QUASIREFLEXIVE LOCALLY CONVEX SPACES WITHOUT BANACH SUBSPACES

A. N. Plichko and V. K. Maslyuchenko

UDC 517.98

James' known quasireflexive space, as well as its various Banach variations and generalizations, have been investigated by several authors. In [1] one has considered quasireflexive, locally convex spaces. We give an example of a quasireflexive, locally convex space, not isomorphic to any Banach space. One can take, for example, the direct sum of some reflexive topological vector space, not isomorphic to a Banach space (for example, the space $\boldsymbol{R}^\infty$ of all sequences with the topology of coordinatewise convergence), and James' quasireflexive space. In connection with this, M. I. Kadets, at the extended session of the Western Scientific Center of the Academy of Sciences of the Ukrainian SSR, has formulated the question of the existence of quasireflexive locally convex spaces, not containing any infinite-dimensional Banach subspaces. In this paper we consider two natural methods for the construction of such spaces.

We give the definition and some required properties of James spaces. We fix a number $1 < p < \infty$. The James space J_p consists of all sequences $x = (\xi_1, \ldots, \xi_k, \ldots)$, converging to zero and having a finite p-th variation

$$\|x\|_{J_p} = \sup\left(\sum_1^m |\xi_{k_i} - \xi_{k_{i-1}}|^p + |\xi_{k_m}|^p\right)^{1/p}, \tag{1}$$

where the supremum is taken over all possible increasing finite collections $k_0, \ldots, k_m$. The elements $e_k = (0, \ldots, 0, \underset{k}{1}, 0, \ldots)$, $k \in N$, form a basis of the space J_p, while the coordinate functionals p_k, biorthogonal to them, form a basis of the dual space J_p^*; the second dual J_p^{**} consists of sequences of $\tilde{J}_p$ with finite p-th variation (1) and the canonical imbedding $\pi\colon J_p \to J_p^{**}$ coincides with the natural imbedding $J_p \hookrightarrow \tilde{J}_p$. For $p = 2$ these statements can be found, for example, in [2]; for an arbitrary $1 < p < \infty$ the proof can be carried over word by word.

<u>LEMMA.</u> Let $a = (\alpha_j)_{j=1}^\infty \in l_p$, let $(k_j)_{j=0}^\infty$ be some increasing sequence of indices with $k_0 = 1$ and let $(y_j)_{j=1}^\infty$ be a sequence of elements of the form $y_j = \sum_{k_{j-1}}^{k_j - 1} \eta_k e_k$ with $\|y_j\|_{J_p} \leqslant 1$. Then the series $\sum_1^\infty \alpha_j y_j$ converges in the space J_p and

$$\left\| \sum_1^\infty \alpha_j y_j \right\|_{J_p} \leqslant 3 \|a\|_{l_p}.$$

<u>Proof</u>. For the sake of brevity, we denote by S the set of all possible finite increasing collections $s = (s_0, \ldots, s_m)$ and for a sequence $x = (\xi_k)$ and a collection $s = (s_i)_{i=0}^m \in S$ we set

$$v_p(x, s) = \sum_1^m |\xi_{s_i} - \xi_{s_{i-1}}|^p + |\xi_{s_m}|^p \text{ and } v_p(x) = \sup_{s \in S} v_p(x, s).$$

Obviously, the sequence $y = (\alpha_1\eta_1, \ldots, \alpha_1\eta_{k_1-1}, \alpha_2\eta_{k_1}, \ldots, \alpha_2\eta_{k_2-1}, \ldots)$ is the coordinatewise sum of

Translated from Teoriya Funktsii, Funktsional'nyi Analiz i Ikh Prilozheniya, No. 44, pp. 78-84, 1985. Original article submitted April 24, 1983.

the series $\sum_1^\infty \alpha_j y_j$. We estimate the norm of the vector $z_n = y - \sum_1^n \alpha_j y_j$. For an arbitrary variational sum $v_p(z_n, s)$ we have an inequality of the form

$$v_p(z_n, s) \leqslant \sum_1^{R_1} v_p(\alpha_{j_r} y_{j_r}, s^{(r)}) + \sum_1^{R_2} |\alpha_{j'_r} \eta_{k'_r} - \alpha_{j''_r} \eta_{k''_r}|^p,$$

where $s^{(r)} \in S$, (j_r), (j'_r), (j''_r) are increasing sequences of indices, greater than n, and, moreover, $j'_r < j''_r$, while (k'_r) and (k''_r) are appropriate increasing sequences of indices, for which $k_{j'_r} \leqslant k'_r < k_{j'_r+1}$ and $k_{j''_r} \leqslant k''_r < k_{j''_r+1}$ for $r = 1, \ldots, R_2$.

Since

$$\sum_1^{R_1} v_p(\alpha_{j_r} y_{j_r}, s^{(r)}) = \sum_1^{R_1} |\alpha_{j_r}|^p v_p(y_{j_r}, s^{(r)}) \leqslant \leqslant \sum_1^{R_1} |\alpha_{j_r}|^p \leqslant \sum_{j>n} |\alpha_j|^p,$$

$$\sum_1^{R_2} |\alpha_{j'_r} \eta_{k'_r} - \alpha_{j''_r} \eta_{k''_r}|^p \leqslant ((\sum_1^{R_2} |\alpha_{j'_r} \eta_{k'_r}|^p)^{1/p} + (\sum_1^{R_2} |\alpha_{j''_r} \eta_{k''_r}|^p)^{1/p})^p \leqslant ((\sum_1^{R_2} |\alpha_{j'_r}|^p)^{1/p} + (\sum_1^{R_2} |\alpha_{j''_r}|^p)^{1/p})^p \leqslant 2^p \sum_{j>n} |\alpha_j|^p,$$

it follows that $v_p(z_n) \leqslant (1 + 2^p) \sum_{j>n} |\alpha_j|^p$.

In particular, for $n = 0$ we obtain $v_p(y) \leqslant (1 + 2^p) \sum_1^\infty |\alpha_j|^p$, from where there follows that $y \in J_p$ and $\|y\|_{J_p} \leqslant (1 + 2^p)^{1/p} \|a\|_{l_p} \leqslant 3 \|a\|_{l_p}$. In addition,

$$\|y - \sum_1^n \alpha_j y_j\|_{J_p} \leqslant 3 (\sum_{j>n} |\alpha_j|^p)^{1/p},$$

and, therefore, the series $\sum_1^\infty \alpha_j y_j$ converges to y in J_p.

Proposition 1. Each closed infinite-dimensional subspace X of the space J_p contains a subspace Y that is isomorphic to l_p.

Proof (compare with [3, pp. 165-167]). From the infinite-dimensionality of X there follows that for each index N there exists in X and element $x = (\xi_k)$ with $\|x\|_{J_p} = 1$, for which $\xi_1 = \cdots \xi_N = 0$.

Let $k_0 = 1$ and let $x_1 = (\xi_{1k}) \in X$ be an arbitrary element with $\|x_1\|_{J_p} = 1$ and $\xi_{11} = 0$. We select an index $k_1 > k_0$, such that the norm of the element $z_1 = \sum_{k > k_1} \xi_{1k} e_k$ in the space J_p is less than 1/4 and we select in X a vector $x_2 = (\xi_{2k})$ with $\|x_2\|_{J_p} = 1$, for which $\xi_{2k} = 0$ for $k = 1, \ldots, k_1$. Then we find $k_2 > k_1$, such that the norm of the element $z_2 = \sum_{k > k_2} \xi_{2k} e_k$ in the space J_p is less than 1/16 and a vector $x_3 = (\xi_{3k}) \in X$ with $\|x_3\|_{J_p} = 1$, for which $\xi_{3k} = 0$ for $k = 1, \ldots, k_2$. Continuing this process indefinitely, we obtain an increasing sequence of indices $(k_j)_{j=0}^\infty$ with $k_0 = 1$ and a corresponding sequence of vectors $x_j = (\xi_{jk}) \in X$, such that for arbitrary $j = 1, 2, \ldots$ one has

$$\|x_j\|_{J_p} = 1, \ \xi_{jk} = 0 \text{ for } k = 1, \ldots, k_{j-1}$$

and the norm of the element $z_j = \sum_{k > k_j} \xi_{jk} e_k$ in the space J_p is less than $1/4^j$.

We set $y_j = x_j - z_j = \sum_{k_{j-1}+1}^{k_j-1} \xi_{jk} e_k$. For an arbitrary sequence $a = (\alpha_j) \in l_p$ we have

$$\|\sum_1^\infty \alpha_j z_j\|_{J_p} \leqslant \sum_1^\infty \|\alpha_j\| \|z_j\|_{J_p} \leqslant \|a\|_{l_p} \sum_1^\infty 4^{-j} = 3^{-1} \|a\|_{l_p}$$

and, by virtue of the lemma,

$$\|\sum_1^\infty \alpha_j y_j\|_{J_p} \leqslant 3 \|a\|_{l_p}.$$

On the other hand, for each $j = 1, 2, \ldots$ we have $\| y_j \|_{J_p} \geqslant \| x_j \|_{J_p} - \| z_j \|_{J_p} \geqslant 1 - 1/4^j > 1/2$, and, therefore, there exists a variational sum $v_p(y_j, s^{(j)})$, larger than $1/2^p$. For every m, the number $\sum_1^m v_p(\alpha_j y_j, s^{(j)})$ does not exceed some variational sum $v_p(y, s)$ of the sequence $y = \sum_1^\infty \alpha_j y_j$. Therefore,

$$v_p(y) \geqslant \sum_1^\infty v_p(\alpha_j y_j, s^{(j)}) = \sum_1^\infty |\alpha_j|^p v_p(y_j, s^{(j)}) \geqslant 2^{-p} \sum_1^\infty |\alpha_j|^p,$$

from where $\| y \|_{J_p} \geqslant 2^{-1} \| a \|_{l_p}$. Then

$$(1/2 - 1/3) \| a \|_{l_p} \leqslant \left\| \sum_1^\infty \alpha_j x_j \right\|_{J_p} \leqslant (3 + 1/3) \| a \|_{l_p}.$$

Consequently, by the formula $\psi(a) = \sum_1^\infty \alpha_j x_j$ one defines a continuous, linear mapping of the space l_p into X, bounded from below. This mapping is an isomorphism of l_p onto the subspace $Y = \psi(l_p) \subseteq X$, which is necessarily closed.

We recall that a separated locally convex space X is said to be quasireflexive if it is a closed subspace of finite deficiency in its second dual X^{**}, endowed with the strong topology $\beta(X^{**}, X^*)$, and the topology $\beta(X, X^*)$ coincides with the initial topology of the space X.

For $1 \leqslant p < \infty$ we set $\tilde{J}_{p+0} = \bigcap_{q>p} \tilde{J}_q$, while for $1 < p \leqslant \infty$ we define $\tilde{J}_{p-0} = \bigcup_{1<q<p} \tilde{J}_q$. We provide the space $\tilde{J}_{p+0}$ with the projective topology relative to the family of imbeddings $\tilde{J}_{p+0} \hookrightarrow \tilde{J}_q \ (q > p)$, while the space $\tilde{J}_{p-0}$ with the inductive topology relative to the family of imbeddings $\tilde{J}_q \hookrightarrow \tilde{J}_{p-0} \ (1 < q < p)$. Since the spaces $\tilde{J}_q$ increase with the increase of q, and their norms decrease, it follows that the space $\tilde{J}_{p+0}$ (resp. $\tilde{J}_{p-0}$) can be considered as a projective (resp. inductive) limit of the sequence of spaces $\tilde{J}_{q_n}$ where $q_n \downarrow p$ (resp. $q_n \uparrow p$). The space $\tilde{J}_{p+0}$ is a Fréchet space, while $\tilde{J}_{p-0}$ is a barreled Mackey space.

We also set $J_{p+0} = \bigcap_{q>p} J_q$ for $1 \leqslant p < \infty$ and $J_{p-0} = \bigcup_{1<q<p} J_q$ for $1 < p \leqslant \infty$. The space $J_{p+0}(J_{p-0})$ is provided with the topology induced from $\tilde{J}_{p+0}(\tilde{J}_{p-0})$, which coincides with the natural projective (inductive) limit topology on $J_{p+0}(J_{p-0})$. The space J_{p+0} (resp. J_{p-0}) is a closed subspace of deficiency 1 of the space $\tilde{J}_{p+0}$ (resp. $\tilde{J}_{p-0}$). Indeed, as one can easily see, both spaces $\tilde{J}_{p+0}$ and $\tilde{J}_{p-0}$ are continuously imbedded in the space c of all convergent sequences with the uniform convergence topology and, consequently, the functional $l(x) = \lim_{k \to \infty} \xi_k$ is continuous on $\tilde{J}_{p+0}$ and on $\tilde{J}_{p-0}$. The spaces J_{p+0} and J_{p-0} are exactly the kernels of the functinal in the corresponding spaces. We also mention that the elements $e_k = (0, \ldots, 0, \underset{k}{1}, 0, \ldots) \ (k = 1, 2, \ldots)$ form a basis in both spaces J_{p+0} and J_{p-0}, while the coordinate functionals p_k are continuous on J_{p+0} and J_{p-0} and, together with e_k form a biorthogonal system of vectors and functionals.

<u>Proposition 2.</u> The mapping $\varphi: F \to (F(p_k))_{k=1}^\infty$ is an isomorphism of the space J_{p+0}^{**} with the strong topology $\beta(J_{p+0}^{**}, J_p^*)$ (resp. of J_{p-0}^{**} with the topology $\beta(J_{p-0}^{**}, J_{p-0}^*)$) onto the space $\tilde{J}_{p+0}$ (resp. $\tilde{J}_{p-0}$), and, moreover, the diagram

$$\begin{array}{ccc} & J_{p+0} & \\ \swarrow^{\pi} & & \searrow^{j} \\ J_{p+0}^{**} & \xrightarrow{\varphi} & \tilde{J}_{p+0} \end{array} \quad \left(\text{respectively} \quad \begin{array}{ccc} & J_{p-0} & \\ \swarrow^{\pi} & & \searrow^{j} \\ J_{p-0}^{**} & \xrightarrow{\varphi} & \tilde{J}_{p-0} \end{array} \right)$$

is commutative.

(Here π is the canonical while j is the natural imbedding of the corresponding spaces.)

Proof.

1) For the space J_{p+0}.

Let $F \in J_{p+0}^{**}$. Then there exists a bounded set B in J_{p+0} and a constant $\gamma > 0$, such that $|F(f)| \leqslant \gamma$ for all $f \in B^0$, where the polar is taken with respect to the duality $\langle J_{p+0}, \tilde{J}_{p+0}^{*}\rangle$. We denote by B_q the closed unit ball of the space J_q. For each $q > p$ there exists $\gamma_q > 0$ such that $B \subseteq \gamma_q B_q$. Then $B^0 \supset \gamma_q^{-1}(B_q \cap J_{p+0})^0$, and, therefore, $|F(f)| \leqslant \gamma\gamma_p$ for all $f \in (B_q \cap \overline{J_{p+0}})^0$. Since the closure of the set $B_q \cap J_{p+0}$ in the space J_q coincides with the ball B_q, it follows that the polar $(B_q \cap J_{p+0})^0$ consists of all possible restrictions of the functionals from the closed unit ball of the space J_q^* on the space J_{p+0}. Consequently, the functional F is continuous on J_q^* (we identify the functionals from J_q^* with their restrictions). Then from the description of J_q^{**} there follows that $(F(p_k))_{k=1}^{\infty} \in \tilde{J}_q$. Since $q > p$ is arbitrary, this proves that $\varphi(F) \in \tilde{J}_{p+0}$.

Let $x = (\xi_k) \in J_{p+0}$. Then $\pi(x)(p_k) = p_k(x) = \xi_k$, i.e, $\varphi(\pi(x)) = x$. Thus, $j = \varphi \circ \pi$ and $\operatorname{im}\varphi \supseteq J_{p+0}$. For the proof of the surjectivity of φ we show that the vector $e = (1, 1, \ldots, 1, \ldots)$ is in $\operatorname{im}\varphi$.

Let $f \in J_{p+0}^*$. Then f is continuous on J_{p+0} as on a subspace of J_q for some $q > p$. Since $e \in \tilde{J}_q$, the series $\sum_1^\infty f_k$, where $f_k = f(e_k)$, converges and, consequently, the functional $F_0(f) = \sum_1^\infty f_k$ is defined on J_{p+0}^*. We show that F_0 is continuous in the strong topology of the space J_{p+0}^*.

We set $B = \bigcap_{q>p} B_q$. From a consequence of the bipolar theorem [4] we obtain that the polar B^0 is the weak closure of the union $\bigcup_{q>p}(B_q \cap J_{p+0})^0$. We fix an index n and a functional $f \in (B_q \cap J_{p+0})^0$ for some $q > p$. We have $\sum_1^n f_k = f(\sum_1^n e_k)$, and, therefore, $|\sum_1^n f_k| \leqslant \|f\|_{J_q^*}$ $\|\sum_1^n e_k\|_{J_q} \leqslant 1$. Thus, for each $n = 1, 2, \ldots$ and for an arbitrary $f \in \bigcup_{q>p}(B_q \cap J_{p+0})^0$ we have the inequality $|\sum_1^n f_k| \leqslant 1$. Taking in this inequality the limit, first with respect to f and then with respect to n, we obtain that $|\sum_1^\infty f_k| \leqslant 1$ for all $f \in B^0$, from here we obtain that $F_0 \in J_{p+0}^{**}$. Clearly, $\varphi(F_0) = e$, and, therefore, φ is surjective. Since the linearity and the injectivity of φ are obvious, it follows that φ is an algebraic isomorphism.

In order to prove that J_{p+0}^{**} and $\tilde{J}_{p+0}$ coincide topologically, first we mention that, since J_{p+0} is bornological, the canonical imbedding $\pi: J_{p+0} \to J_{p+0}^{**}$ is an isomorphism [4, p. 125 of the Russian edition]. The space J_{p+0} is closed in the in the strong topology of the space J_{p+0}^{**} as well as in the space $\tilde{J}_{p+0}$, since the strong topology on J_{p+0}^{**} majorizes its uniform topology, induced from the space c.

Indeed, the set $U = \{F \in J_{p+0}^{**} : \sup_k |F(p_k)| \leqslant 1\}$ is absolutely convex and $\sigma(J_{p+0}^{**}, J_{p+0}^*)$ is closed; therefore, $U = U^{00}$, where the polars are taken with respect to the duality $\langle J_{p+0}^{**}, J_{p+0}^*\rangle$, and, in addition, $U \cap J_{p+0}$ is a neighborhood of zero in J_{p+0}, consequently, the polar U^0 contained in $(U \cap J_{p+0})^0$ is compact relative to $\sigma(J_{p+0}^*, J_{p+0})$ and, thus, also a strongly bounded set.

Further, the codimension of the space J_{p+0} in the spaces J^{**}_{p+0} and $\tilde{J}_{p+0}$ is equal to 1. Consequently, each of the spaces J^{**}_{p+0} and is topologically isomorphic to the direct sum of the space J_{p+0} and a one-dimensional space E_0 with basis vector e_0, and the isomorphisms of $J_{p+0}\oplus E_0$ onto J^{**}_{p+0} and onto $\tilde{J}_{p+0}$ are the mappings $\varphi_1(x, te_0)=\pi(x)+tF_0$ and $\varphi_2(x, te_0)=x+te$, respectively. Since $\varphi(\varphi_1(x, te_0))=\varphi(\pi(x)+tF_0)=x+te=\varphi_2(x, te_0)$, we have $\varphi=\varphi_2\circ\varphi_1^{-1}$, and, therefore, φ is a topological isomorphism.

2) For the space J_{p-0}.

As it follows from the properties of the inductive topology, a linear functional f is continuous on the space J_{p-0} if and only if its restriction to each space $J_q (1<q<p)$ is continuous. Therefore, identifying each functional $f\in J^*_{p-0}$ with all of its restrictions to the spaces $J_q (1<q<p)$, we can write $J^*_{p-0}=\bigcap_{1<q<p}J^*_q$. It is easy to see that the ball of the space $J_q (1<q<p)$ is a closed set in the space J_{p-0} (it is even closed in the topology of coordinatewise convergence). Consequently, as this follows from a result of B. M. Makarov [5], the inductive limit J_{p-0} is regular, i.e., each bounded set in J_{p-0} lies and it is bounded in some J_q with $1<q<p$. From here we obtain easily that the strong topology on the space J^*_{p-0} coincides with the topology defined by the family of norms $\|\cdot\|_{J^*_q} (1<q<p)$.

Assume now that $F\in J^{**}_{p-0}$. Then F is continuous with respect to at least one norm $\|\cdot\|_{J^*_q}$ $(1<q<p)$ on J^*_{p-0}, consequently, there exists its extension $\bar{F}\in J^{**}_q$. Then $(F(p_k))_{k=1}^{\infty}=(\bar{F}(p_k))_{k=1}^{\infty}\in\tilde{J}_q$, from where there follows that $\varphi(F)\in\tilde{J}_{p-0}$.

Conversely, if $x\in\tilde{J}_{p-0}$, then there exist a number $1<q<p$ and a functional $\bar{F}\in J^{**}_q$, such that $(F(p_k))_{k=1}^{\infty}=x$. Then the restriction F of the functional $\bar{F}$ to the space J^*_{p-0} belongs to the space J^{**}_{p-0} and satisfies the equality $\varphi(F)=x$.

The remaining statements in this case are proved in the same way as in the previous case.

THEOREM. The spaces J_{p+0} and J_{p-0} are quasireflexive, locally convex spaces, not containing infinite-dimensional Banach spaces.

Proof. The quasireflexivity of the spaces J_{p+0} and J_{p-0} is a direct consequence of Proposition 2. We prove their second property, formulated in the theorem.

1) For the space J_{p+0}.

Let X be an infinite-dimensional subspace of J_{p+0}, such that the topology induced on X from J_{p+0}, is generated by some Banach norm $\|\cdot\|_X$. There exists $q_1>p$ and a constant γ_1, such that $\|x\|_X\leqslant\gamma_1\|x\|_{J_{q_1}}$ for all $x\in X$. We select $p<q_2<q_1$. For it one can find γ_2, such that $\|x\|_{J_{q_2}}\leqslant\gamma_2\|x\|_X$. In addition, $\|x\|_{J_{q_1}}\leqslant\|x\|_{J_{q_2}}$. Consequently, on the space X all the norms $\|\cdot\|_X$, $\|\cdot\|_{J_{q_1}}$, $\|\cdot\|_{J_{q_2}}$ are equivalent. Then X is a closed subspace of both J_{q_1}, and J_{q_2}. From Proposition 1 there follows that in X one can select a subspace Y, isomorphic to l_{q_2}, and in it, in turn, a subspace Z, isomorphic to l_{q_1}. The composition of the isomorphism $l_{q_1}\to Z$, the imbedding $Z\hookrightarrow Y$, and the isomorphism $Y\to l_{q_2}$ is an isomorphism of l_{q_1} onto some subspace of l_{q_2}. But, as it is known [6, p. 115], each operator acting from l_{q_1} into l_{q_2} with $1\leqslant q_2<q_1<\infty$ is compact and, therefore, such an isomorphism cannot exist.

2) For the space J_{p-0}.

Let X be an infinite-dimensional subspace of J_{p-0}. From category considerations or from the regularity of the inductive limit J_{p-0} there follows that there exists $1 < q_2 < p$, such that $X \subseteq J_{q_2}$. Let $q_2 < q_1 < p$. Since $X \subset J_{q_2}$, from the closed graph theorem there follows that the imbedding $\overline{X} \hookrightarrow J_{q_2}$ is continuous; therefore, there exists γ_2, such that $\|x\|_{J_{q_2}} \leqslant \gamma_2 \|x\|_X$ for all $x \in X$. On the other hand, the topology of the norm $\|\cdot\|_{J_{q_1}}$ on the space X majorizes the topology of the norm $\|\cdot\|_X$, in fact, the latter is the trace of the inductive topology. Consequently, there exists γ_1, such that $\|x\|_X \leqslant \gamma_1 \|x\|_{J_{q_1}}$. Then one applies exactly the same arguments as in the previous case.

LITERATURE CITED

1. A. N. Plichko, "Reflexivity and quasireflexivity conditions of topological vector spaces," Ukr. Mat. Zh., 27, No. 1, 24-32 (1975).
2. J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces. I. Sequence Spaces, Springer-Verlag, Berlin (1977).
3. S. S. Banakh (S. Banach), A Course in Functional Analysis [in Ukrainian], Radyans'ka Shkola, Kiev (1948).
4. A. P. Robertson and W. J. Robertson, Topological Vector Spaces, Cambridge Univ. Press, Cambridge (1964).
5. B. M. Makarov, "On inductive limits of normed spaces," Vestn. Leningr. Univ. Mat. Mekh. Astron., 13, No. 3, 50-58 (1965).
6. V. D. Mil'man, "The geometric theory of Banach spaces, Part 2," Usp. Mat. Nauk, 26, No. 6, 73-149 (1971).

SOLUTION OF A SINGULAR INTEGRAL EQUATION ON A SYSTEM OF INTERVALS

T. S. Polyanskaya

UDC 517.968

1. A series of mixed boundary value problems of mathematical physics lead to the following singular integral equation on a system of intervals [1]:

$$\int_E \frac{F(y)}{y-x}\,dy + \int_E K(x, y) F(y)\,dy = f(x),\ x \in E, \tag{1}$$

where $E = \bigcup_{k=1}^{m} (a_k, b_k)$, $-\infty < a_1 < b_1 < \cdots < a_m < b_m < \infty$, $f(x)$, $x \in \overline{E}$ μ times continuously differentiable and $f^{(\mu)}(x) \in H(\gamma)$, $(f(x) \in C_{\overline{E}}^{\mu,\gamma})$, $K(x, y) \in C_{\overline{E}}^{\mu,\gamma}$ with respect to each of the variables, uniformly with respect to the other variable.

The solution $F(y)$, $y \in \overline{E}$ is sought in the class of functions that can be represented in the form

$$F(y) = \Phi(y) \prod_{k=1}^{m} |b_k - y|^{\alpha_k} |y - a_k|^{\beta_k},\ y \in E,$$

where $\alpha_k = \pm\frac{1}{2}$, $\beta_k = \pm\frac{1}{2}$, $(k = 1, \ldots, m)$ are given numbers, $\Phi(y) \in H$, $y \in \overline{E}$. We denote by $\bar{\varkappa} = (\varkappa_1, \ldots, \varkappa_m)$ the index of equation (1). If $\varkappa_k = 1$, then we require that condition

Translated from Teoriya Funktsii, Funtsional'nyi Analiz i Ikh Prilozheniya, No. 44, pp. 84-88, 1985. Original article submitted November 12, 1983.

 0090-4104/90/4803-0312$12.50