NORMALIZING AND STRONGLY NORMALIZING SUBSPACES OF A CONJUGATE BANACH SPACE

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Let X be a Banach space. The subspace F of the conjugate space X* is called normalizing, if its Diksmay characteristic $r(F) = \sup\{r : rB(X^*) \subset \mathrm{cl}^* B(F)\} > 0$, where B(X) is the unit sphere in the space X, and cl* A is the weak* closure of the subset $A \subset X$ [1 (see [1, p. 29]). It is known (see [1, p. 39]) that a numerical region of characteristic $\Re(X) = \{\lambda : \exists F \subset X^*, r(F) = \lambda\}$, where F is a total proper subspace of X*, closed in norm, is an interval $(0, a)$, whose ends may or may not be included. In this article we study the relationship between the numerical region of characteristic of the space X and its subspace Y, the possible values of the number a, and we also introduce the concept of a strongly normalizing subspace, and we show that for improperly defined problems on linear finite-dimensional Tikhonov regularizability (and its equivalent functional-analytic properties), this concept fills the role of a normalizing subspace for simple regularizability.

1. The numerical region of characteristic was studied in [1-4]. In particular, it follows from the result of [4] that for a space X with unconditional orthogonal spanning basis, we always have $\Re(Y) \subset \Re(X)$ for any subspace $Y \subset X$. In connection with this, the following question naturally arises: do we always have $\Re(Y) \subset \Re(X)$, if Y is a subspace of X. The following statement shows that when Y is well complemented in X, the conclusion holds.

THEOREM 1. Let Y and Z be closed, mutually complementary subspaces of the Banach space $X : X = Y \oplus Z$, and let the norm of the sum have the "lattice property": if $\|y_1\| \leqslant \|y_2\|$ and $\|z_1\| \leqslant \|z_2\|$ then $\|y_1 + z_1\| \leqslant \|y_2 + z_2\|$ for arbitrary $y_i \in Y,\ z_i \in Z,\ i = 1, 2$. Then $\Re(Y) \subset \Re(X)$ and $\Re(Z) \subset \Re(X)$.

Proof. It is sufficient to verify the inclusion $\Re(Y) \subset \Re(X)$. It follows from the conditions of the theorem that the norms of the natural projectors X → Y and X → Z are equal to one. It is known (see, for example, [5]), and is easily verified, that in this case X* = $Z^\perp \oplus Y^\perp$, where $Z^\perp$ and $Y^\perp$ are the annihilators in X* of the corresponding subspaces, and moreover $Z^\perp$ is isometric to Y*, and $Y^\perp$, to Z*; the isometry is defined in the natural way. Therefore we shall identify Y* with $Z^\perp$, and Z* with $Y^\perp$. It is also easily verified that if the norm of the sum $Y \oplus Z$ has the "lattice property," then the same is also true for the annihilators: for arbitrary $f_i \in Z^\perp$ and $g_i \in Y^\perp$, i = 1, 2, it follows from $\|f_1\| \leqslant \|f_2\|$ and $\|g_1\| \leqslant \|g_2\|$ that $\|f_1 + g_1\| \leqslant \|f_2 + g_2\|$.

Let M be a subspace of Y* = $Z^\perp$ of characteristic a. We show that the characteristic of the subspace $F = M \oplus Y^\perp \subset X^*$ is not less than a. Take an arbitrary element $u = f + g \in X^*$, $f \in Z^\perp, g \in Y^\perp$, with norm $\|u\| = a$. Then $\|f\| \leqslant a$, and by the definition of characteristic there exists a net $\{f_\alpha\}_{\alpha \in A} \subset \|f\| B(M)/a$, which converges weakly* to f. Then the net $\{f_\alpha + g\}_{\alpha \in A} \subset F$ weakly* converges to u. Since $\|f_\alpha + g\| \leqslant \| \|f\| a^{-1} f + g\| \leqslant \|a^{-1} f + a^{-1} g\| = 1$, then $r(F) \geqslant a$. The inclusion $\Re(Y) \subset \Re(X)$ is proved.

We introduce an example of a Banach space X, for which there exists a subspace Y such that $\Re(Y) \not\subset \Re(X)$. Moreover Y has a one-dimensional complement Z and the norms of the natural projectors X → Y and X → Z are equal to 1.

Example 1. For any set M, the notation $c_0(M)$, $l_1(M)$, $l_\infty(M)$ has the standard sense (see [1, p. 7]). Denote by X the subspace $l_\infty(M)$, where $M = \{-1\} \cup \{0\} \cup N$, where N is the natural numbers, spanned on $c_0(N)$, and consider the elements e_0: $e_0(m) = 1,\ m \in M$ and $e_{-1} : e_{-1}(-1) = 1,\ e_{-1}(0) = -1,\ e_{-1}(n) = 0,\ n \in N$. We show that X* = $l_1(M)$. Duality is defined by the formula $f(x) = \Sigma_{-1}^{\infty} f(m)\, x(m),\ f \in l_1(M),\ x \in X$. Since $|f(x)| \leqslant \|x\| \Sigma_{-1}^{\infty} |f(m)|$, then $\|f\|_{X^*} \leqslant \Sigma_{-1}^{\infty} |f(m)|$. For each $\varepsilon > 0$, it is easy to select an element $x \in B(X)$, for which $f(x) > \Sigma_{-1}^{\infty} |f(m)| - \varepsilon$. Therefore

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$X^* \supset l_1(M)$. Since $c_0(N)^* = l_1(N)$, and $c_0(N)$ and $l_1(N)$ differ from X and $l_1(M)$ by precisely two dimensions, then $X^* = l_1(M)$.

The subspace $Y = c_0(N) + [e_0] \subset X$ is isometric to the space of convergent sequences c ([A] denotes the norm-closed linear hull of the set A), and therefore $\mathfrak{R}(Y) = [0, 1]$ (see [2]). Moreover, the subspace Y has in X complement $Z = [e_{-1}]$, and the norms of the corresponding projectors are equal to one.

We show that $\mathfrak{R}(X) \subset [0, 2/3]$. To do this we note that $l_1(M)^* = l_\infty(M)$, and for one of the equivalent definitions of the characteristic [1, p. 30] it is sufficient to show that for an arbitrary element $\varphi \in l_\infty(M)$

$$\inf\{\|\lambda\varphi - x\| : x \in X, \|x\| = 1, \lambda \in \mathbb{R}\} \leqslant 2/3. \tag{1}$$

As usual we identify the space X with its canonical image in the second conjugate.

Take any element $\varphi \in l_\infty(M)$ with norm 2/3. If $\sup\{|\varphi(n)| : n \in N\} \geqslant 1/3$, then for any $\varepsilon > 0$ there exists $k \geqslant 1$ such that $|\varphi(k)| > 1/3 - \varepsilon$. Let $e_k \in X$ be a unit vector: $e_k(k) = 1$, $e_k(m) = 0$, $m \neq k$. Then $\|e_k\| = 1$, $\|\varphi - \operatorname{sign}\varphi(k) e_k\| \leqslant 2/3 + \varepsilon$ and for such a φ, inequality (1) is satisfied. If, however, $|\varphi(n)| < 1/3, n \geqslant 0$, then either $|\varphi(-1)| = 2/3$ or $|\varphi(0)| = 2/3$. The space X contains the elements $x = (e_0 + e_{-1})/2$ and $y = (e_0 - e_{-1})/2$, $\|x\| = \|y\| = 1$. If $|\varphi(-1)| = 2/3$, then $\|2^{-1}\varphi - \operatorname{sign}\varphi(1) \times x\| \leqslant 2/3$, i.e., inequality (1) is also satisfied in this case.

<u>THEOREM 2.</u> Let $(X, \|\cdot\|)$ be a Banach space and let $R(X, \|\cdot\|) = \sup\{\lambda : \lambda \in \mathfrak{R}(X)\} = a > 0$. Then for an arbitrary number $b \in [a, 1]$, there exists a norm $|||\cdot|||$ on X, equivalent to the original norm, for which $R(X, |||\cdot|||) = b$.

<u>Proof.</u> Let $F \subset (X, \|\cdot\|)^*$ be any proper normalizing subspace, one of which must exist by the conditions of the theorem. Then the norm $\|x\|_1 = \sup\{|f(x)| : f \in F, \|f\| \leqslant 1\}$ is equivalent to the original norm and $R(X, \|\cdot\|_1) = 1$. For any $\lambda \in [0, 1]$ we define a norm on the space X: $\|x\|_\lambda = \lambda\|x\|_1 + (1 - \lambda)\|x\|$. Clearly, it is equivalent to the original norm, and the Banach–Mazur distance $d((X, \|\cdot\|_\mu), (X, \|\cdot\|_\lambda)) \to 0$ as $\mu \to \lambda$. Since for arbitrary Banach spaces X and Y $R(X) \leqslant d(X, Y)R(Y)$ (see [3]), the mapping φ, which associates with each $\lambda \in [0, 1]$ a number $R(X, \|\cdot\|_\lambda)$, is continuous, and $\varphi(0) = a$, $\varphi(1) = 1$. Since a continuous function takes all its intermediate values, the theorem is proved.

While the conditions under which the left boundary of the numerical region $\mathfrak{R}(X)$ is contained in $\mathfrak{R}(X)$ are fully described (see [1, p. 78]), it is not clear when $R(X) \in \mathfrak{R}(X)$. We introduce an example showing that for an arbitrary number $1/2 \leqslant a \leqslant 1$ there exists a Banach space X with $\mathfrak{R}(X) = [0, a]$.

<u>Example 2.</u> Let $0 \leqslant \alpha \leqslant 1$. Consider the subspace $c_\alpha = c_0(N) + [e_\alpha] \subset l_\infty(N \cup \{0\})$, where e_α is defined thus: $e_\alpha(0) = 1$, $e_\alpha(n) = \alpha$, $n \geqslant 1$, and we show that $\mathfrak{R}(c_\alpha) = [0, (1 + \alpha^2)/2]$.

We note that $c_\alpha^* = l_1(N \cup \{0\})$ with the natural duality, and $c_\alpha^{**} = l_\infty(N \cup \{0\})$. We first prove the inequality $R(c_\alpha) \leqslant (1 + \alpha^2)/2$, i.e., for an arbitrary element $\varphi \in l_\infty(N \cup \{0\})$ we verify the relation

$$\inf\{\|\lambda\varphi - x\| : x \in c_\alpha, \|x\| = 1, \lambda \in \mathbb{R}\} \leqslant (1 + \alpha^2)/2. \tag{2}$$

If for some index $k > 0$ $|\varphi(k)| \geqslant (1 - \alpha^2)/(1 + \alpha^2)$, then $\|2^{-1}(1 + \alpha^2)\varphi - \operatorname{sign}\varphi(k) e_k\| \leqslant \max\{(1 + \alpha^2)/2, 1 - (1 + \alpha^2)(1 - \alpha^2)/2(1 + \alpha^2)\} = (1 + \alpha^2)/2$, where e_k is the k-th unit vector of the space $c_0(N)$. Let $|\varphi(n)| < (1 - \alpha^2)/(1 + \alpha^2)$ for any $n > 0$. Then $|\varphi(0)| = 1$; we may assume that $\varphi(0) = -1$. In the case $\overline{\lim}_n |\varphi(n)| = a < (1 - \alpha)/(1 + \alpha)$, i.e., when $|\varphi(n)| > (1 - \alpha)/(1 + \alpha)$ for only a finite number of positive indices $n_1, \ldots, n_m$, the norm of the element $x = e_\alpha - \sum_{i=1}^{m} 2^{-1}(1 - \alpha^2)\varphi(n_i) e_{n_i}$ is equal to 1, and considering separately the values of the function $2^{-1}(1 - \alpha^2)\varphi + x$ at the points 0, n_i, $i = \overline{1, m}$ and at the remaining points, we obtain $\|2^{-1}(1 - \alpha^2)\varphi + x\| \leqslant (1 + \alpha^2)/2$.

Finally, when $a > (1 - \alpha)/(1 + \alpha)$, we choose an arbitrary ε and index $k > 0$ such that $a - \varepsilon < |\varphi(k)| < a + \varepsilon$. Let $n, \ldots, n_m$ be a finite number of positive indices for which $|\varphi(n_i)| > a + \varepsilon$. Set $\lambda = (1 + \alpha)/(2a + \alpha(1 + a))$, $b = (1 - a)/(2a + \alpha(1 + a))$. Then the norm of the element $y = be_\alpha - (b\alpha + \operatorname{sign}\varphi(k)) e_k - \sum_{i=1}^{m} \lambda\varphi(n_i) e_{n_i}$ is equal to 1, and considering separately the values of the function $\lambda\varphi + y$ at the point 0, k, n_i, $i = \overline{1, m}$ and at the other points, we obtain $\|\lambda\varphi - y\| \leqslant (\alpha + a)/(2a + \alpha(1 + a)) + \varepsilon(1 + \alpha)/(2a + \alpha(1 + a)) \leqslant (1 + \alpha)^2/2 + \varepsilon(1 + \alpha)^2/2$. Inequality (2) follows from the fact that ε was chosen arbitrarily.

We show that for the element $\varphi\in c_\alpha^{**}:\varphi(0)=-1, \varphi(n)=(1-\alpha)/(1+\alpha)$, $n > 0$, the characteristic of the subspace $\varphi^\perp\subset c_\alpha^*$ is exactly equal to $(1+\alpha^2)/2$, i.e., for any $x\in c_\alpha$, $\|x\|=1$ and $\lambda>0$ $\|\lambda\varphi+x\|\geqslant(1+\alpha^2)/2$. Suppose that for some x and λ

$$\|\lambda\varphi+x\|<(1+\alpha^2)/2. \quad (3)$$

If $|x(k)| = 1$ for some k, then since $\lambda\varphi(k)<0$ it makes sense to consider only the case $x(k) = -1$. By inequality (3) $-\lambda\varphi(k)-x(k)<(1+\alpha^2)/2$; therefore, $\lambda>(1+\alpha^2)/2$. Substituting this estimate in the relation $-\lambda\varphi(0)-x(0)<(1+\alpha^2)/2$, we obtain $x(0) > \alpha$. Substitute the values for $x(0)$ and λ in the inequality $\overline{\lim}_n(\lambda\varphi(n)+x(n))>(1+\alpha^2)/2$; we obtain the contradictory relation $(1+\alpha^2)(1-\alpha)/2(1+\alpha)+\alpha\alpha<(1+\alpha^2)/2$. If, however, $|x(n)| < 1$ for all $n > 0$, then $|x(0)| = 1$; it makes sense to consider only the case $x(0) = 1$. By (3), $\lambda\varphi(0)+x(0)<(1+\alpha^2)/2$; therefore $\lambda>(1-\alpha^2)/2$. Substituting this estimate for λ in the inequality $\lim_n(\lambda\varphi(n)+x(n))<(1+\alpha^2)/2$, we again obtain a contradiction. Therefore $r(\varphi^\perp)=(1+\alpha^2)/2$, and then $(0, (1+\alpha^2)/2]\subset\Re(c_\alpha)$ (see [1], p. 39). It is easily verified that the space c_α is isomorphic to c_0 (the isomorphism is established in the same way as between c and c_0). Therefore c_α is nonquasireflexive and contains a total subspace of zero characteristic (see [1, p. 78]); therefore $\Re(c_\alpha)=[0, (1+\alpha^2)/2]$.

2. Let E be a Banach space and let M be a total linear subspace of E*. We say that it is strongly λ-normalizing ($0 < \lambda \leqslant 1$) if for any finite-dimensional subspaces $X\subset E$ and $F\subset E^*$ $\inf\max\{\|T\|, \|T^{-1}\|\}\leqslant\lambda^{-1}$, where the infimum is taken over all operators $T:F\to M$, for which $\langle x, f\rangle = \langle x, Tf\rangle$ for any $x\in X, f\in F$, and if λ is the maximal of those numbers for which this inequality is satisfied. We call the number λ the strong characteristic sr(M) of the subspace M. In the case when the value of the constant λ is not important, we shall use the terminology "strongly normalizing subspace."

This definition is evoked by the principle of local reflexivity [6], which states that $E\subset(E^*)^*$ is strongly 1-normalizing. Clearly, if the subspace $N\subset M$, then $sr(N) \leqslant sr(M)$ and $sr(M) \leqslant r(M)$ [r(M) is the usual characteristic of the subspace M]. It is also easily verified that $sr(\overline{M}) = sr(M)$ for the closure $\overline{M}$ in norm of the subspace M.

THEOREM 3. Let E be a Banach space, let M be a strongly normalizing subspace of E*, let $a > 0$ and let J be a factor-mapping of E** into $E^{**}/M^\perp = M^*$. Let N be a subspace of M with the following property: for any finite-dimensional subspace $Z\subset JE$, there exist norm-closed subspaces $\Phi\supset Z$ and $\Psi\supset N^{(\perp M)}$ in M*, and a projector $P:M^*\to\Phi$ parallel to Ψ with $\|P\|\leqslant a$, where $N^{(\perp M)}$ is the annihilator in M*. Then N is strongly normalizing.

Proof. Since under norm closure, strong normalizability is not affected, then without loss of generality we may assume that the subspaces M and N are norm-closed. Let X and F be finite-dimensional subspaces of E and E*, let $\varepsilon > 0$ and let $Y\supset X$ be a finite-dimensional subspace of E such that for each $f\in F$ $\|f\|<(1+\varepsilon)\sup\{f(y):y\in B(Y)\}$.

Let $T:F\to M$ be an operator with $\langle y, f\rangle = \langle y, Tf\rangle$ for arbitrary $y\in Y$ and $f\in F$, and let $\max\{\|T\|, \|T^{-1}\|\}\leqslant\lambda^{-1}$, $\lambda>\mathrm{sr}M-\varepsilon$. Let $Z = JY$ and let Φ, Ψ and P be the objects mentioned in the conditions of the theorem. The conjugate operator P* maps M** into $\Psi^\perp\subset(N^{(\perp M)})^\perp$, parallel to $\Phi^\perp$. Identifying $(N^{(\perp M)})^\perp$ with N**, we may assume that P* maps M** into N**. Then for $y\in Y$ and $f\in F$, denoting by J' the factor-mapping of M* into $M^*/N^{(\perp M)}=N^*$, we have

$$\langle y, f\rangle=\langle y, Tf\rangle=\langle Jy, Tf\rangle=\langle Jy, P^*Tf\rangle=\langle J'Jy, P^*Tf\rangle. \quad (4)$$

By the condition $\|P^*\| = \|P\| \leqslant a$. For the restriction $P^*|_{TF}$ we estimate the norm of the inverse. Let $g\in TF, \|g\|=1, f=T^{-1}g$ and let $y\in B(Y)$ be an element for which $\|f\|<(1+\varepsilon)f(y)$. Then $\|Jy\|\leqslant 1$ and $\langle P^*g, Jy\rangle=\langle Tf, PJy\rangle=\langle Tf, Jy\rangle=\lambda/(1+\varepsilon)$, since $\|(P^*|_{TF})^{-1}\|<\lambda^{-1}(1+\varepsilon)$. Applying the principle of local reflexivity to the space N and the subspaces $G=P^*TF\subset N^{**}$ and $J'Z\subset N^*$, we obtain a subspace $H\subset N$ and an operator $R:G\to H$, for which $\max\{\|R\|, \|R^{-1}\|\}<1+\varepsilon$ and $\langle z, g\rangle = \langle z, Rg\rangle$ for $z\in J'Z$ and $g\in G$. Set $S = RP^*T$. It is easily seen that this operator maps F into N. Moreover, for $x\in X\subset Y$ and $f\in F$, by (4) we have $\langle x, f\rangle = \langle J'Jx, P^*Tf\rangle = \langle JJ'x, RP^*Tf\rangle = \langle x, Sf\rangle$.

Finally, there exist constants bounding R, $P^*|_{TF}$ and their inverses, which do not depend on the subspaces X and F. Therefore there exists a number μ such that $\max\{\|S\|, \|S^{-1}\|\}<\mu$, i.e., the subspace $N\subset E^*$ is strongly normalizing.

COROLLARY 1. If M is a strongly normalizing subspace of E* and $N \subset M$ is a total subspace on E with $\dim M/N < \infty$, then N is strongly normalizing. In particular, if N is a total subspace of finite defect in E*, then N is also strongly normalizing.

In fact, since $\dim M/N < \infty$, then the annihilator $N^{(\perp M)} \subset M^*$ is finite-dimensional. From the totality of N we have $N^{(\perp M)} \cap JE = 0$. Moreover, the restriction $J|_E$ is an isomorphism, and therefore the subspace $JE \subset M^*$ is closed. Therefore it can be extended to a closed subspace Φ, complementary to $N^{(\perp M)}$, and we can apply Theorem 3 with $\Psi = N^{(\perp M)}$.

We recall that the Banach space E is called quasireflexive if $\dim E^{**}/E < \infty$.

COROLLARY 2. The Banach space E is quasireflexive if and only if any total subspace $M \subset E^*$ is strongly normalizing.

For a nonquasireflexive space, there exists a total normalizing subspace $M \subset E^*$ (see [1, p. 78]), which of course is not strongly normalizing. Since each norm-closed total subspace of M, conjugate to the quasireflexive E, has finite defect in E*, then Corollary 2 follows from Corollary 1.

The Banach space E is called a $\mathfrak{L}_\infty$ -space, if there exists a number $\lambda > 0$ such that for any finite-dimensional subspace $X \subset E$, there exists a finite-dimensional subspace $Y \supset X$ and a linear bijective operator $T: Y \to l_\infty^{(n)}$ with norm $\max\{\|T\|, \|T^{-1}\|\} < \lambda$.

COROLLARY 3. Let E be a $\mathfrak{L}_\infty$ -space. Then any normalizing subspace $M \subset E^*$ is strongly normalizing.

Proof. Let $X \subset E$ be a finite-dimensional subspace and let Y, λ and T be the objects in the definition of a $\mathfrak{L}$ -space. Since M is normalizing, then in the space $E + M^\perp \subset E^{**}$ there exists a continuous projector $q: E + M^\perp \to E$ parallel to $M^\perp$. Consider the mapping TQ: $E + M^\perp \to l_\infty^{(n)}$. By p. 247 of [7], this can be extended to an operator $S: E^{**} \to l_\infty^{(n)}$, preserving the norm. Set $P = T^{-1}S$. The operator P projects the space E** onto $Y \supset X$ parallel to $\operatorname{Ker} P \supset M$; thus we can apply Theorem 3, setting M = E*, N = M, Φ = Y, Ψ = Ker P.

We recall that the Banach space E has the λ-metric approximation property (the λ-MAP), if for any finite-dimensional subspace $X \subset E$ and any $\varepsilon > 0$, there exists a finite-dimensional linear operator $R_{X,\varepsilon}: E \to E$ with $\|R_{X,\varepsilon}\| \leqslant \lambda$ and $\|R_{X,\varepsilon}x - x\| \leqslant \varepsilon\|x\|$ for $x \in X$. If the space X has the λ-MAP for some $\lambda \geqslant 1$, then we say that it has the bounded approximation property (the BAP). We call the set $\mathfrak{R} = \{R_{X,\varepsilon}\}$ the set of λ-approximating operators. For a given E, there may of course be many such sets $\mathfrak{R}$. Set $M_\mathfrak{R} = \operatorname{lin}\{R^*E^*: R \subset \mathfrak{R}\}$.

THEOREM 4. Let the Banach space E have the λ-MAP. Then:

1) $\operatorname{sr}(M_\mathfrak{R}) \geqslant \lambda^{-1}$;

2) if for the linear subspace $M \subset E^*, \operatorname{sr}(M) > \mu^{-1}$, then in the space E there exists a set $\mathfrak{R}_1$ of $\lambda\mu$-approximating operators, such that $M_{\mathfrak{R}_1} \subset M$.

Proof. 1) Let $X \subset E$ and $F \subset E^*$ be finite-dimensional subspaces and let ε, $\delta > 0$. Choose a finite-dimensional subspace $X \subset Y \subset E$, such that for any $f \in F$

$$\|f\| < (1+\delta)\sup\{f(y): y \in B(Y)\}. \tag{5}$$

Let $R_{Y,\varepsilon} \in \mathfrak{R}$ be an approximating operator corresponding to the subspace Y and the number ε. By Lemma 2.4 of [6], we can choose an operator R_1 with $R_1|_Y = I|_Y$, $\|R_1 - R\| < (1-\varepsilon)^{-1}\varepsilon\lambda\dim Y$ and $R_1^*E^* = R^*E^* \subset M_\mathfrak{R}$. Then for $y \in Y$ and $f \in F$, $\langle y, R_1^*f\rangle = \langle R_1y, f\rangle = \langle y, f\rangle$. Moreover, $\|R_1\| = \|R_1\| \leqslant \|R - R_1\| + \|R\| \leqslant (1-\varepsilon)^{-1}\varepsilon\lambda\dim Y + \lambda$. For $f \in F$, $\|R_1^*f\| = \sup\{\langle e, R_1^*f\rangle: e \in B(E)\} = \sup\{\langle R_1e, f\rangle: e \in B(E)\} \geqslant \sup\{\langle R_1y, f\rangle: y \in B(Y)\} = \sup\{f(y): y \in B(Y)\} \geqslant \|f\|/(1+\delta)$. In the last inequality we are using formula (5). Thus $\|f\| < (1+\delta)\|R_1^*f\|$, i.e., the norm of the restriction $\|(R_1^*|_F)^{-1}\| < 1+\delta$. Inequality 1) follows from the fact that ε and δ were chosen arbitrarily.

2) Let $\mathfrak{R}$ be the family of λ-approximating operators, let X be a finite-dimensional subspace of E, let $\varepsilon > 0$, let $R_{X,\varepsilon} \in \mathfrak{R}$ be the operator corresponding to them, and let S be the operator associated with the mapping R by Lemma 2.4 of [6]. Let $T: S^*E^* \to M$ be an operator for which $\langle x, Tf\rangle = \langle x, f\rangle$ for $x \in X$, $f \in S^*E^*$ and $\|T\| < \mu$. Set $T_1 = TS^*$. Then T_1 is a finite-dimensional operator mapping E* into M, where M, $\|T_1\| \leqslant \|T\|\|S^*\| \leqslant \mu[(1-\varepsilon)^{-1}\lambda\varepsilon\dim X + \lambda]$ and for $x \in X$ and $f \in E^*$, $\langle T_1^*x, f\rangle = \langle x, T_1f\rangle = \langle x, TS^*f\rangle = \langle x, S^*f\rangle = \langle Sx, f\rangle = \langle x, f\rangle$. Therefore $T_1^*x = x$ (E is considered as a subspace of E**). By Lemma 3.1 of [6], there exists a weakly* continuous

operator $S_1:E^* \to T_1E$ such that a) $\|f_1\| \leqslant \|T_1\|(1+\varepsilon)$ and b) $S_1^*y = T_1^*y$, only if $T_1y \in E$. The operator R_1 into E, to which S_1 is conjugate, is bounded by the expression $\mu[(1-\varepsilon)^{-1}\lambda\varepsilon \dim X + \lambda](1+\varepsilon)$, $R_1x = x$ and $R_1^*E^* \subset M$. The number ε may be chosen sufficiently small so that $\|R_1\| < \lambda\mu$.

In the case of a separable space E, the set of approximating operators $\mathfrak{R}$ becomes a sequence R_n, pointwise converging to the unit operator I. We recall that for each of the equivalent definitions [8], the operator inverse to the linear continuous injective operator $A:X \to Y$, where X and Y are normed spaces, is called linearly finite-dimensionally regularizable, if there exists a sequence of linear continuous finite-dimensional operators $B_n:Y \to X$, for which for any $y \in AX$, $\|B_ny - A^{-1}y\| \to 0$ as $n \to \infty$.

THEOREM 5. Let A be a linear continuous injective operator from the separable Banach space X into the normed space Y. The following statements are equivalent:

1) A^{-1} is linearly finite-dimensionally regularizable;

2) X has the BAP, and for the subspace $M = A^*Y^* \subset X^*$, $\mathrm{sr}(M) > 0$.

Proof. If B_n are the operators approximating A^{-1}, then the operators B_nA approximate the unit operator, and therefore are bounded in total, so that $(B_nA)^* \subset M$. Then by Theorem 4 sr(M) > 0; therefore we have established the implication 1) ⇒ 2). It follows from 2) and the first part of Theorem 4 that there exists a sequence of linear continuous finite-dimensional operators $R_n:X \to X$, converging to I, for which $R_n^*X^* \subset M$. This condition ensures that A^{-1} is linearly finite-dimensionally regularizable [8].

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