REFLEXIVITY AND QUASIREFLEXIVITY OF TOPOLOGICAL VECTOR SPACES

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Suppose E is a separated locally convex topological vector space, E' its dual, and E'' its bidual. Recall that E is said to be reflexive if \( K(E) = E'' \), where \( K(E) \) is the image of the canonical embedding of E into E', and the strong topology \( \beta(E, E') \) coincides with the original topology of E. In the sequel we will identify E with its image \( K(E) \).

A separated locally convex topological vector space E is called quasireflexive if it is a closed subspace of finite deficiency in its bidual E'' and the strong topology \( \beta(E, E') \) coincides with the original topology of E.

An example of a quasireflexive topological vector space that is not isomorphic to a Banach space is the Cartesian product \( B \times E \) of a quasireflexive Banach space B and a reflexive topological vector space E that is not isomorphic to any Banach space.

This paper contains several generalizations of the results of [1]-[3] to topological vector spaces.

1. THEOREM 1. For a separated locally convex topological vector space E to be reflexive it is necessary and sufficient that it be barreled and that any bounded closed convex set \( V \subset E \) be closed in any separated locally convex topology \( \Gamma \) on E that is comparable with the original topology of E.

Necessity. If E is reflexive, then it is barreled [4, Chap. IV, §3, Theorem 2]. Suppose \( \Gamma \) is a separated locally convex topology on E that is weaker than the original. Denote by \( E_{\Gamma} \) the space E with the topology \( \Gamma \), and by \( E'_{\Gamma} \) its dual. It is obvious that each linear functional defined on E that is continuous in the topology \( \Gamma \) will also be continuous in the original topology of E. Thus, \( E'_{\Gamma} \subset E' \). Since E is reflexive, any bounded closed convex set \( V \subset E \) is compact in the weak topology \( \sigma(E, E') \), hence also in the topology \( \sigma(E, E'_{\Gamma}) \). Thus the set V is closed in the topology \( \sigma(E, E'_{\Gamma}) \), hence also in the topology \( \Gamma \).

Sufficiency. Suppose E is nonreflexive. If it is not barreled, then sufficiency is proved. Assume that E is barreled. Then there exists a bounded closed convex subset \( V \subset E \), \( \emptyset \notin V \), that is not compact in the weak topology \( \sigma(E, E') \) [4, §3, Theorem 2]. Since the closure \( \overline{V} \) of V in the topology \( \sigma(E'', E') \) is compact [4, §2, Corollary 2], it follows that \( \overline{V} \) contains an element \( x_0 \in E \). Take \( x_0 \in E \), \( x_0 \notin V \), and \( \hat{x}_0 = x_0 - x_0 \). Let \( \mathcal{M}^{x_0} = \{ x' \in E' : \langle x', \hat{x}_0 \rangle = 0 \} \). We introduce on E the weak topology \( \sigma(E, \mathcal{M}^{x_0}) \) defined by the duality between E and \( \mathcal{M}^{x_0} \). It is easy to verify that \( \sigma(E, \mathcal{M}^{x_0}) \) is a separated locally convex topology on E that is weaker than the original, and the closure of V in this topology contains the element \( x_0 \).

The theorem is proved.

Note that the conditions of the theorem are independent. The existence of nonreflexive barreled spaces is obvious. An example of a nonreflexive space E in which any bounded closed convex set \( V \subset E \) is closed in any separated locally convex topology \( \Gamma \) on E that is comparable with the original is an infinite-dimensional reflexive Banach space with the weak topology \( \sigma(E, E') \).

2. We now turn to the study of quasireflexivity of topological vector spaces.

Definition. Suppose E is a separated locally convex topological vector space, and \( M' \) a subspace of the dual space \( E' \) that is everywhere dense in the weak topology \( \sigma(E', E) \). Denote by \( \beta(E, M') \) the topology
on E with a neighborhood base of zero consisting of the polars of the sets \( V' \cap M' \), where \( V' \) ranges over the bounded subsets of \( E' \). We say that the subspace \( M' \) has characteristic zero if the topology \( \beta(E, M') \) is weaker than the strong topology \( \beta(E, E') \).

**THEOREM 2.** If a space \( E \) is quasireflexive, then the dual space \( E' \) contains no subspace of characteristic zero that is everywhere dense in the weak topology \( \sigma(E', E) \).

**Proof.** Suppose \( M' \) is a subspace of \( E' \) that is everywhere dense in the weak topology \( \sigma(E', E) \). Since \( E \) is quasireflexive, it has the form

\[
M' = \{ x' \in E' : (x', x'_i) = 0, \ldots, (x', x'_k) = 0 \},
\]

where \( x'_1, \ldots, x'_k \) is a finite set of elements of \( E'' \). Denote by \( M^0 \) the linear hull of the elements \( x'_1, \ldots, x'_k \). Let \( x_1, \ldots, x_k \) be a set of nonzero elements of \( E' \) for which \( |(x_i, x'_i)| \leq 1 \), \( i = 1, \ldots, k \). Consider the neighborhood \( W^0_0 = \{ x'' \in E' : |(x''_i, x'_i)| \leq 1 \}, i = 1, \ldots, k \} \) and its polar in \( E' \). The polar of \( W^0_0 \) is a bounded subset of \( E' \), hence for any bounded subset \( U' \subset E' \) the union \( U' = W^0_0 \cup U' \) will also be bounded. If the subsets \( U'_{\alpha}, \alpha \in I \) form a base for the bounded sets of \( E' \), then the subsets \( V'_{\alpha} = W^0_0 \cup U'_{\alpha}, \alpha \in I \), will also form a base for the bounded sets of \( E' \). Consider the polar of the set \( V' \cap M' \) in \( E' \). Since \( M^0 \) is locally compact, \( V' \) and \( M^0 \) are closed in the weak topology \( \sigma(E', E) \), and \( V' \subset W^0_0 \), we have

\[
(V' \cap M^0)^0 = V^0 + M^0.
\]

If the subsets \( V'_{\alpha} = W^0_0 \cup U'_{\alpha}, \alpha \in I \), form a base for the bounded sets, then their polars in \( E \) form a neighborhood base of zero in the strong topology \( \beta(E, E') \), and the subsets \( (V'_{\alpha} \cap M')_K = (V^0 + M^0) \cap E \) a neighborhood base of zero of the space \( E \) in the topology \( \beta(E, M') \). Assume that the topology \( \beta(E, M') \) is weaker than the topology \( \beta(E, E') \). Then there exists a neighborhood \( V^0_0 \) of zero such that for any \( \alpha \in I \) there exists \( x_{\alpha} \in (V^0_{\alpha} + M^0) \cap E \) and \( x_{\alpha} \in V^0_{\alpha} \). We have

\[
x_{\alpha} = y_{\alpha} + z_{\alpha}, \quad y_{\alpha}, z_{\alpha} \in V^0_{\alpha}, \quad z_{\alpha} \in M^0,
\]

and since the net \( \{ y_{\alpha} \} \) tends to zero, it follows that from some \( \alpha_0 \in I \) we have \( z_{\alpha} \in \cap (1/2)V^0_{\alpha} \) for \( \alpha > \alpha_0 \).

Select from the net \( \{ x_{\alpha} \} \) the subnet \( \{ x_{\alpha_{\lambda}} \} = \{ x_{\alpha} : \alpha > \alpha_0, \alpha \in I \} \). Since \( V^0_{\alpha} \) is balanced, it follows that if \( 0 \leq \lambda \leq 1 \), then \( \lambda x_{\alpha_{\lambda}} \in V^0_{\alpha_{\lambda}} \). For each \( z_{\alpha_{\lambda}} \) we select \( \lambda_{\alpha_{\lambda}} \in [0,1] \) so that

\[
\lambda_{\alpha_{\lambda}} z_{\alpha_{\lambda}} \in V^0_{\alpha_{\lambda}}, \quad \lambda_{\alpha_{\lambda}} z_{\alpha_{\lambda}} \in V^0_{\alpha_{\lambda}} - \frac{1}{2} V^0_{\alpha_{\lambda}}.
\]

Since \( V^0_{\alpha} \subset W^0_0 \) and the set \( W^0_0 \cap M^0 \) is bounded, the net \( \{ \lambda_{\alpha_{\lambda}} x_{\alpha_{\lambda}} \} \) has a subnet converging in the strong topology \( \beta(E, E') \) to a nonzero element \( z_0 \). But the net \( \{ \lambda_{\alpha} y_{\alpha} \} \) converges to zero, hence from the net \( \{ \lambda_{\alpha} x_{\alpha} \} \) we can choose a subnet converging to the element \( z_0 \in M^0 \) in the strong topology \( \beta(E, E') \), which contradicts the fact that the space \( E \subset E'' \) is closed. The theorem is proved.

3. We will now prove several propositions which will be needed later.

**LEMMA 1.** Suppose \( E \) is a barreled space and \( F \) a barreled subspace of \( E \). If the dual space \( F' \) contains a subspace \( M' \) of characteristic zero that is everywhere dense in the weak topology \( \sigma(F', F) \), then the dual space \( E' \) contains a subspace of characteristic zero that is everywhere dense in the weak topology \( \sigma(E', E) \).

**Proof.** As is well known, the subspace \( F' \) can be identified with the quotient space \( E'/F_0 \). Consider the predual \( K^{-1}(M') \) of the subspace \( M' \subset E'/F_0 \) under the canonical embedding \( E' \rightarrow E'/F_0 \). It is easy to see that the subspace \( K^{-1}(M') \) is everywhere dense in \( E' \) in the weak topology \( \sigma(E', E) \). If the topologies \( \beta(E, E') \) and \( \beta(E, K^{-1}(M')) \) were equivalent, then the restrictions of these topologies to \( F \) would also be equivalent. Since \( E \) and \( F \) are barreled, the restriction of the topology \( \beta(E, E') \) to \( F \) coincides with the strong topology \( \beta(F, F') \). Since the canonical embedding \( K \) is continuous, the restriction of the topology \( \beta(E, K^{-1}(M')) \) to \( F \) is no stronger than the topology \( \beta(F, M') \). The topology \( \beta(F, F') \) is stronger than the topology \( \beta(F, M') \); hence the topologies \( \beta(E, E') \) and \( \beta(E, K^{-1}(M')) \) are not equivalent. The lemma is proved.

**LEMMA 2.** Any complete nonreflexive barreled space \( E \) contains a bounded sequence \( \{ x_n \} \) having no limit point in the weak topology \( \sigma(E, E') \).

**Proof.** It follows from Theorem 2 of [4, Chap. IV, §3] that \( E \) contains a bounded closed subset \( W \) that is not compact in the weak topology \( \sigma(E, E') \). It follows from [4, Chap. IV, §2, Exercise 15(b)] that \( W \) contains a sequence \( \{ x_n \} \) having no limit point in the weak topology \( \sigma(E, E') \). The lemma is proved.
LEMMA 3. Any complete nonquasireflexive barreled space $E$ contains a closed separable subspace $F$ such that $\dim F''/F = \infty$.

**Proof.** We will construct a sequence of closed separable subspaces $F_n \subseteq E$ such that $F_{n+1} \supsetneq F_n$ and $\dim F''/F_n = n$. Then the closure $F = \bigcup F_n$ of their union will be the desired subspace. We take as $F_1$ the closed linear hull of the sequence $\{x_i\}$ of Lemma 2. It is easy to see that $\dim F_1''/F_1 = 1$.

Suppose we have constructed subspaces $F_j (j = 1, \ldots, n)$ with the required properties. If $\dim F_n''/F_n = \infty$, we may regard the construction as finished by putting $F_{n+1} = F_n$ for $j > n$. Thus, suppose $\dim F_n''/F_n = k < \infty$. We will show that the quotient space $E/F_n$ is nonreflexive. The bidual of $E/F_n$ is $E''/F_n''$, where $F_n''$ coincides with the closure of $F_n$ in the weak topology $\sigma(E'', E')$. If $E/F_n$ coincided with its bidual $E''/F_n''$, then each element $x'' + F_n''$ of $E''/F_n''$ would have the form $x + F_n$, $x \in E$, hence each element $x'' \in E''$ could be represented in the form $x'' = x + \hat{x}'$, $x, \hat{x} \in E$, $\hat{x}' \in F_n''$. But $\dim F_n''/F_n = k < \infty$, which contradicts the fact that $E$ is nonquasireflexive. Hence $E/F_n$ is a nonreflexive barreled space. By Lemma 2, we can choose in it a bounded sequence $x_i + F_n$ having no limit point in the weak topology $\sigma(E/F_n, (E/F_n)'')$. In view of the completeness of the bidual of $E''/F_n''$ in the weak topology $\sigma(E'', E''')$, this sequence has a limit point $x'' + F_n''$ in $E''/F_n''$. Choose a representative $x'' + F_n''$ of the class $x'' + F_n''$ and representatives $x_j$ of the classes $x_j + F_n$.

LEMMA 4. Any complete nonreflexive barreled space $E$ contains a balanced convex neighborhood $V_0$ of zero such that for any finite set $x_1', \ldots, x_r'$ in $E'$

$$\{x \in E : (x, x_1') = 0, \ldots, (x, x_r') = 0\} \cap V_0 \neq \emptyset \neq \{x \in E : (x, x_1') = 0, \ldots, (x, x_r') = 0\}.$$

**Proof.** The proof follows immediately from the fact that the original topology of $E$ does not coincide with the topology $\sigma(E, E')$.

We introduce the following notation: $V^{00}$ is the closure of the neighborhood $V \subseteq E$ in the weak topology $\sigma(E'', E')$,

$$M_{x_1', \ldots, x_r'} = \{x' \in E' : (x', x_1') = 0, \ldots, (x', x_r') = 0\}$$

and $p_V(x)$ is the gauge function of $V$.

LEMMA 5. Suppose $E$ is a complete nonquasireflexive barreled space, $x_1''$, $\ldots$, $x_r''$ a finite set of elements in $E''$, $x_1'$, $\ldots$, $x_r'$ a finite set of elements in $E'$, the subspace $M_{x_1'' \ldots x_r''}$ is everywhere dense in $E'$ in the weak topology $\sigma(E', E)$, $V$ is a neighborhood of zero in $E$, and $V_0$ is the neighborhood whose existence was established in Lemma 4. Then there exists an element $x'' + 1 \in (x_1'' \ldots x_r'')$ such that $x'' + 1 \in E + L (x_1', \ldots, x_r')$, $x'' + 1 \in x_1'' \ldots x_r''$ and the choice of $V_0$ that there exists an element $x \in 2V_0$, $x \in (x_1'' \ldots x_r'') \cap E$. We can obviously choose $\epsilon > 0$ so small that the element $x'' + 1 = x + \epsilon x''$ will possess the required properties. The lemma is proved.

**Proof.** Since $E$ is nonquasireflexive, the subspace $H_{x_1'' \ldots x_r''}$ contains an element $x''$ not belonging to $E + L (x_1', \ldots, x_r')$. It follows from the finite-dimensionality of $E''/H_{x_1'' \ldots x_r''}$ and the choice of $V_0$ that there exists an element $x \in 2V_0$, $x \in (x_1'' \ldots x_r'') \cap E$. We can obviously choose $\epsilon > 0$ so small that the element $x'' + 1 = x + \epsilon x''$ will possess the required properties. The lemma is proved.

LEMMA 6. Suppose $E$ is a barreled space, $H$ a linear subspace of $E'$ containing $E$, $x_1''$, $\ldots$, $x_r''$ a finite set of elements in $E''$, $V$ a bounded closed convex balanced subset of $E'$, and $V^{00}$ its polar in $E''$. If $\inf \{p_V'(y'' - Y') : y'' \in H, p_V'(y'') = 1, x'' \in L(x_1', \ldots, x_r')\} = a, a \neq 0$, then the closure of the set
Proof. Since the set \((1/a)V' \cap M_{x_1} \ldots x_n \cap M_{y_1} \ldots y_k\) is convex, it suffices to show that there exists no element \(x'_0 \in V' \cap M_{y_1} \ldots y_k\) that can be strongly separated from the set \((1/a)V' \cap M_{x_1} \ldots x_n \cap M_{y_1} \ldots y_k\) by a hyperplane that is closed in the weak topology \(\sigma(E', E)\). Without loss of generality, we may assume that \(p_{\gamma_0}(x'_0) = 1\). Assume there exists an element \(x'_0 \in E\) for which

\[
\sup \left\{ (x, x'_0) : x' \in \frac{1}{a}V' \cap M_{x_1} \ldots x_n \cap M_{y_1} \ldots y_k \right\} < 1,
\]

\[
(x, x'_0) = 1.\]

Then the linear manifold \(N' = \{x' \in E' : (x, x'_0) = 1\} \cap M_{x_1} \ldots x_n \cap M_{y_1} \ldots y_k\) strongly separates \(x'_0\) and \((1/a)V' \cap M_{x_1} \ldots x_n \cup M_{y_1} \ldots y_k\) in the subspace \(M_{y_1} \ldots y_k\). Since the set \((1/a)V' \cap M_{x_1} \ldots x_n \cap M_{y_1} \ldots y_k\) is closed and convex in the strong topology \(\beta(E', E)\) and does not contain \(x'_0\), then, by the Hahn–Banach theorem, \(N'\) can be extended to a hyperplane \(\tilde{N}'\) which is closed in the strong topology \(\beta(E', E)\) and strongly separates \(x'_0\) and \((1/a)V' \cap M_{x_1} \ldots x_n \cup M_{y_1} \ldots y_k\). Since \(N'\) is closed in the topology \(\sigma(E', E + L(y_1, \ldots, y_k))\) and has finite deficiency, the hyperplane \(\tilde{N}'\) is also closed in the topology \(\sigma(E', E + L(y_1, \ldots, y_k))\), i.e. has the form

\[
N' = \{x' : (x', y'_{0}) = 1\},
\]

where \(y_0 = \lambda_0 x_0 + \sum_{i=1}^{k} \lambda_i y_i^n\). It is easy to see that \(\sup\{e_{i0}, y'_{0}\} = 1\), \(\sup\{e_{i0}, y'_{0}\} : x' \in (1/a)V' \cap M_{x_1} \ldots x_n \cap M_{y_1} \ldots y_k\} < 1\), or \(\sup\{e_{i0}, y'_{0}\} : x' \in V' \cap M_{x_1} \ldots x_n \cap M_{y_1} \ldots y_k\} < a\). Consider the restriction of the functional \(y'_{0}\) to the subspace \(M_{x_1} \ldots x_n \cap M_{x_1} \ldots x_n\). By the Hahn–Banach theorem, this restriction can be extended to a linear functional \(\tilde{y}'_{0}\) defined on all of \(E\), in such a way that \(\sup\{e_{i0}, y'_{0}\} : x' \in V'\} < a\). Since the subspace \(M_{x_1} \ldots x_n\) has finite deficiency, the functional \(\tilde{y}'_{0}\) is continuous. The element \(x'' = y''_{0} - y'_{0}\) obviously belongs to the subspace \(L(x_1, \ldots, x_n)\),

\[
\sup\{e_{i0}, y'_{0}\} : x'' \in V'\} \geq (x'_0, y'_{0}) = 1 \text{ and } y''_{0} \in H.
\]

Since \(p_{\gamma_0}(y''_{0}) = \sup\{e_{i0}, y'_{0}\} : y'' \in H\}, \text{ we obtain inf } \{p_{\gamma_0}(y''_{0}) : y'' \in H\} = a\), \(x'_{0} \in L(x_1, \ldots, x_n)\), \(x'' \in L(x_1, \ldots, x_n)\), \(x'' \in L(x_1, \ldots, x_n)\), which contradicts the hypothesis of the lemma. The lemma is proved.

From now on we will assume that \(E\) is a Fréchet space, \(V_1, \ldots, V_n\), a countable base of closed convex balanced neighborhoods of zero in \(E\) such that \(V_1 \supseteq V_1 + 1\). It is easy to see that their closures \(V''_1\) in the weak topology \(\sigma(E', E')\) form a countable neighborhood base of zero in the strong topology \(\sigma(E', E')\).

**Theorem 3.** For any nonquasireflexive Fréchet space \(E\) the dual space \(E'\) contains a subspace \(M'\) of characteristic zero that is everywhere dense in the weak topology \(\sigma(E', E)\).

Proof. A Fréchet space is barreled, and a closed subspace of a Fréchet space is again a Fréchet space. Hence, in view of Lemmas 1 and 3, we may assume that \(E\) is separable. By Proposition 3 of [4, Chap. IV, 32], the dual space \(E'\) is the union of bounded sets \(V''_n\) that are metrizable and separable in the weak topology \(\sigma(E', E)\). Let \(x_1, \ldots, x_n\) be a countable set that is everywhere dense in \(E\) in the weak topology \(\sigma(E', E)\). Without loss of generality, we may assume that \(x_1 \in V''_1\). The construction of the subspace \(M'\) will be carried out inductively.

1. There exists an element \(x''_1 \in E''\) satisfying the following conditions: \(x''_1 \in V''_1\), \(x''_1 \in M_{x_1}\); \(x''_1 \in E\); there exists an element \(x_1 \in E\) such that \(x''_1 - x_1 \in V''_1\). To demonstrate the existence of \(x''_1\) it suffices to apply Lemma 5. The element \(x''_1\) possesses the following properties.

a) There exists an element \(x''_1 \in E''\) such that \((x''_1, x''_1) \neq 0\). This follows from the fact that \(x''_1 \notin E\).

b) There exists a neighborhood \(W_1 \subset V_1\) of zero such that

\[
a_i = \inf \{p_{\gamma_0}(x''_1 - y') : y' \in H_{x_1}, p_{\gamma_0}(y') \geq 1, x''_1 \in L(x_1)\} \neq 0.
\]

Indeed, we can take as \(W_1\) any neighborhood of zero whose closure in the topology \(\sigma(E''_1, E'')\) is contained in the set \(V''_1 \cap \{x'' \in E'' : (x''_1, x''_1) \leq 1\}\). Lemma 6 implies that the closure of \((1/a_i)W'_1 \cap M_{y_1} \cap M_{y_2} \ldots y_k\) contains \(W'_1 \cap M_{y_1} \cap y''_k\) for any set \(y_1, \ldots, y_k\) in \(H_{y_1}^{y_k}\).

c) We introduce on the bounded set \(W_1\) a metric \(\rho_1\) equivalent to the weak topology \(\sigma(E', E)\). The previous property implies the existence of an element \(x''_1 \in (1/a_i)W'_1\) for which \(\rho_1(x''_1, x''_1) = 1\).

We will give two illustrations of the second step of the inductive construction.
2. There exists an element \( x_0^* \in E^* \) satisfying the following conditions: \( x_0^* \in V_{\theta}^0 \); \( x_1^1 \in M_{x_0^*}x_0^*; x_2^2 \in H_{x_1^1}; \{x_1^1, x_2^2\} \subset M_{x_0^*}; x_0^* \in E + L(x_1^1) \); there exists an element \( x_0 \in E \) such that \( x_0^* - x_0 \in V_{\theta}^0 \). To demonstrate the existence of \( x_0^* \) it suffices to apply Lemma 5.

The element \( x_0^* \) possesses the following properties.

a) There exists \( x_0^{1*} \in \{(1/a_0)W_0^0 \cap M_{x_0^*}x_0^* \) for which \( \rho_1(x_0^{1*}, x_0^*) \leq 1/2 \). The existence of such an element follows from Lemma 6, the role of the set \( y_1^* \), \( \ldots, y_k^* \) being played here by \( x_0^* \).

b) There exists an element \( x_2^2 \in E^0 \) for which \( \langle x_2^2, x_0^* \rangle = 0, \langle x_2^2, x_0^* \rangle \neq 0 \). This follows from the fact that \( x_2^2 \subset E + L(x_1^1) \) and \( E^0 \) is infinite-dimensional.

c) There exists a neighborhood \( W_2 \subset V_2 \) of zero for which

\[
a_2 = \inf \{\rho_{w_2^0}(x^* - y^*) : y^* \in H_{x_1^1, x_2^2}; \rho_{w_2^0}(y^*) > 1, x^* \in L(x_1^1, x_2^2)\} \neq 0.
\]

Indeed, we can take as \( W_2 \) any neighborhood of zero whose closure in the topology \( \sigma(E^*, E') \) is contained in the set \( V_{\theta}^0 \cap \{x^* \in E^* : \langle x^*, x_0^* \rangle \leq 1, \langle x^*, x_0^* \rangle \leq 1\} \). Lemma 6 implies that the closure of \( \{(1/a_2)W_2^0 \cap M_{x_1^1}x_1^1 \cap M_{x_2^2}x_2^2 \} \) in the weak topology \( \sigma(E^*, E) \) contains \( W_2^0 \cap M_{y_1^*} \ldots y_k^* \) for any set \( y_1^* \), \( \ldots, y_k^* \) in \( H_{x_1^1, x_2^2} \).

d) We introduce on the bounded set \( W_2^0 \) a metric \( \rho_2 \) equivalent to the weak topology \( \sigma(E^*, E) \). The previous property implies the existence of an element \( x_2^2 \in \{(1/a_2)W_2^0 \cap M_{x_1^1}x_1^1 \) for which \( \rho_2(x_2, x_2^2) \leq 1/2 \).

Consider the \( n \)-th step of the inductive construction. Suppose that after the \((n-1)\)-st step of the inductive construction we have sets \( \{x_1, \ldots, x_{n-1}\} \subset E; \{x_1, \ldots, x_{n-1}\} \subset E^0; \{x_1^1, x_2^2, \ldots, x_1^1; x_2^2, \ldots, x_{n-1}^1; \ldots, x_{n-1}^1 \} \subset E^0; \) neighborhoods \( W_1 \subset V_1, \ldots, W_{n-1} \subset V_{n-1}; \) numbers \( a_1, \ldots, a_{n-1} \) unequal to zero; and metrics \( \rho_1, \ldots, \rho_{n-1} \) equivalent to the weak topology \( \sigma(E^*, E) \) on the bounded sets \( \{(1/a_q)W_q^0 \cap M_{x_1^1} \ldots x_q^q \cap M_{y_1^*} \ldots y_k^* \) for any set \( y_1^* \) \) \( \ldots, y_k^* \) in \( H_{x_1^1, \ldots, x_q^q} \) \((q = 1, n-1); \)

1) \( x_i^0 \in V_{\theta}^0, x_i^0 \in V_{\theta}^0, i = 1, \ldots, n-1; \)
2) \( \{x_1^1, \ldots, x_{n-1}^1; x_2^2, \ldots, x_{n-1}^2; \ldots, x_{n-1}^2 \} \subset M_{x_i^*}x_i^* \); \( x_0^* \in H_{x_1^1, x_2^2}; \quad x_0^* \in E + L(x_1^1, \ldots, x_{n-1}^1), i = 1, \ldots, n-1; \)
3) the closure of the set \( \{(1/a_q)W_q^0 \cap M_{x_1^1} \ldots x_q^q \cap M_{y_1^*} \ldots y_k^* \) contains \( W_q^0 \cap M_{y_1^*} \ldots y_k^* \) for any set \( y_1^* \) \) \( \ldots, y_k^* \) in \( H_{x_1^1, \ldots, x_q^q} \) \((q = 1, n-1); \)

4) \( \rho_1(x_1, x_1^1) \leq 1; \rho_1(x_1^1, x_1^2) \leq 1/2; \ldots; \rho_1(x_{n-1}^1, x_1^2) \leq \frac{1}{n-1}; \)
\[
\rho_2(x_1^2, x_2^2) \leq \frac{1}{2}; \rho_2(x_2^2, x_2^{2q}) \leq \frac{1}{n-1}; \]
\[
\rho_{n-1}(x_{n-1}^2, x_{n-1}^{2q}) \leq \frac{1}{n-1}.
\]

Then there exists an element \( x_n^* \in E^* \) satisfying the following conditions: \( x_n^* \in V_{\theta}^0, \{x_1^1, \ldots, x_{n-1}^1; x_2^2, \ldots, x_{n-1}^2; \ldots, x_{n-1}^2 \} \subset M_{x_1^1} \ldots x_{n-1}^2 \); \( \{x_1^1, \ldots, x_{n-1}^1 \} \subset M_{x_1^1} \ldots x_{n-1}^2 \); \( x_n^* \in H_{x_1^1} \ldots x_{n-1}^2 \); \( x_n^* \in E + L(x_1^1, \ldots, x_{n-1}^1) \); there exists an element \( x_n \in E \) such that \( x_n^* - x_n \in V_{\theta}^0 \). To demonstrate the existence of \( x_n^* \) it suffices to apply Lemma 5.

The element \( x_n^* \) possesses the following properties.

a) There exists an element \( x_n^q * \in \{(1/a_q)W_q^0 \cap M_{x_1^1} \ldots x_q^q \) for which \( \rho_q(x_n^q, x_n^{q*}) \leq 1/n \) \((q = 1, n-1)\). The existence of such elements follows from Lemma 6.

b) There exists an element \( x_n^* \in E^0 \) for which \( \langle x_n^*, x_0^* \rangle = 0, \ldots, \langle x_n^*, x_{n-1}^* \rangle = 0, \langle x_n^*, x_0^* \rangle \neq 0 \). This follows from the fact that \( x_n^* \subset E + L(x_1^1, \ldots, x_{n-1}^1) \) and \( E^0 \) is infinite-dimensional.

c) There exists a neighborhood \( W_n \subset V_n \) of zero for which

\[
a_n = \inf \{\rho_{w_0}(x^* - y^*) : y^* \in H_{x_1^1 \ldots x_n^q}; \rho_{w_0}(y^*) \geq 1, x^* \in L(x_1^1 \ldots, x_n^q)\}, \quad x^* \in L(x_1^1 \ldots, x_n^q).
\]

Indeed, we can take as \( W_n \) any neighborhood of zero whose closure in the topology \( \sigma(E^*, E') \) is contained in the set \( V_{\theta}^0 \cap \{x^n \in E^* : \langle x^n, x_0^* \rangle \leq 1, \ldots, \langle x^n, x_{n-1}^* \rangle \leq 1\} \). Lemma 6 implies that the closure
of the set \((1/an)W_0^n \cap Mx_1^n \cap \ldots \cap Mx_k^n \cap \ldots \cap My_1^n \cap \ldots \cap My_k^n\) in the weak topology \(\sigma(E', E)\) contains \(W_0^n \cap Mx_1^n \cap \ldots \cap Mx_k^n \cap \ldots \cap My_1^n \cap \ldots \cap My_k^n\) for any set \(y_1^n, \ldots, y_k^n\) in \(Hx_1^n \cap \ldots \cap Hx_k^n\).

d) We introduce on the bounded set \(W_0^n\) a metric \(\rho_n\) equivalent to the weak topology \(\sigma(E', E)\). The previous property implies the existence of an element \(x_0^n \in (1/an)W_0^n \cap Mx_1^n \cap \ldots \cap Mx_k^n \cap \ldots \cap My_1^n \cap \ldots \cap My_k^n\) for which \(\rho_n(x_0^n, x_k^n) \leq 1/n\).

Thus, it is possible to carry out the inductive step, i.e. to replace \(n-1\) by \(n\) in the set of properties 1)-4). We will show that \(M' = \bigcap_{i=1}^{\infty} Mx_i^{n}\) is a subspace of characteristic zero that is everywhere dense in the weak topology \(\sigma(E', E)\). Property 2) implies that

\[
\{x_1^n, x_2^n, \ldots, x_1^n, \ldots; x_1^n, x_2^n, \ldots, x_1^n; \ldots; x_1^n, \ldots\} \subseteq M'.
\]

The sequences \(x_0^n, x_1^n, x_2^n, \ldots\) converge to elements \(x^n\) in the weak topology \(\sigma(E', E)\), so that the closure of \(M'\) in the weak topology \(\sigma(E', E)\) contains the elements \(x_1^n, \ldots, x_k^n, \ldots\), i.e. it coincides with \(E'\). It follows from the construction of \(\{x_n\}\) that zero is not a limit point in the strong topology \(\beta(E, E')\) of the sequence \(\{x_n\}\). But \(\{x_n\}\) converges to zero in the topology \(\beta(E, M')\). Indeed, consider any neighborhood \(V_1\) of zero in the fundamental system \(\{V_n\}\) of neighborhoods of zero of \(E\). Then

\[
(V_1^n \cap M')^0 \Rightarrow (V_1^n \cap M')^0 \Rightarrow V_1^n \Rightarrow L(x_1^n).
\]

But \(x_1^n - x_1 \in V_1^n\), hence \(x_1 \notin (V_1^n \cap M')^0 \cap E\).

Thus, the topologies \(\beta(E, E')\) and \(\beta(E, M')\) are not equivalent. The theorem is proved.

Using Theorems 2 and 3, we can formulate the following criterion for the quasireflexivity of a Fréchet space.

**THEOREM 4.** For a Fréchet space \(E\) to be quasireflexive it is necessary and sufficient that the dual space \(E'\) not contain a subspace of characteristic zero that is everywhere dense in the topology \(\sigma(E', E)\).

**LITERATURE CITED**