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BASES AND COMPLEMENTS IN NONSEPARABLE BANACH SPACES

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Any Banach space has a total biorthogonal system (see [1, p. 638]). As Bessag noted [1, p. 599], a Banach space for which dens*E* = dens E has a subspace of weight E with projective basis (precise definitions will be given later). The following statement combines these results.

THEOREM 1. Let E be a Banach space and let α_0 be the first ordinal of weight dens*E*. For any $\varepsilon > 0$, the space E has a total biorthogonal system $(x_\alpha, f_\alpha, 1 \leqslant \alpha < \alpha_0)$, bounded by the number $4 + \varepsilon$, and moreover the elements x_α form a projective basis in their closed linear hull with basis constant $\leqslant 1 + \varepsilon$.

The proof is much shorter and of a more standard form than the known proof of the existence of a total biorthogonal system. In the space l_∞ there is no fundamental and total biorthogonal system (M-basis) (Johnson [1, p. 692]) and for noncountable Γ, $l_\infty(\Gamma)$ is not isomorphic to a subspace of a space with an M-basis (Dyer [1, p. 639]). But we do have the following:

THEOREM 2. A Banach space with a weakly* separable conjugate (in particular, l_∞) is isometric to a complemented subspace of a space with an M-basis.

This gives an answer to questions of Troyanskii, Distel, Ion, Sisler, and Singer [1, pp. 692, 832, and 837]. Later on we shall study some properties of the Enflo–Rosenthal basis, and in particular we show that it is a norming M-basis, and we give an example of a projective basis x_α in a Banach space X (which may even be separable), which is not an ordinary basis for any permutation of the indices, but for any $x \in X$ there exists a permutation σ of the natural series such that $x = \Sigma_1^\infty a_{\sigma(i)} x_{\sigma(i)}$. This answers a question of Bessag and Singer [1, p. 575]. Finally we shall be concerned with problems on Borel complements. If X and Y are Borel complementary subspaces of a separable Banach space, then they are closed [2]. This result follows from the Borel graph theorem and is closely connected with it. In the well-known formulations of the Borel graph theorem [3, 4, 2], the condition of separability appears in one form or another. Godefroy posed the question [2] of the validity of these statements for nonseparable spaces. We show that even for other fairly stringent requirements, both the Borel complement theorem and the Borel graph theorem are not true unless we stipulate separability.

1. Total Biorthogonal Systems. Denote by dens E the weight, i.e., the least cardinality of the dense subsets, of the space E. For the conjugate space E*, the symbol dens* E* denotes the least cardinality of the weakly* dense subsets of E*. The transfinite sequence x_α, $1 \leqslant \alpha < \alpha_0$ is called a projective basis of length α_0 of the space X, if the closed linear hull

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$[x_\alpha : 1 \leqslant \alpha < \alpha_0] = X$ and the projectors $P_\alpha : X \to [x_\beta : \beta < \alpha]$ parallel to $[x_\beta : \beta \geqslant \alpha]$ exist and are jointly bounded by the basis constant $x = \sup_\alpha \|P_\alpha\|$. The system $x_i, f_i, i \in I, x_i \in E, f_i \in E^*, i \in I$ (where I is some set) is called a total biorthogonal system, if $f_i(x_j) = \delta_{ij}$ (where δ is the Kronecker symbol) and the linear hull of $[f_i : i \in I]_* = E^*$ is weakly* closed. A system for which $\sup_i \|x_i\| \|f_i\| \leqslant c$ is said to be bounded by the number c. For a projective basis x_α, the functionals f_α exist and are defined uniquely, and (x_α, f_α) form a bounded total biorthogonal system [1, p. 585].

LEMMA 1. In any Banach space E, there exists a subspace of weight dens* E* with a projective whose basis constant is ~1.

We note that in the separable case the lemma only guarantees that such a subspace is one-dimensional. The proof does not differ in any way from the above-mentioned result of Bessag [1, p. 599]. For a subspace X of a Banach space E, we identify X* with the factor-space $E^*/X^\perp$. Denote by φ the canonical mapping $E^* \to E^*/X^\perp$, and by cl* M the weak* closure of the set M, let B(X) be the unit ball, and let S(X) be a sphere in the space X.

LEMMA 2. Let E be a Banach space, let X be a subspace of E with projective basis x_α, $1 \leqslant \alpha < \alpha_0$, and let $f_\alpha \in E^*$ be any functionals biorthogonal to x_α. Then there exists a subset $G \subset X^\perp$ of cardinality $\leqslant \overline{\overline{\alpha}}_0$ ($\overline{\overline{\alpha}}$ is the cardinality of the ordinal α), for which

$$\varphi([G, f_\alpha, 1 \leqslant \alpha < \alpha_0]_*) = X^*.$$

Proof. We show that for any α there exists a subset $G_\alpha \subset X^\perp$ of cardinality $\leqslant \overline{\overline{\alpha}}$, such that $\varphi([G_\alpha, F_\alpha]_*) = P^*_\alpha X^*$, where $F_\alpha = (f_\beta : 1 \leqslant \beta < \alpha)$ and P_α are natural projectors in the space X. For $\alpha = \alpha_0$ this gives us the statement of the lemma. For $\alpha = 1$ $P_1 \equiv 0$, and we may set $G_1 = 0$. If for all $\beta < \alpha$ the subsets G_β have been chosen and α is not a limiting ordinal, then $P^*_\alpha X^* = P^*_{\alpha-1} X^* + [\varphi(f_{\alpha-1})]$ and we may set $G_\alpha = G_{\alpha-1}$. If the ordinal α is limiting, then (see [5]) the set $\bigcup_{\beta<\alpha} B(P^*_\beta X^*)$ is weakly* dense in some ball $aB(P^*_\alpha X^*)$. In each ball $\rho e B(P^*_\beta X^*)$ we choose a weakly* dense set H_β of cardinality not more than $\overline{\overline{\alpha}}$, and let $H_\beta \subset B(E^*)$ be some set of representatives of the classes $\hat{h} \in \hat{H}_\beta$. Since $\varphi([G_\beta, F_\beta]_*) = P^*_\beta X^*$, then for any $h \in H_\beta$ there exists an element $g_h \in X^\perp$ such that $h = f + g_h$ for some $f \in [G_\beta, F_\beta]_*$. Set $G = \bigcup_{\beta<\alpha} \bigcup_{h \in H_\beta} g_h$. Then $\mathrm{cl}^*(\bigcup_{\beta<\alpha} H_\beta) \subset [G, G_\beta, F_\beta, \beta < \alpha]_*$. By the Banach–Alaoglu theorem, the set $\mathrm{cl}^*(\bigcup_{\beta<\alpha} H_\beta)$ is weakly* compact, and therefore its image $\varphi(\mathrm{cl}^* \bigcup_{\beta<\alpha} H_\beta)$ is weakly* closed and thus contains the ball $aB(P^*_\alpha X^*)$. Therefore, for the set $G_\alpha = G \cup \bigcup_{\beta<\alpha} G_\beta$

$$\varphi([G_\alpha, F_\alpha]_*) = P^*_\alpha X^*.$$

Its cardinality is $\operatorname{card} G_\alpha \leqslant 2 \times \overline{\overline{\alpha}} \times \overline{\overline{\alpha}} = \alpha$. ■

COROLLARY 1. Let E be a Banach space, let α_0 be the first ordinal of cardinality dens* × E*, let X be a subspace of E with projective basis $x_\alpha : 1 \leqslant \alpha < \alpha_0$, and let $f_\alpha \in E^*$ be any functionals biorthogonal to x_α. Then in $X^\perp$ there exists a subset H of cardinality $\leqslant \overline{\overline{\alpha}}_0$, such that $[H, f_\alpha, 1 \leqslant \alpha < \alpha_0]_* = E^*$.

Proof. Let M be a weakly* dense subset of E* of cardinality $\overline{\overline{\alpha}}_0$ and let G be a subset of $X^\perp$ as defined in Lemma 2. For any $m \in M$, we choose an element $h_m \in X^\perp$, for which $m = f + h_m$ for some $f \in [G, f_\alpha : 1 \leqslant \alpha < \alpha_0]_*$. The set $H = G \cup \bigcup_{m \in M} h_m$ satisfies the condition of the corollary. ■

Proof of Theorem 1. The following arguments are fairly standard (see [1], p. 690). Any infinite-dimensional Banach sapce has an (ordinary) basis sequence with basis constant less than $1 + \varepsilon$ [1, p. 49]. If $\alpha_0 > \omega$, then the cardinality of a projective basis in a space of weight α_0 is equal to $\overline{\overline{\alpha}}_0$. Hence, and from Lemma 1, it follows that in the space E there exists a subspace X with projective basis x_α, $1 \leqslant \alpha < \alpha_0$, $\|x_\alpha\| = 1$, whose projective basis constant is $\leqslant 1 + \varepsilon$. Let $h_\alpha \in E^*$ be functionals biorthogonal to x_α with the norm $<(1 + \varepsilon)(2 + \varepsilon) \approx 2 + \varepsilon$, which, as we have already mentioned, exist [1, p. 585]. We enumerate the sequence (x_α, h_α) with the double index $(x^n_\alpha, h^n_\alpha : n = 1, \infty, 1 \leqslant \alpha < \alpha_0)$; for each α, let $\bar{h}_\alpha$ be some weakly* limiting point of the set $(h^n_\alpha)^\infty_{n=1}$.

Clearly, $\bar{h}_\alpha \in X^\perp$ and $\|\bar{h}_\alpha\| \leqslant \sup_n \|h^n_\alpha\| < 2 + \varepsilon$. By Corollary 1, there exists a subset $(g_\alpha : 1 \leqslant \alpha < \alpha_0) \subset (2 + \varepsilon) B(X^\perp)$, including $(\bar{h}_\alpha : 1 \leqslant \alpha < \alpha_0)$, such that $[g_\alpha, h_\alpha : 1 \leqslant \alpha < \alpha_0]_* = E^*$. For any α, set $f_\alpha^n = h_\alpha^n - \bar{h}_\alpha + \varepsilon g_\alpha$. Since $\varepsilon g_\alpha - \bar{h}_\alpha \in X^\perp$, the sequence x_α^n, f_α^n is biorthogonal. The norm $\|f_\alpha^n\| \leqslant 2+\varepsilon+2+\varepsilon+\varepsilon(2+\varepsilon) \approx 4+\varepsilon$. Since 0 is a weakly* limiting point of the sequence $(h_\alpha^n - \bar{h}_\alpha)_{n=1}^\infty$, then εg_α is a weakly* limiting point of the sequence $(f_\alpha^n)_{n=1}^\infty$. Thus $(g_\alpha : 1 \leqslant \alpha < \alpha_0) \subset [f_\alpha^n : n = \overline{1, \infty}, 1 \leqslant \alpha < \alpha_0]_*$, and therefore $h_\alpha^n = f_\alpha^n + \bar{h}_\alpha - \varepsilon g_\alpha \in [f_\alpha^n : n = \overline{1, \infty}, 1 \leqslant \alpha < \alpha_0]_*$ and the system x_α^n, f_α^n is total. Enumerating it again with one index, we obtain the statement of the theorem. ■

2. M-Bases. An M-basis is a total biorthogonal system for which $[x_i : i \in I] = E$.

Proof of Theorem 2. Let the space Y have a weakly* separable conjugate. Denote by U the direct l_1-sum of dens Y copies of the space c_0: $U = \underset{l_1}{\oplus} \Sigma_{i \in I} c_0^i$, $\operatorname{card} I = \operatorname{dens} Y$, where c_0^i is the i-th copy of the space c_0. The conjugate U* consists of sequences $(a_n^i, i \in I, n = \overline{1, \infty})$, for which $\max_i \Sigma_n |a_n^i| < \infty$. Set $X = U \oplus Y$ with the norm $\|u + y\| = \|u\| + \|y\|$, $y \in Y$, $u \in U$. The conjugates U*, Y* are now identified with the annihilators $Y^\perp$ and $U^\perp$ in X*. Let $u_n^i \in U$ be an element with all coordinates zero except the n-th coordinate in the subspace c_0^i, and let (f_n^i) be functionals biorthogonal to (u_n^i). Let y^i, $i \in I$ be a dense subset of the unit sphere of the space Y, and let $(e_m, g_m)_1^\infty$ be a total biorthogonal system in the space Y, bounded by the number 4 + ε, which exists in view of Theorem 1. Set $y_n^i = y^i - \sum_{m=1}^{n} g_m(y^i) e_m$ and $x_n^i = u_n^i + y_n^i$.

For fixed m and each i, $g_m(y_n^i) = 0$ for $n \geqslant m$ and $|g_m(y_n^i)| < 4 + \varepsilon$ for $n < m$. Therefore the element h_m with coordinates $(g_m(y_n^i), i \in I, n = \overline{1, \infty})$ belongs to U*. Setting $g_m' = g_m - h_m$, we show that the set $(x_n^i, f_n^i, i \in I, n = \overline{1, \infty}) \cup (e_m, g_m', m = \overline{1, \infty})$ is an M-basis of the space X.

Since $g_m'(x_n^i) = g_m(y_n^i) - h_m(u_n^i) = 0$, this system is biorthogonal.

The elements $u_n^i + y^i = x_n^i + \sum_{m=1}^{n} g_m(y^i) e_m$ belong to the linear hull $\operatorname{lin}(x_m^i, e_m)$. Since $\left\| \sum_{n=1}^{k} (u_n^i + y^i)/k - y^i \right\| = 1/k \to 0$, then $y^i \in [x_n^i, e_m]_{i,n,m}$, and therefore $u_n^i = x_n^i - y^i \in [x_n^i, e_m]_{i,n,m}$ and $[x_n^i, e_m] = X$.

The annihilator of the set f_n^i is a subspace of Y, and the functionals g_m', which coincide with the functionals g_m on Y, are total on the subspace Y. Thus there exists an M-basis in the space X. ■

The M-basis $x_i, f_i, i \in I$, is called *norming*, if the subspace $F = [f_i : i \in I]$ is norming, i.e., the norm $|||x||| = \sup\{f(x) : f \in F, \|f\| \leqslant 1\}$ is equivalent to the original norm $\| \ \|$ of the space X. If the subspace $\{f \in X^* : \operatorname{supp} f$ is countable$\}$ is norming, where $\operatorname{supp} f = \{i \in I : f(x_i) \neq 0\}$, then the M-basis is called *countably norming*. Any norming M-basis is countably norming. If for any subset $J \subset I$ the subspace $[x_j : j \subset J]$ coincides with the annihilator $(f_i : i \notin J)^\perp$, the M-basis is called *strong*. We say that the norm of the space x is *strictly convex*, if for any linearly independent $x, y \in X$ $\|x + y\| < \|x\| + \|y\|$, and if it follows from the relations, $\|x\| = \|x_n\| = 1$ and $\|x_n + x\| \to 2$ that $\|x_n - x\| \to 0$, then the norm is called *locally uniformly convex*.

THEOREM 3. There exists a Banach space X with the following properties:

1) it has an M-basis and a complementary subspace (isometric to l_∞) without an M-basis;

2) it has no equivalent uniformly convex norm, nor a countably norming or strong M-basis.

Proof. Take the space X constructed in the proof of Theorem 2 with $Y = l_\infty$. Condition 1) is satisfied since l_∞ has no M-basis.

The space l_∞ has no equivalent locally uniformly convex norm [6, p. 93], and therefore neither has X. If the Banach space X has a countable norming M-basis, then for each separable subspace $Z \subset X$ there exists a separable complementary space $Z' \supset Z$ [7]. Take as Z the subspace $c_0 \subset Y$, and construct the corresponding subspace Z'. The restriction of the (separable-valued) projector $X \to Z'$ on Y is identical on Z, and therefore is not weakly compact, which by Grothendieck [8] is impossible.

Suppose that the space X has a strong M-basis x_i, f_i, $i \in I$. Then there exists a countable subset $J \subset I$, such that the closed linear hull $X_J = [x_j : j \in J]$ contains the subspace $c_0 \in Y$. In the factor-space X_J/c_0, choose an M-basis $(\hat{y}_m, g_m)_1^\infty$, $g_m \in c_0^\perp$ and denote by $\hat{x}_i$, $i \notin J$, the images of the corresponding elements under the factor-mapping $X \to X/c_0$. Since $X_J = (f_i : i \notin J)^\perp$, the elements $(\hat{x}_i, f_i : i \notin J) \cup (\hat{y}_m, g_m)_1^\infty$ form an M-basis of the space X/c_0. Therefore

X/c_0 has an equivalent strictly convex norm [1, p. 692]. But its subspace Y/c_0 cannot have an equivalent strictly convex norm [9]. Contradiction. ∎

3. An Enflo–Rosenthal basis is a set x_i, $i \in I$, in a Banach space with $[x_i : i \in I] = X$, of which each countable subset can be ordered so that it becomes a basis in its closed linear hull. There exist functionals $f_i \in X^*$, biorthogonal to x_i, where the set $x_i, f_i, i \in I$, is a bounded M-basis (Singer [1, Sec. 17]). We introduce the ER-basis constant of an Enflo–Rosenthal basis. For any sequence $(i_n)_{n=1}^{\infty} \subset \dot{I}$, let

$$P(i_n) = \sup_k \|P_k\|,$$

where P_k are projectors in the space $[x_{i_n}]_{n=1}^{\infty}$ onto the linear hull of the first k elements parallel to the rest. Let $Q(i_n) = \inf P(i_{\sigma(n)})$, where inf is taken over all permutations $\sigma(n)$ of the natural series. Finally,

$$ER(x_i) = \sup Q(i_n,\ n = 1, \infty),$$

where the supremum is taken over all countable subsets $(i_n,\ n = 1, \infty)$ in $\dot{I}$.

THEOREM 4. The ER-basis constant of an Enflo–Rosenthal basis is finite.

Proof. If not, there would exist a sequence J_k of subsets of $\dot{I}$, for which $Q(J_k) \to \infty$ as $k \to \infty$. Since for $I_1 \subset I_2$ $Q(I_1) \leqslant Q(I_2)$, then $Q(\cup_k J_k) = \infty$. But by the definition of an Enflo–Rosenthal basis the set $x_i : i \in \cup_k J_k$, for some ordering of the indices, forms a (Schauder) basis, i.e., the corresponding projectors are jointly bounded. Contradiction. ∎

LEMMA 3. For any element $x \in S(X)$, there exists a countable subset $J \subset I$, such that

$$d(x,\ [x_i : i \notin J] \geqslant 1/ER(x_i). \tag{1}$$

Proof. First suppose that $x \in \operatorname{lin}(x_i : i \in I)$, i.e., $x = a_1 x_{j_1} + \ldots + a_m x_{j_m}$. If the inequality is not satisfied for this x, this means that there exist a sequence $(i_n)_{n=1}^{\infty} \subset \dot{I}$ and a number $\varepsilon > 0$ such that for any k, $d\left(x, [x_{i_n}]_{n=k}^{\infty}\right) < 1/(ER(x_i) + \varepsilon)$. Then $Q\left(x_{j_1}, \ldots, x_{j_m}, x_{i_1}, \ldots, x_{i_n}, \ldots\right) > ER(x_i) + \varepsilon$, which is impossible. If now the element $x \in S(X)$ is arbitrary and the sequence $y_n \in S(\operatorname{lin}(x_i : i \in I))$ is such that $\|y_n - y\| \to 0$ as $n \to \infty$, then let J_n be a set for which condition (1) is satisfied for the element y_n. Then the union $J = \cup_n J_n$ is the required union.

LEMMA 4. For each countable subset $J_0 \subset I$, there exists a countable subset $J_0 \subset J \subset I$, such that the projector $P : X \to [x_j : j \in J]$, parallel to the subspace $[x_j : j \notin J]$, exists, and its norm is not greater than $ER(x_i)$.

Proof. We construct an increasing sequence of countable subsets $J_n \subset \dot{I}$ such that for any n

$$d(S[x_j : j \in J_n],\ [x_j : j \notin J_{n+1}]) \geqslant 1/ER(x_i).$$

For the set J_n, let $(y_m)_1^{\infty}$ be a dense subset of the sphere $S[x_j : j \in J_n]$. For each y_m, from Lemma 3 there exists a countable set $J^m \subset I$, such that $d(y_m,\ [x_i : i \notin J^{(m)}]) \geqslant 1/ER(x_i)$. Set $J_{n+1} = J_n \cup \cup_m J^m$. The set $J = \bigcup_{n=0}^{\infty} J_n$ satisfies the conditions of the lemma. ∎

THEOREM 5. An Enflo–Rosenthal basis is a norming M-basis.

Proof. Since an Enflo–Rosenthal basis is an M-basis, for any element x there exists a countable subset $J_0 \subset I$ such that $x \in [x_j : j \in J_0]$. By Lemma 4, there exists a countable subset $J_0 \subset J \subset \dot{I}$ such that the norm of the projector $P : X \to [x_j : j \in J]$, parallel to $[x_i : i \notin J]$, is not greater than $ER(x_i)$. We order the set $(x_j : j \in J)$ so that the norms of the projectors P_k: $[x_n]_1^{\infty} \to [x_n]_1^k$ $\left((x_n)_1^{\infty} = (x_j : j \in J)\right)$ are not greater than $ER(x_i)$. Then for the projectors $Q_k = P_k P$ and $y \in Q_k X$,

$$\|y\| = \sup\{f(y) : f \in B(X^*)\} = \sup\{f(Q_k y) : f \in B(X^*)\} = \sup\{Q_k^* f(y) : f \in B(X^*)\} \leqslant \sup\{f(y) : f \in \|Q_k\| B(Q_k^* X^*)\}.$$

Thus $\|x\| \leqslant \sup\{f(x) : f \in (ER(x_i))^2 B(\operatorname{lin}(f_j : j \in J)\}$ and the subspace $[f_i : i \in I] \subset X^*$ is $1/(ER(x_i))^2$ norming.

LEMMA 5. Let X be a Banach space with projective basis $x_\alpha : 1 \leqslant \alpha < \alpha_0$ and biorthogonal functionals f_α.

For any $x \in X$ and $\varepsilon > 0$, there exists a finite number of ordinals $\alpha_1, \ldots, \alpha_n$, such that $\left\| x - \sum_{i=1}^{n} f_{\alpha_i}(x) x_{\alpha_i} \right\| < \varepsilon$.

Proof. For finite ordinals α_0 the statement is obvious. Let P_α be natural projectors in the space X. Suppose that for all spaces with projective basis of length less than α_0, the lemma is proved. If α_0 is not a limiting ordinal, then $x = P_{\alpha_0-1}x + f_{\alpha_0-1}(x) x_{\alpha_0-1}$. Since $(x_\beta : \beta < \alpha_0 - 1)$ is a projective basis in the space $P_{\alpha_0-1}X$, then we choose ordinals $\alpha_1, \ldots, \alpha_n$ such that $\left\| P_{\alpha_0-1}x - \sum_{i=1}^{n} f_{\alpha_i}(P_{\alpha_0-1}x) x_{\alpha_i} \right\| < \varepsilon$. Since $f_{\alpha_i}(P_{\alpha_0-1}x) = f_\alpha(x)$, then $\left\| x - \sum_{i=1}^{n} f_{\alpha_i}(x) x_{\alpha_i} - f_{\alpha_0-1}(x) x_{\alpha_0-1} \right\| < \varepsilon$. If, however, α_0 is a limiting ordinal, we then choose $\beta - \alpha_0$ such that $\|P_\beta x - x\| < \varepsilon/2$, and for the element $P_\beta x$, by the inductive hypothesis there exists ordinals $\alpha_1, \ldots, \alpha_n < \beta$ for which $\left\| \sum_{i=1}^{n} f_{\alpha_i}(P_\beta x) x_{\alpha_i} - P_\beta x \right\| < \varepsilon/2$. Then $\left\| x - \sum_{i=1}^{n} f_{\alpha_i}(x) x_{\alpha_i} \right\| < \varepsilon$. ■

THEOREM 6. Let X be a space with projective basis x_α and functionals f_α biorthogonal to it. For any element $x \in X$, there exists a sequence of ordinals β_i, $i = 1, \infty$, such that $\left\| x - \sum_{i=1}^{n} f_{\beta_i}(x) x_{\beta_i} \right\| \to 0$ as $n \to \infty$.

Proof. Let the sequence of numbers $\varepsilon_k \to 0$, $\varepsilon_k > 0$. By Lemma 5 we can choose ordinals $(\beta_i)_1^{n_1}$ such that $\|x_1\| < \varepsilon_1$, where $x_1 = x - \sum_{i=1}^{n_1} f_{\beta_i}(x) x_{\beta_i}$. We then choose ordinals $(\beta_i)_{n_1+1}^{n_2}$ such that $\|x_2\| < \varepsilon_2$, where $x_2 = x - \sum_{i=1}^{n_2} f_{\beta_i}(x) x_{\beta_i}$, and so on. As a result we obtain a sequence of ordinals β_i and an increasing sequence of numbers n_k, such that

$$\|x_k\| < \varepsilon_k,$$

where $x_k = x - \sum_{i=1}^{n_k} f_{\beta_i}(x) x_{\beta_i}$. We shall assume that each set $(\beta_i,\ i = n_k + 1,\ n_{k+1})$ is ordered by the increase of ordinals, i.e., $\beta_i < \beta_j$ if $i < j$. Let c be the projective constant of the projective basis x_β and let $n_k < j < n_{k+1}$. Then $\left\| \sum_{i=n_k+1}^{j} f_{\beta_i}(x) x_{\beta_i} \right\| \leqslant c \|x_k - x_{k+1}\|$, and therefore

$$\left\| x - \sum_{i=1}^{j} f_{\beta_i}(x) x_{\beta_i} \right\| = \left\| x_k + \sum_{i=n_k+1}^{j} f_{\beta_i}(x) x_{\beta_i} \right\| \leqslant \|x_k\| + c\|x_k - x_{k+1}\| \leqslant \varepsilon_k + c(\varepsilon_k + \varepsilon_{k+1}). \ ■$$

Denote by $C[\alpha]$ the space of order continuous functions on the interval of the ordinals $[1, \alpha]$. The elements $x_\beta(\xi) = 1$ for $\xi \leqslant \beta$, and $x_\beta(\xi) = 0$ for $\xi > \beta$, form a natural projective basis there with biorthogonal functionals $f_\beta(x) = x(\beta) - x(\beta + 1)$ (see [1, p. 590]).

LEMMA 6. In the (separable) space $C[\omega^2]$, the elements (x_β) do not form a basis for any permutation of the indices.

Proof. This fact can be deduced, for example, from the fact that functionals biorthogonal to (x_β) span a nonnorming subspace [10]. For the convenience of the reader, we shall give a direct proof. Let $(x_n)_1^\infty$ be some permutation of the elements x_β, $1 \leqslant \beta \leqslant \omega^2$. Choose from this elements $(x_{n_i})_{i=1}^{k}$ such that $x_{n_i} \in (x_{\beta_i} : i\omega \leqslant \beta < (i+1)\omega)$. From the set $(x_n : n > n_k)$, choose elements x_{m_i}, $i = 1, k$ such that $x_{m_i} \in (x_\beta : \beta_i \leqslant \beta < (i+1)w)$, where β_i is an ordinal such that $x_{\beta_i} = x_{n_i}$. Then $\left\| \sum_{i=1}^{k} x_{n_i} \right\| = k$, and $\left\| \sum_{i=1}^{k} x_{n_i} - \sum_{i=1}^{k} x_{m_i} \right\| = 1$, which contradicts the criterion that the basis be projective.

From Theorem 6 and Lemma 6, we have:

COROLLARY 2. For any $x \in C[\omega^2]$ there exists a permutation x_{β_n} of the natural projective basis x_β, such that $x = \sum_{n=1}^{\infty} a_n x_{\beta_n}$, but x_β does not form a basis for any permutation of the indices.

4. Borel Components. THEOREM 7. Let X be an infinite-dimensional Banach space with Hamel basis $(x_\gamma^\cdot : \gamma \in \Gamma)$, and let $e_\gamma^\cdot : \gamma \in \Gamma$ be the natural unit vectors in the space $l_1(\Gamma)$. The linear mapping $A(\Sigma a_\gamma^\cdot x_\gamma^\cdot) = \Sigma a_\gamma^\cdot e_\gamma$ from X into $l_1(\Gamma)$ has the following properties:

1) AX is a nonclosed Borel subset of the space $l_1(\Gamma)$, with closed complement there;

2) the graph G(A) is a Borel subset of the product $X \times l_1(\Gamma)$;

3) $A^{-1}|_{AX}$ is continuous;

4) A is injective and discontinuous.

Proof. The nonclosure of the subspace AX, the continuity of the restriction $A^{-1}|_{AX}$, and the injectivity and discontinuity of the operator A, are obvious. The kernel Z_0 of a continuous extension of the operator $A^{-1}|_{AX}$ is a closed complement of AX. The subset $Z_n = \{z \in l_1(\Gamma) : \mathrm{card\,supp}\, z \leqslant n\} \subset l_1(\Gamma)$ is closed, and therefore $AX = \cup_n Z_n$ is Borel. The graph G(A) = $G(A^{-1})$ of the continuous operator $A^{-1}|_{AX}$ is a closed subset of the product X × AX, which is a Borel subset of X × Y. Therefore it is also Borel. ■

Remark. If as the space X we take, for example, l_2, then the sunspace Z_0 does not have any closed complement in $l_1(\Gamma)$, although it has a Borel complement. This follows from the fact that $l_1(\Gamma)$ does not contain an infinite-dimensional subspace isomorphic to a Hilbert space.

THEOREM 8. There exist a Banach space Y and a linear injective discontinuous operator from Y to $l_1(\Gamma)$, with Borel graph.

Proof. Let A, X, Z_0, and Z_n be as in Theorem 7 and let Y_0 be some space isometric to Z_0 with the isometry $\dot{I}: Y_0 \to Z_0$. Set $Y = X + Y_0$ with the norm $\|y\| = \|x\| + \|y_0\|$, $y = x + y_0$, $x \in X$, $y_0 \in Y_0$. Define a mapping $B: Y \to l_1(\Gamma)$ by the equation

$$B(x + y_0) = Ax + \dot{I}y_0.$$

For the proof we need only to show that the graph G(B) is Borel. Clearly, $G(B) = \bigcup_{n=1}^{\infty} G_n$, where $G_n = \{(x + y_0,\ Ax + \dot{I}y_0) : Ax \in Z_n\}$. Let $\left\|(x^k + y_0^k,\ Ax^k + \dot{I}y_0^k) - (y, z)\right\| \to 0$ for $k \to \infty$, $(x^k + y_0^k,\ Ax^k + \dot{I}y_0^k) \in G_n$. This means that $\|x^k + y_0^k - y\| \to 0$ and $\|Ax^k + \dot{I}y_0^k - z\| \to 0$. Since X and Y_0 are closed complementary subspaces of the Banach space Y, x^k tends to some $x \in X$, where y_0^k tends to some $y_0 \in Y_0$, where $y = x + y_0$. Then $\dot{I}y_0^k \to \dot{I}y_0$ and $Ax^k \to z - \dot{I}y_0$. Since the subset $Z_n \subset l_1(\Gamma)$ is closed, $z - \dot{I}y_0 \in Z_n$. The operator A is continuous, and therefore $Ax = z - \dot{I}y_0$. Thus, $B(y) = B(x + y_0) = Ax + \dot{I}y_0 = z - \dot{I}y_0 + \dot{I}y_0 = z$, where $z - \dot{I}y_0 \in Z_n$. We have proved that the sets G_n are closed, and therefore the graph G(B) is Borel. ■

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