1} \in Y_n^{\perp}$.

Take $x_1 \in S(X)$ and $f_1 \in S(X^*)$ such that $f_1(x_1) = 1$. Set $Y_1 = x_1$ and $G_1 = f_1$. If the objects are constructed for $n - 1$, then by the Krein–Krasnosel'skii–Mil'man theorem on orthogonal elements [9], there exists an element $x_n \in S(G_n^{\perp})$, orthogonal to $[Y_{n-1}]$ (i.e., an element for which $\inf\{\|x_n - y\|: y \in [Y_{n-1}]\} = 1$), and by the Hahn–Banach theorem, there exists a functional $f_n \in S(X^*)$ with $f_n(x_n) = 1$ and $f_n(Y_{n-1}) \equiv 0$. Take finite sets $Y_n \supset (x_n, Y_{n-1})$ and $G_n \supset (f_n, G_{n-1})$, which λ-norm $[f_i]_1^n$ and $[x_i]_1^n$, respectively. Since for any n $[x_i]_{n+1}^{\infty} \subset G_n^{\perp}$, and G_n λ-norms $[x_i]_1^n$ (respectively, $[f_i]_{n+1}^{\infty} \subset Y_n^{\perp}$ and Y_n λ-norms $[f_i]_1^n$), then by part C of the lemma, x_n, f_n form an orthonormal bibasic sequence, and moreover the norms of the corresponding projectors are not greater than λ^{-1}.

An M-basis $(x_n)_1^{\infty}$ is called a *basis with brackets*, if there exists an increasing sequence of natural numbers $(n_i)_{i=1}^{\infty}$, such that the natural projectors $P_{n_i}: X \to [x_n]_1^{n_i}$ parallel to $[x_n]_{n_i+1}$ are totally bounded.

Proposition 3. A standard projectional basis x_β: $1 \leqslant \beta < \omega^2$ of the space $C[\omega^2]$ can never be divided into a finite number of sequences, such that each sequence is a basis with brackets in its closed linear hull.

Proof. Let $(\dot{I}_k)_1^p$ be some partition of the set of ordinals $[1, \omega^2]$. Clearly, at least one of these sets, say $\dot{I}_1$, contains a subset $J = (k_n\omega + l_m)$, m, n = $\overline{1, \infty}$, where k_n and l_m are strictly increasing sequences of natural numbers. A subset of a basis with brackets is a basis with brackets in its closed linear hull. Thus it is sufficient to show that the set x_β: $\beta \in J$ is not a basis with brackets in its closed linear hull for any permutation of the indices. Since one element does not change anything, we shall assume that $\omega^2 \in J$. The mapping $x_{k_n\omega+l_m} \to x_{n\omega+m}$, $x_{\omega^2} \to x_{\omega^2}$ defines an isometry from $[x_\beta: \beta \in J]$ to $C[\omega^2]$. Therefore it is sufficient to establish that the projective basis x_β of the space $C[\omega^2]$ can never be made into a basis with brackets by any permutation of the indices. Suppose that x_n is a permutation of (x_β) and is a basis with brackets, let P_{n_i} be the corresponding projectors, and let f_n be functions biorthogonal to x_n. Since the projectors P_{n_i} are totally bounded, then the subspace $\bigcup_i P_{n_i}^*(C[\omega^2])^* = \text{lin}(f_n)_1^{\infty} \subset C[\omega^2]$ is norming (see [2, p. 277]). But this is not so [10]. ■

Remark. If the biorthogonal functionals to an M-basis in a separable space span a norming subspace, then this M-basis can be divided into 2 bases with brackets (see [2, p. 458]).

We say that the subset A of the Banach space X is *cusped*, if for any $\varepsilon > 0$ there exists $x \in A$, $x \notin \overline{\text{conv}}(A \setminus \{y: \|y - x\| < \varepsilon\})$. The set A is *inherently cusped*, if each subset is cusped. A Banach space which is the closed linear hull of its closed convex inherently cusped set is called an *RNPG-space*.

Proposition 4. Any space with an M-basis is an RNPG-space.

Proof. Let x_i, $i \in \dot{I}$, be an M-basis of the space X, let $\|x_i\| = 1$, and let f_i be functionals biorthogonal to x_i. The operator R: $l_1(\dot{I}) \to X$ defined by the formula $R(\sum a_i e_i) = \sum a_i x_i$

[e_i are the standard unit vectors of the space $l_1(\dot{I})$] is injective, continuous, and the image $R(B)$ of the sphere B of the space $l_1(\dot{I})$ generates X. Since $l_1(\dot{I}) = c_0(\dot{I})^*$, then the sphere B is compact in the topology $\sigma(l_1(I), c_0(I))$. Since $R^*(\mathrm{lin}(f_i)) \subset c_0(I) \subset l_1(I)^*$, then the image $R(B)$ is $\sigma(X, \mathrm{lin}(f_i))$-compact, and therefore is closed in norm. The space $l_1(\dot{I})$ has the Radon–Nikodym property [8, p. 174], i.e., B is an inherently cusped set. A closed set is inherently cusped if and only if any closed subset has an extremal point [11], and therefore $R(B)$ is also an inherently cusped set. ∎

COROLLARY 1. Let X^* be weakly* separable and nonreflexive (in particular, $X = l_\infty$). Then there exist an RNPG-space Y and a nonweakly compact operator of embedding $T: X \to Y$.

This is a simple combination of Theorem 2 of [1] and Proposition 4.

COROLLARY 2. The property of being an RNPG-space is not inherited by complemented subspaces.

In fact, the James tree JT [12] is separable and does not contain a subspace isomorphic to l_1, but the conjugate space $Y = JT^*$ is not separable. A space Y with such properties is not RNPG [13]. The conjugate Y^* is weakly* separable, and therefore by Theorem 2 of [1] the space Y can be embedded complementedly in the space X with an M-basis, which by Proposition 4 is an RNPG-space.

Remark. The classes of Banach spaces, generated by closed inherently cusped and closed convex inherently cusped sets are distinct. Since any minimal system is an inherently cusped set, then any space with a complete minimal system is generated by an inherently cusped set. Any Banach space with a Hilbert factor-space of the same weight [6], for example l_∞ or JT^* [12], has a complete minimal system. JT^* is not generated by a closed convex inherently cusped set. In particular, the closed convex hull of an inherently cusped set is not always inherently cusped.

The Banach space X has a weakly* Angel conjugate, if the weak* and weak* sequential closures of each bounded subset of X^* coincide.

Proposition 5. Let X^* be weakly* Angel. For any subspace $Y \subset X$, there exist a subspace $Y \subset Z \subset X$ of weight not greater than $m = \max(\mathrm{dens}Y, c)$ and a projector $P: X \to Z$ with $\|P\| = 1$.

Proof. We construct a transfinite sequence of subspaces $Z_\alpha \subset X$, $F_\alpha \subset X^*$, $1 \leqslant \alpha < \omega_1$ of weight not greater than m, such that for any α

1) $Z_1 = Y$, $Z_\alpha \subset Z_\beta$, $F_\alpha \subset F_\beta$ for $\alpha < \beta$;

2) F_α 1-norms Z_α and $Z_{\alpha+1}$ 1-norms F_α;

3) Z_α is norm-closed, $\mathrm{cl}^* B(F_\alpha) \subset F_{\alpha+1}$.

Let G_1 be a subset of the sphere $S(X^*)$ of cardinality not greater than m, 1-norming $Z_1 = Y$; this exists in view of part B of the lemma. Set $F_1 = [G_1]$. If for all $\beta < \alpha$ the subspaces Z_β and F_β have been constructed and α is a nonlimiting ordinal, then by part B of the lemma we can choose a subset $Y_{\alpha-1} \subset S(X)$ of cardinality not greater than m, which 1-norms the subspace $F_{\alpha-1}$, and set $Z_\alpha = [Z_{\alpha-1}, Y_{\alpha-1}]$. Clearly, the weight of the subspace Z_α is not greater than $m \times m = m$. Take a set $G_\alpha \subset S(X^*)$ of cardinality not greater than m, 1-norming Z_α, and set $F_\alpha = [\mathrm{cl}^* B(F_{\alpha-1}), G_\alpha]$. Since X^* is weakly* Angel, then each element of the closure $\mathrm{cl}^* B(F_{\alpha-1})$ is the limit of a sequence in the sphere $B(F_{\alpha-1})$, and therefore $\mathrm{dens}(\mathrm{cl}^* B(F_{\alpha-1})) \leqslant (\mathrm{dens}\, F_{\alpha-1})^{\aleph_0} \leqslant m^{\aleph_0} = m$. Thus $\mathrm{dens}\, F_\alpha \leqslant m$. If moreover α is a limiting ordinal, then set $Z_\alpha = [Z_\beta\colon \beta < \alpha]$ and $F_\alpha = [F_\beta\colon \beta < \alpha]$.

Thus we have constructed a sequence of spaces with the properties 1-3. Let $Z = \bigcup_{\alpha<\omega_1} Z_\alpha$, $F = \bigcup_{\alpha<\omega_1} F_\alpha$. Then $\mathrm{dens}\, Z \leqslant \overline{\overline{\omega_1}} \times m = m$. The subspaces Z and F 1-norm each other. If x_n is a fundamental sequence in Z, $x_n \in Z_{\alpha_n}$, then for $\alpha = \sup_n \alpha_n$ $x_n \in Z_\alpha = [Z_\alpha]$ and $x = \lim x_n \in Z$. Thus the subspace Z is closed. In exactly the same way, using condition 3, we can show that the unit sphere $B(F)$ is weakly* sequentially closed. Therefore it is weakly* closed, and the subspace F is weakly* closed. All the conditions of the lemma are satisfied, and therefore the projector $P: X \to Z$ parallel to the annihilator $F^\perp$ exists, and its norm is equal to 1. ∎

Remark. As an example of a space X with weakly* Angel conjugate and separable subspace $Y \subset X$, for which there is no complemented separable subspace $Y \subset Z \subset X$, consider the conjugate of the James tree [14].

Remark. Proposition 5 is false for arbitrary Banach spaces. Let Γ be a set with the cardinality of the continuum, and let $m_0(\Gamma)$ be the subspace of $m(\Gamma)$ consisting of all the bounded functions on Γ with countable carrier. The weight of the subspace $m(\Gamma)$ is equal to c. If we assume that the continuum hypothesis is valid, then $m_0(\Gamma)$ is the set of functions with carriers of cardinality less than card Γ. It was in fact noted in [15] that then, from a set-theoretical result of Serpinskii, it follows that $m(\Gamma)/m_0(\Gamma)$ contains a subspace isometric to $c_0(\Delta)$, Card Δ > Card Γ. Suppose that there exists a closed subspace $m_0(\Gamma)\subset Z\subset m(\Gamma)$, which has closed complement U in $m(\Gamma)$. The factor-space $m(\Gamma)/m_0(\Gamma)$ is isomorphic to $U\oplus Z/m_0(\Gamma)$. Since U is a subspace of $m(\Gamma)$, and dens $Z/m_0(\Gamma) \leqslant c$, then on $m(\Gamma)/m_0(\Gamma)$, and therefore also on $c_0(\Gamma)$, there exists a total set of linear continuous functionals of cardinality not greater than c. But this is impossible.

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