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A study of regularizability of linear inverse problems by linear methods was begun in [1] and [2]. We turn to the question of regularizability of linear inverse problems by one special form of linear methods useful in applications. Let us give the main definitions.

Consider the equation

\[ Ax = y, \]

where \( A : X \rightarrow Y \) is a continuous linear operator acting in the pair of normed linear spaces \( X \) and \( Y \), i.e., \( A \in B(X, Y) \). If \( \text{Ker} \ A = \{0\} \), then we write \( A \in B_0(X, Y) \).

**Definition 1** ([3], p. 44). A sequence \( \{R_n\} \) of operators \( R_n \in B(Y, X), n \in N \), is called a linear regularizer for the equation (1) if

\[ \forall \varepsilon \in X \exists x_0 \in X : R_n Ax \rightarrow x_0, n \rightarrow \infty \land x - x_0 \in \text{Ker} \ A. \]

In the case where \( \text{Ker} \ A = \{0\} \), the relation (2) is replaced by the condition that \( R_n Ax \rightarrow x \ as \ n \rightarrow \infty \, for \ all \ x \in X. \)

Equation (1) (and the inverse operator \( A^{-1} \) in the case where \( \text{Ker} \ A = \{0\} \)) is said to be linearly \( (L) \) regularizable if it has at least one linear regularizer \( \{R_n\} \). If, in addition, the operators \( R_n \) are finite-dimensional for each \( n \in N \), then equation (1) (the operator \( A^{-1} \)) is said to be linearly finite-dimensionally \( (\text{LFD}) \) regularizable.

In [4] Maslov observed that a Tikhonov regularizing family of operators has the property that its convergence at an element \( y_0 \in Y \) is equivalent to solvability of the equation \( A^*Ax = A^*y_0 \) and, consequently, if \( \text{R}(A) = Y \), to solvability of the equation \( Ax = y_0 \). In a number of publications [5]–[12] this result was then generalized and complemented.

In Chapter II, §3.8, of the monograph [13] of Lavrent’ev, Romanov, and Shishat-skii it is noted that if \( \text{Ker} \ A = \{0\} \), then convergence of every linear regularizer \( \{R_n\} \) such that \( R_n A = A R_n, n \in N \), at an element \( y_0 \in Y \) is equivalent to solvability of the equation \( Ax = y_0 \).

**Definition 2** [9]. A linear regularizer \( \{R_n\} \) for (1) is said to be solvable if \( M(\{R_n\}) = \text{R}(A), \) where \( M(\{R_n\}) = \{y \in Y | \exists x \in X : R_n Ay \rightarrow x, n \rightarrow \infty \} \) is the set of convergence of the regularizer \( \{R_n\} \).

Thus, a Tikhonov regularizer is solvable under the condition \( \text{R}(A) = Y \), and all regularizers \( \{R_n\} \) such that \( A R_n = R_n A, n \in N \) (these include regularizers of the operator-valued function type; [7]–[9]) are solvable in the case of \( \text{Ker} \ A = \{0\} \). Examples of solvable regularizers can be found also in classical mathematics. A well-known theorem in the theory of integral equations is that of Picard on solvability of the integral equation

\[ K x \int_a^b k(s, t)x(t) dt = y(s), \]

where \( K \in B(L_1[a; b], L_2[a; b]) \), and the kernel \( k(s, t) \) is continuous, symmetric, and closed. For the convenience of the subsequent arguments we give its formulation.

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Equation (3) is solvable for \( y = y_0 \) if and only if \( \sum_{k=1}^{\infty} \lambda_k^2(y_0, \varphi_k)^2 \) converges, where \( \{\lambda_k\} \) is the system of eigenvalues of the kernel \( k(s,t) \), and the \( (y_0, \varphi_k) \) are the Fourier coefficients of the element \( y_0 \) with respect to a complete orthonormal system \( \{\varphi_k\} \) of eigenelements of \( K \) corresponding to the system \( \{\lambda_k\} \). Moreover, if \( y_0 \in R(K) \), then the representation \( x_0 = \sum_{k=1}^{\infty} \lambda_k(y_0, \varphi_k) \varphi_k \) is valid for an exact solution \( x_0 \) of (3) when \( y = y_0 \).

Picard's theorem can be reformulated in terms of the properties of the sequence \( \{R_n\} \), where the operators \( R_n : L_2[a;b] \rightarrow L_2[a;b], \ n \in N, \) are defined by the formula

\[
R_ny = \sum_{k=1}^{n} \lambda_k(y, \varphi_k) \varphi_k.
\]

V. K. Ivanov noted that the sequence (4) forms a linear regularizer for (3) and used this fact to devise a method for solving (3) approximately (pp. 87–92 in [3]). It is not hard to see that Picard's theorem can be reformulated as follows: the sequence \( \{R_n\} \) of operators (4) is a solvable regularizer for (3). Thus, the construction of each finite-dimensional solvable regularizer leads directly to a generalization or analogue of Picard's theorem.

Consider linear regularizers \( \{R_n\} \) for (1) of the form

\[
R_ny = \sum_{k=1}^{n} \epsilon_k(y) \varphi_k, \quad n \in N,
\]

where \( \{\epsilon_k\} \) is some basis in \( X \), and \( \{\varphi_k\} \) is a corresponding system of functionals in \( Y^* \). For example, in Picard's theorem \( \epsilon_k = \varphi_k \), \( k \in N \), is a basis of eigenelements of the operator \( K \).

Maslov's result is supplemented by the following theorem.

**Theorem 1.** Let \( X \) and \( Y \) be Banach spaces, where \( X \) has a basis and is reflexive, and let \( A \in B_0(X,Y) \) be such that \( R(A) = Y \). Then for equation (1) there exists a solvable regularizer of the form (5).

Many known linear regularizers are solvable. It might seem that for triples \( (X, A, Y) \) with the condition \( R(A) = Y \) all regularizers are solvable. However, this is not so. What is more, for any linear regularizer it is possible to construct a linear regularizer without the property of solvability.

**Theorem 2.** Suppose that \( X \) and \( Y \) are Banach spaces, \( A \in B_0(X,Y), A^{-1} \) is unbounded, and the sequence \( \{R_n\} \) is a linear regularizer for (1). Then there exists a sequence \( \{P_n\} \) of operators \( P_n : Y \rightarrow Y, n \in N \), such that the sequence \( \{R'_n\} \) with \( R'_n = R_nP_n, n \in N \), forms a linear regularizer for (1) that is not solvable.

We say that (1) is linearly solvable (LS) regularizable if there exists at least one solvable regularizer for (1).

**Theorem 3.** Suppose that equation (1) is LS regularizable.

1) \( Y^* \) is weak-* separable if \( X^* \) is.
2) If \( Y \) is a WCG-space, then the weak-* separability of \( X^* \) implies the separability of \( X \) and \( Y \).
3) If \( Y \) is a reflexive Banach space, then \( Y^* \) is separable if \( X^* \) is.

Among the equations (1) satisfying the conditions of Theorem 3 there are LFD regularizable equations. For example, for an equation (1) with an operator \( A \in B_0(L_1[0;1], H) \) acting from \( L_2[0;1] \) to a nonseparable Hilbert space \( H \) there are no solvable regularizers, although there are linear (including finite-dimensional) regularizers. Picard's theorem cannot be extended to such equations with preservation of the form of its statement. It
is of interest to determine whether there is a linearly regularizable equation (1) that does not have a solvable regularizer in the case where $X$ and $Y$ are separable.

Generalizations of Picard's theorem are known in which solvability of the equation $Ax = y_0$ is described by two conditions, or which one is convergence of a corresponding linear regularizer (sometimes of a series) at the element $y_0$. Another condition, for example, for equations (1) with a compact operator $A: X \to Y$ acting in a pair of Hilbert spaces $X$ and $Y$ is the condition that $y_0 \in (\text{Ker } A^*)^\perp$ (see [14]).

The following theorem can, in our view, serve as a starting point for getting generalizations or analogues of Picard's theorem.

**Theorem 4.** Let $A \in B(X, Y)$. For any linear regularizer $\{R_n\}$ for (1) and for every linear set $P \subset Y$ such that $P \supset R(A)$ and $P \cap M_A(R_n) = \{0\}$ the equality $M(R_n) \cap P = R(A)$ holds, where $M_A(R_n) = \{y \in Y|\exists x \in \text{Ker } A: R_n y = x, n \to \infty\}$.

Thus, under the conditions of Theorem 4 the equation $Ax = y_0$ is solvable if and only if $y_0 \in M(R_n)$ and $y_n \in P$. In the case of a solvable regularizer $\{R_n\}$ we can take $P = Y$, this situation is very simple. In the general case $P$ can be chosen, for example, in the form $P = \{y \in Y|AR_n y \to y, n \to \infty\}$. We present two interesting solidifications of Theorem 4.

**Theorem 5.** Suppose that in the Banach space $X$ the bases $\{e_k\}$ and $\{e'_k\}$ are such that $BAe_k = \beta_k e'_k$, $k \in N$, for operators $A \in B_0(X, Y)$ and $B \in B(Y, X)$ having the property that $\text{Ker } B \cap R(A) = \{0\}$.

Then the equation $Ax = y_0$ is solvable if and only if the following conditions hold:

1) $\sum_{k=1}^{\infty} \frac{1}{\beta_k} f_k'(B y_0) e_k = y_0$ in the sense of the norm in $Y$.
2) $\sum_{k=1}^{\infty} f_k'(B y_0) e_k/\beta_k$ converges, where $\{f_k\}$ is a dual system for the basis $\{e'_k\}$.

Moreover, an exact solution $x_0$ of the equation $Ax = y_0$ has the representation

$$x_0 = \sum_{k=1}^{\infty} \frac{1}{\beta_k} f_k'(B y_0) e_k.$$

**Theorem 6.** Let $A \in B_0(X, Y)$, where $X$ and $Y$ are Hilbert spaces over the field $R$, with $X$ separable. The equation $Ax = y_0$ is solvable if and only if the following conditions hold:

1) $y_0 \in (\text{Ker } B)^\perp$,
2) $\sum_{k=1}^{\infty} b_k^2 (B y_0, \varphi_k)^2 < +\infty$, where $B \in B(Y, X)$ is any operator such that $BA$ is selfadjoint and compact, and $\text{Ker } B \subset \text{Ker } A^*$, $\{\varphi_k\}$ is a complete orthonormal system of eigenelements of $BA$, and $\{b_k\}$ is the corresponding system of eigenvalues. Moreover, an exact solution $x_0$ of the equation $Ax = y_0$ has the representation

$$x_0 = \sum_{k=1}^{\infty} b_k (B y_0, \varphi_k) \varphi_k.$$

Linear regularizers corresponding to Theorems 5 and 6 are defined by the formulas

$$R_n y = \sum_{k=1}^{n} \frac{1}{\beta_k} f_k'(B y) e_k, \quad R_n y = \sum_{k=1}^{n} b_k (B y, \varphi_k) \varphi_k, \quad n \in N.$$

The result in Theorem 6 for $B = E$ (the identity operator) coincides with Picard's theorem, and for $B = A^*$ it is contained in [14].

We remark that equation (1) has linear regularizers of the form (5) for any operator $A \in B_0(X, Y)$ acting from a reflexive Banach space $X$ with a basis into $Y$.

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