REFLEXIVITY AND QUASIREFLEXIVITY OF TOPOLOGICAL VECTOR SPACES

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Suppose E is a separated locally convex topological vector space, E' its dual, and E" its bidual. Recall that E is said to be reflexive if K(E) = E", where K(E) is the image of the canonical embedding of E into E", and the strong topology $\beta(E, E')$ coincides with the original topology of E. In the sequel we will identify E with its image K(E).

A separated locally convex topological vector space E is called quasireflexive if it is a closed subspace of finite deficiency in its bidual E" and the strong topology $\beta(E, E')$ coincides with the original topology of E.

An example of a quasireflexive topological vector space that is not isomorphic to a Banach space is the Cartesian product $B \times E$ of a quasireflexive Banach space B and a reflexive topological vector space E that is not isomorphic to any Banach space.

This paper contains several generalizations of the results of [1]-[3] to topological vector spaces.

1. THEOREM 1. For a separated locally convex topological vector space E to be reflexive it is necessary and sufficient that it be barreled and that any bounded closed convex set $V \subset E$ be closed in any separated locally convex topology Γ on E that is comparable with the original topology of E.

Necessity. If E is reflexive, then it is barreled [4, Chap. IV, §3, Theorem 2]. Suppose Γ is a separated locally convex topology on E that is weaker than the original. Denote by E_Γ the space E with the topology Γ, and by E'_Γ its dual. It is obvious that each linear functional defined on E that is continuous in the topology Γ will also be continuous in the original topology of E. Thus, $E'_\Gamma \subset E'$. Since E is reflexive, any bounded closed convex set $V \subset E$ is compact in the weak topology $\sigma(E, E')$, hence also in the topology $\sigma(E, E'_\Gamma)$. Thus the set V is closed in the topology $\sigma(E, E'_\Gamma)$, hence also in the topology Γ.

Sufficiency. Suppose E is nonreflexive. If it is not barreled, then sufficiency is proved. Assume that E is barreled. Then there exists a bounded closed convex subset $V \subset E$, $\theta \in V$, that is not compact in the weak topology $\sigma(E, E')$ [4, §3, Theorem 2]. Since the closure $\overline{V}$ of V in the topology $\sigma(E'', E')$ is compact [4, §2, Corollary 2], it follows that $\overline{V}$ contains an element $x''_0 \notin E$. Take $x_0 \in E$, $x_0 \notin V$, and $\hat{x}'' = x_0 - x''_0$. Let $M_{\hat{x}''} = \{x' \in E' : \langle x', \hat{x}'' \rangle = 0\}$. We introduce on E the weak topology $\sigma(E, M_{\hat{x}''})$ defined by the duality between E and $M_{\hat{x}''}$. It is easy to verify that $\sigma(E, M_{\hat{x}''})$ is a separated locally convex topology on E that is weaker than the original, and the closure of V in this topology contains the element x_0.

The theorem is proved.

Note that the conditions of the theorem are independent. The existence of nonreflexive barreled spaces is obvious. An example of a nonreflexive space E in which any bounded closed convex set $V \subset E$ is closed in any separated locally convex topology Γ on E that is comparable with the original is an infinite-dimensional reflexive Banach space with the weak topology $\sigma(E, E')$.

2. We now turn to the study of quasireflexivity of topological vector spaces.

Definition. Suppose E is a separated locally convex topological vector space, and M' a subspace of the dual space E' that is everywhere dense in the weak topology $\sigma(E', E)$. Denote by $\beta(E, M')$ the topology

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on E with a neighborhood base of zero consisting of the polars of the sets $V' \cap M'$, where V' ranges over the bounded subsets of E'. We say that the subspace M' has characteristic zero if the topology $\beta(E, M')$ is weaker than the strong topology $\beta(E, E')$.

THEOREM 2. If a space E is quasireflexive, then the dual space E' contains no subsapce of characteristic zero that is everywhere dense in the weak topology $\sigma(E', E)$.

Proof. Suppose M' is a subspace of E' that is everywhere dense in the weak topology $\sigma(E', E)$. Since E is quasireflexive, it has the form

$$M' = \{x' \in E' : \langle x', x_1'' \rangle = 0, \ldots, \langle x', x_k'' \rangle = 0\},$$

where $x_1'', \ldots, x_k''$ is a finite set of elements of E''. Denote by M'^0 the linear hull of the elements $x_1'', \ldots, x_k''$. Let $x_1', \ldots, x_k'$ be a set of nonzero elements of E' for which $|\langle x_i', x_i'' \rangle| \leq 1$, $i = 1, \ldots, k$. Consider the neighborhood $W_0'^0 = \{x'' \in E'' : |\langle x_i', x'' \rangle| \leq 1, i = 1, \ldots, k\}$ and its polar in E'. The polar of W_0' is a bounded subset of E', hence for any bounded subset $U' \subset E'$ the union $V' = W_0' \cup U'$ will also be bounded. If the subsets U_α' $(\alpha \in I)$ form a base for the bounded sets of E', then the subsets $V_\alpha' = W_0' \cup U_\alpha'$, $\alpha \in I$, will also form a base for the bounded sets of E'. Consider the polar of the set $V' \cap M'$ in E''. Since M'^0 is locally compact, V'^0 and M'^0 are closed in the weak topology $\sigma(E'', E')$, and $V'^0 \subset W_0'^0$, we have

$$(V' \cap M')^0 = V'^0 + M'^0.$$

If the subsets $V_\alpha' = W_0' \cup U_\alpha'$, $\alpha \in I$, form a base for the bounded sets, then their polars in E form a neighborhood base of zero in the strong topology $\beta(E, E')$, and the subsets $(V_\alpha' \cap M')_E^0 = (V_\alpha'^0 + M'^0) \cap E$ a neighborhood base of zero of the space E in the topology $\beta(E, M')$. Assume that the topology $\beta(E, M')$ is weaker than the topology $\beta(E, E')$. Then there exists a neighborhood $V_0'^0$ of zero such that for any $\alpha \in I$ there exists $x_\alpha \in (V_\alpha'^0 + M'^0) \cap E$ and $x_\alpha \notin V_0'^0$. We have

$$x_\alpha = y_\alpha'' + z_\alpha'', \quad y_\alpha'' \in V_\alpha'^0, \quad z_\alpha'' \in M'^0,$$

and since the net $\{y_\alpha\}$ tends to zero, it follows that from some α_0 on we have $z_\alpha'' \notin (1/2)V_0'^0$ for $\alpha > \alpha_0$. Select from the net $\{x_\alpha\}$ the subnet $\{x_{\bar\alpha}\} = \{x_\alpha : \alpha > \alpha_0, \alpha \in I\}$. Since $V_\alpha'^0$ is balanced, it follows that if $0 \leq \lambda \leq 1$, then $\lambda y_{\bar\alpha}'' \in V_\alpha'^0$. For each $z_{\bar\alpha}''$ we select $\lambda_{\bar\alpha} \in [0, 1]$ so that

$$\lambda_{\bar\alpha} z_{\bar\alpha}'' \in V_0'^0, \quad \lambda_{\bar\alpha} z_{\bar\alpha}'' \notin \frac{1}{2} V_0'^0.$$

Since $V_0'^0 \subset W_0'^0$ and the set $W_0'^0 \cap M'^0$ is bounded, the net $\{\lambda_{\bar\alpha} z_{\bar\alpha}''\}$ has a subnet converging in the strong topology $\beta(E'', E')$ to a nonzero element z_0''. But the net $\{\lambda_\alpha y_\alpha''\}$ converges to zero, hence from the net $\{\lambda_\alpha x_\alpha\}$ we can choose a subnet converging to the element $z_0'' \in M'^0$ in the strong topology $\beta(E'', E')$, which contradicts the fact that the space $E \subset E''$ is closed. The theorem is proved.

3. We will now prove several propositions which will be needed later.

LEMMA 1. Suppose E is a barreled space and F a barreled subspace of E. If the dual space F' contains a subspace M' of characteristic zero that is everywhere dense in the weak topology $\sigma(F', F)$, then the dual space E' contains a subspace of characteristic zero that is everywhere dense in the weak topology $\sigma(E', E)$.

Proof. As is well known, the subspace F' can be identified with the quotient space E'/F^0. Consider the preimage $K^{-1}(M')$ of the subspace $M' \subset E'/F^0$ under the canonical embedding $E' \xrightarrow{K} E'/F^0$. It is easy to see that the subspace $K^{-1}(M')$ is everywhere dense in E' in the weak topology $\sigma(E', E)$. If the topologies $\beta(E, E')$ and $\beta(E, K^{-1}(M'))$ were equivalent, then the restrictions of these topologies to F would also be equivalent. Since E and F are barreled, the restriction of the topology $\beta(E, E')$ to F coincides with the strong stopology $\beta(F, F')$. Since the canonical embedding K is continuous, the restriction of the topology $\beta(E, K^{-1}(M'))$ to F is not stronger than the topology $\beta(F, M')$. The topology $\beta(F, F')$ is stronger than the topology $\beta(F, M')$; hence the topologies $\beta(E, E')$ and $\beta(E, K^{-1}(M'))$ are not equivalent. The lemma is proved.

LEMMA 2. Any complete nonreflexive barreled space E contains a bounded sequence $\{x_n\}$ having no limit point in the weak topology $\sigma(E, E')$.

Proof. It follows from Theorem 2 of [4, Chap. IV, §3] that E contains a bounded closed subset W that is not compact in the weak topology $\sigma(E, E')$. It follows from [4, Chap. IV, §2, Exercise 15(b)] that W contains a sequence $\{x_n\}$ having no limit point in the weak topology $\sigma(E, E')$. The lemma is proved.

LEMMA 3. Any complete nonquasireflexive barreled space E contains a closed separable subspace F such that $\dim F''/F = \infty$.

Proof. We will construct a sequence of closed separable subspaces $F_n \subset E$ such that $E_{n+1} \supset F_n$ and $\dim F_n''/F_n \geq n$. Then the closure $F = \bigcup_n F_n$ of their union will be the desired subspace. We take as F_1 the closed linear hull of the sequence $\{x_i\}$ of Lemma 2. It is easy to see that $\dim F_1''/F_1 \geq 1$.

Suppose we have constructed subspaces F_j ($j = 1, \ldots, n$) with the required properties. If $\dim F_n''/F_n = \infty$, we may regard the construction as finished by putting $F_j = F_n$ for $j > n$. Thus, suppose $\dim F_n''/F_n = k < \infty$. We will show that the quotient space E/F_n is nonreflexive. The bidual of E/F_n is E''/F_n'', where F_n'' coincides with the closure of F_n in the weak topology $\sigma(E'', E')$. If E/F_n coincided with its bidual E''/F_n'', then each element $x'' + F_n''$ of E''/F_n'' would have the form $x + F_n''$, $x \in E$, hence each element $x'' \in E''$ could be represented in the form $x'' = x + \tilde{x}''$, $x \in E$, $\tilde{x}'' \in F_n''$. But $\dim F_n''/F_n = k < \infty$, hence $\dim E''/E = k < \infty$, which contradicts the fact that E is nonquasireflexive. Hence E/F_n is a nonreflexive barreled space. By Lemma 2, we can choose in it a bounded sequence $x_i + F_n$ having no limit point in the weak topology $\sigma(E/F_n, (E/F_n)')$. In view of the completeness of the bidual of E''/F_n'' in the weak topology $\sigma(E''/F_n'', (E/F_n)')$, this sequence has a limit point $\hat{x}'' + F_n''$ in E''/F_n''. Choose a representative $\hat{x}''$ of the class $\hat{x}'' + F_n''$ and representatives x_i of the classes $x_i + F_n$. Let $F_{n+1} = L\{x_i\} + F_n$ be the closure of the sum of the subspaces F_n in the linear hull of the sequence $\{x_i\}$. We will show that

$$\dim F''_{n+1}/F_{n+1} \geqslant \dim F''_n/F_n + 1.$$

Since $F''_{n+1} \supset F''_n$, it suffices to verify that $\hat{x}''$ belongs to F''_{n+1} and does not belong to $F_{n+1} + F''_n$. If $\hat{x}''$ did not belong to F''_{n+1}, then since F''_{n+1} is closed in the weak topology $\sigma(E'', E')$ there would exist an element $x_0' \in E'$ such that $\langle x_0', \hat{x}''\rangle \neq 0$, $x_0' \in F^0_{n+1}$, where F^0_{n+1} is the polar of F_{n+1} in the dual space E'. The element x_0' can be regarded as a continuous linear functional on E/F_n. Then $x_0'(x_i + F_n) = 0$ ($i = 1, \ldots, \infty$), $x_0'(\hat{x}'' + F_n'') \neq 0$, which contradicts the fact that the element $\hat{x}'' + F_n''$ is a limit point of the sequence $x_i + F_n''$ in the weak topology $\sigma(E''/F_n'', (E/F_n)')$. We will now show that $\hat{x} \notin F_{n+1} + F_n$. If $\hat{x}'' = x + x''$, where $x \in F_{n+1}$, $x'' \in F_n$, then $\hat{x}'' + F_n'' = (x + F_n'') + (x'' + F_n'')$ and the sequence $\{x_i + F_n''\}$ has as a limit point the element $(x + F_n'') + (x'' + F_n'')$. But $x'' + F_n'' = F_n''$, hence the element $x + F_n''$ is a limit point of the sequence $\{x_i + F_n\}$, which contradicts the choice of $\{x_i + F_n\}$. The lemma is proved.

LEMMA 4. Any complete nonreflexive barreled space E contains a balanced convex neighborhood V_0 of zero such that for any finite set $x_1', \ldots, x_k'$ in E'

$$\{x \in E : \langle x, x_1'\rangle = 0, \ldots, \langle x, x_k'\rangle = 0\} \cap V_0 \neq \{x \in E : \langle x, x_1'\rangle = 0, \ldots, \langle x, x_k'\rangle = 0\}.$$

Proof. The proof follows immediately from the fact that the original topology of E does not coincide with the topology $\sigma(E, E')$.

We introduce the following notation: V^{00} is the closure of the neighborhood $V \subset E$ in the weak topology $\sigma(E'', E')$,

$$M_{x_1''\ldots x_r''} = \{x' \in E' : \langle x', x_1''\rangle = 0, \ldots, \langle x', x_r''\rangle = 0\},$$

$$H_{x_1'''\ldots x_s'''} = \{x'' \in E'' : \langle x'', x_1'''\rangle = 0, \ldots, \langle x'', x_s'''\rangle = 0\},$$

and $p_V(x)$ is the gauge function of V.

LEMMA 5. Suppose E is a complete nonquasireflexive barreled space, $x_1''', \ldots, x_s'''$ a finite set of elements in E''', $x_1'', \ldots, x_r''$ a finite set of elements in E'', the subspace $M_{x_1''\ldots x_r''}$ is everywhere dense in E' in the weak topology $\sigma(E', E)$, V is a neighborhood of zero in E, and V_0 is the neighborhood whose existence was established in Lemma 4. Then there exists an element $x''_{r+1} \in H_{x_1'''\ldots x_s'''}$ such that $x''_{r+1} \notin E + L(x_1'', \ldots, x_r'')$, $x''_{r+1} \notin V_0^{00}$, $M_{x_1''\ldots x''_{r+1}}$ is everywhere dense in E' in the weak topology $\sigma(E', E)$, and there exists an element $x \in E$ for which $x''_{r+1} - x \in V^{00}$.

Proof. Since E is nonquasireflexive, the subspace $H_{x_1'''\ldots x_s'''}$ contains an element x'' not belonging to $E + L(x_1'', \ldots, x_r'')$. It follows from the finite-dimensionality of $E''/H_{x_1'''\ldots x_s'''}$ and the choice of V_0 that there exists an element $x \notin 2V_0$, $x \in H_{x_1'''\ldots x_s'''} \cap E$. We can obviously choose $\varepsilon > 0$ so small that the element $x''_{r+1} = x + \varepsilon x''$ will possess the required properties. The lemma is proved.

LEMMA 6. Suppose E is a barreled space, H a linear subspace of E'' containing E, $x_1'', \ldots, x_r''$ a finite set of elements in E'', V' a bounded closed convex balanced subset of E', and V'^0 its polar in E''. If $\inf \{p_{V'^0}(x'' - Y'') : y'' \in H, p_{V'^0}(y'') \geq 1, x'' \in L(x_1'', \ldots, x_r'')\} = a$, $a \neq 0$, then the closure of the set

$(1/a)V' \cap M_{x_1''\ldots x_r''} \cap M_{y_1''\ldots y_k''}$ in the weak topology $\sigma(E', E)$ contains $V' \cap M_{y_1''\ldots y_k''}$ for any finite set y_1'', $\ldots$, y_k'' in H.

Proof. Since the set $(1/a)V' \cap M_{x_1''\ldots x_r''} \cap M_{y_1''\ldots y_k''}$ is convex, it suffices to show that there exists no element $x_0' \in V' \cap M_{y_1''\ldots y_k''}$ that can be strongly separated from the set $(1/a)V' \cap M_{x_1''\ldots x_r''} \cap M_{y_1''\ldots y_k''}$ by a hyperplane that is closed in the weak topology $\sigma(E', E)$. Without loss of generality, we may assume that $p_{V'}(x_0') = 1$. Assume there exists an element $x \in E$ for which

$$\sup\left\{\langle x, x'\rangle : x' \in \frac{1}{a} V' \cap M_{x_1''\ldots x_r''} \cap M_{y_1''\ldots y_k''}\right\} < 1,$$

$\langle x, x_0'\rangle = 1$. Then the linear manifold $N' = \{x' \in E' : \langle x, x'\rangle = 1\} \cap M_{y_1''\ldots y_k''}$ strongly separates x_0' and $(1/a)V' \cap M_{x_1''\ldots x_r''} \cap M_{y_1''\ldots y_k''}$ in the subspace $M_{y_1''\ldots y_k''}$. Since the set $(1/a)V' \cap M_{x_1''\ldots x_r''}$ is closed and convex in the strong topology $\beta(E', E)$ and does not contain x_0', then, by the Hahn–Banach theorem, N' can be extended to a hyperplane $\tilde{N}'$ which is closed in the strong topology $\beta(E', E)$ and strongly separates x_0' and $(1/a)V' \cap M_{x_1''\ldots x_r''}$. Since N' is closed in the topology $\sigma(E', E + L(y_1'', \ldots, y_k''))$ and has finite deficiency, the hyperplane $\tilde{N}'$ is also closed in the topology $\sigma(E', E + L(y_1'', \ldots, y_k''))$, i.e. has the form

$$\tilde{N}' = \{x' : \langle x', y_0''\rangle = 1\},$$

where $y_0 = \lambda_0 x_0 + \sum_{i=1}^{k} \lambda_i y_i''$. It is easy to see that $\langle x_0', y_0''\rangle = 1$, $\sup\{\langle x', y_0''\rangle : x' \in (1/a)V' \cap M_{x_1''\ldots x_r''}\} < 1$, or $\sup\{\langle x', y_0''\rangle : x' \in V' \cap M_{x_1''\ldots x_r''}\} < a$. Consider the restriction of the functional y_0'' to the subspace $M_{x_1''\ldots x_r''}$. By the Hahn–Banach theorem, this restriction can be extended to a linear functional $\hat{y}_0''$ defined on all of E', in such a way that $\sup\{\langle x', \hat{y}_0''\rangle : x' \in V'\} < a$. Since the subspace $M_{x_1''\ldots x_r''}$ has finite deficiency, the functional $\hat{y}_0''$ is continuous. The element $x'' = y_0'' - \hat{y}_0''$ obviously belongs to the subspace $L(x_1'', \ldots, x_r'')$,

$$\sup\{\langle x', y_0''\rangle : x' \in V'\} \geqslant \langle x_0', y_0''\rangle = 1 \text{ and } y_0'' \in H.$$

Since $p_{V'^0}(y'') = \sup\{\langle y', y''\rangle : y' \in V'\}$, we obtain $\inf\{p_{V'^0}(x'' - y'') : y'' \in H,\ p_{V'^0}(y'') \geq a,\ x'' \in L(x_1'', \ldots, x_r'')\} \leq p_{V'^0}(\hat{y}_0'') < a$, which contradicts the hypothesis of the lemma. The lemma is proved.

From now on we will assume that E is a Fréchet space, $V_1, \ldots, V_n, \ldots$ a countable base of closed convex balanced neighborhoods of zero in E such that $V_i \supset V_{i+1}$. It is easy to see that their closures V_i^{00} in the weak topology $\sigma(E'', E')$ form a countable neighborhood base of zero in the strong topology of E''.

THEOREM 3. For any nonquasireflexive Fréchet space E the dual space E' contains a subspace M' of characteristic zero that is everywhere dense in the weak topology $\sigma(E', E)$.

Proof. A Fréchet space is barreled, and a closed subspace of a Fréchet space is again a Fréchet space. Hence, in view of Lemmas 1 and 3, we may assume that E is separable. By Proposition 3 of [4, Chap. IV, §2], the dual space E' is the union of bounded sets V_n^0 that are metrizable and separable in the weak topology $\sigma(E', E)$. Let $x'^1, \ldots, x'^n$ be a countable set that is everywhere dense in E' in the weak topology $\sigma(E', E)$. Without loss of generality, we may assume that $x'^i \in V_i^0$. The construction of the subspace M' will be carried out inductively.

1. There exists an element $x_1'' \in E''$ satisfying the following conditions: $x_1'' \in V_0^{00}$; $x_1'' \in M_{x_1''}$; $x_1'' \notin E$; there exists an element $x_1 \in E$ such that $x_1'' - x_1 \in V_1^{00}$. To demonstrate the existence of x_1'' it suffices to apply Lemma 5. The element x_1'' possesses the following properties.

a) There exists an element $x_1''' \in E^0$ such that $\langle x_1'', x_1'''\rangle \neq 0$. This follows from the fact that $x_1'' \notin E$.

b) There exists a neighborhood $W_1 \subset V_1$ of zero such that

$$a_1 = \inf\{p_{W_1^{00}}(x'' - y'') : y'' \in H_{x_1'''},\ p_{W_1^{00}}(y'') \geqslant 1,\ x'' \in L(x_1'')\} \neq 0.$$

Indeed, we can take as W_1 any neighborhood of zero whose closure in the topology $\sigma(E'', E')$ is contained in the set $V_1^{00} \cap \{x'' \in E'' : |\langle x'', x_1'''\rangle| \leq 1\}$. Lemma 6 implies that the closure of $(1/a_1)W_1^0 \cap M_{x_1''} \cap M_{y_1''\ldots y_k''}$ contains $W_1^0 \cap M_{y_1''\ldots y_k''}$ for any set $y_1'', \ldots, y_k''$ in $H_{x_1'''}$.

c) We introduce on the bounded set W_1^0 a metric ρ_1 equivalent to the weak topology $\sigma(E', E)$. The previous property implies the existence of an element $x_1'^1 \in (1/a_1)W_1^0$ for which $\rho_1(x_1'^1, x'^1) \leq 1$.

We will give two illustrations of the second step of the inductive construction.

2. There exists an element $x_2'' \in E''$ satisfying the following conditions: $x_2'' \notin V_0^{00}$; $x_1'^1 \in M_{x_1''x_2''}$; $x_2'' \in H_{x_1'''}$; $\{x'^1, x'^2\} \subset M_{x_2''}$; $x_2'' \notin E + L(x_1'')$; there exists an element $x_2 \in E$ such that $x_2'' - x_2 \in V_2^{00}$. To demonstrate the existence of x_2'' it suffices to apply Lemma 5.

The element x_2'' possesses the following properties.

a) There exists $x_2'^1 \in (1/a_1)W_1^0 \cap M_{x_1''x_2''}$ for which $\rho_1(x'^1, x_2'^1) \leq 1/2$. The existence of such an element follows from Lemma 6, the role of the set $y_1'', \ldots, y_k''$ being played here by x_2''.

b) There exists an element $x_2''' \in E^0$ for which $\langle x_1'', x_2''' \rangle = 0$, $\langle x_2'', x_2''' \rangle \neq 0$. This follows from the fact that $x_2'' \notin E + L(x_1'')$ and E^0 is infinite-dimensional.

c) There exists a neighborhood $W_2 \subset V_2$ of zero for which

$$a_2 = \inf\{p_{W_2^{00}}(x'' - y'') : y'' \in H_{x_1'''x_2'''},\ p_{W_2^{00}}(y'') \geqslant 1,\ x'' \in L(x_1'', x_2'')\} \neq 0.$$

Indeed, we can take as W_2 any neighborhood of zero whose closure in the topology $\sigma(E'', E')$ is contained in the set $V_2^{00} \cap \{x'' \in E'' : |\langle x'', x_1''' \rangle| \leq 1,\ |\langle x'', x_2''' \rangle| \leq 1\}$. Lemma 6 implies that the closure of $(1/a_2)W_2^0 \cap M_{x_1''x_2''} \cap M_{y_1''\ldots y_k''}$ in the weak topology $\sigma(E', E)$ contains $W_2^0 \cap M_{y_1''\ldots y_k''}$ for any set $y_1'', \ldots, y_k''$ in $H_{x_1'''x_2'''}$.

d) We introduce on the bounded set W_2^0 a metric ρ_2 equivalent to the weak topology $\sigma(E', E)$. The previous property implies the existence of an element $x_2'^2 \in (1/a_2)W_2^0 \cap M_{x_1''x_2''}$ for which $\rho_2(x'^2, x_2'^2) \leq 1/2$.

Consider the n-th step of the inductive construction. Suppose that after the (n—1)-st step of the inductive construction we have sets $\{x_1, \ldots, x_{n-1}\} \subset E$; $\{x_1'', \ldots, x_{n-1}''\} \subset E''$; $\{x_1''', \ldots, x_{n-1}'''\} \subset E^0$; $\{x_1'^1, x_2'^1, \ldots, x_{n-1}'^1;\ x_2'^2, x_3'^2, \ldots, x_{n-1}'^2; \ldots;\ x_{n-1}'^{n-1}\} \subset E'$; neighborhoods $W_1 \subset V_1, \ldots, W_{n-1} \subset V_{n-1}$; numbers $a_1, \ldots, a_{n-1}$ unequal to zero; and metrics $\rho_1, \ldots, \rho_{n-1}$ equivalent to the weak topology $\sigma(E', E)$ on the bounded sets $(1/a_1)W_1^0, \ldots, (1/a_{n-1})W_{n-1}^0$ possessing the following properties:

1) $x_i'' \notin V_0^{00}$; $x_i'' - x_i \in V_i^{00}$, $i = 1, \ldots, n-1$;

2) $\{x_1'^1, x_2'^1, \ldots, x_{n-1}'^1; x_2'^2, x_3'^2, \ldots; x_{n-2}'^{n-2}, x_{n-1}'^{n-2}; x_{n-1}'^{n-1}\} \subset M_{x_1''\ldots x_{n-1}''}$; $\{x'^1, \ldots, x'^i\} \subset M_{x_i''}$; $x_i \notin H_{x_i'''}$; $x_i'' \in H_{x_1'''\ldots x_{i-1}'''}$; $x_i'' \notin E + L(x_1'', \ldots, x_{i-1}'')$, $i = 1, \ldots n-1$;

3) the closure of the set $(1/a_q)W_1^0 \cap M_{x_1''\ldots x_q''} \cap M_{y_1''\ldots y_k''}$ contains $W_q^0 \cap M_{y_1''\ldots y_k''}''$ for any set $y_1''\ldots y_k''$ in $H_{x_1'''\ldots x_q''}$ $(q = 1, \overrightarrow{n-1})$;

4)

$$\rho_1(x_1'^1, x'^1) \leqslant 1; \rho_1(x_2'^1, x'^1) \leqslant \frac{1}{2}; \ldots; \rho_1(x_{n-1}'^1, x'^1) \leqslant \frac{1}{n-1},$$

$$\rho_2(x_2'^2, x'^2) \leqslant \frac{1}{2}; \ldots; \rho_2(x_{n-1}'^2, x'^2) \leqslant \frac{1}{n-1},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\rho_{n-1}(x_{n-1}'^{n-1}, x'^{n-1}) \leqslant \frac{1}{n-1}.$$

Then there exists an element $x_n'' \in E''$ satisfying the following conditions: $x_n'' \notin V_0^{00}$; $\{x_1'^1, \ldots, x_{n-1}'^1; x_2'^2, \ldots; x_n'^{n-1}\} \subset M_{x_1''\ldots x_n''}$; $\{x'^1, \ldots, x'^n\} \subset M_{x_n''}$; $x_n'' \in H_{x_1'''\ldots x_{n-1}'''}$; $x_n'' \notin E + L(x_1'', \ldots, x_{n-1})$; there exists an element $x_n \in E$ such that $x_n'' - x_n \in V_n^{00}$. To demonstrate the existence of x_n'' it suffices to apply Lemma 5.

The element x_n'' possesses the following properties.

a) There exists an element $x_n'^q \in (1/a_q)W_q^0 \cap M_{x_1''\ldots x_n''}$ for which $\rho_q(x'^q, x_n'^q) \leq 1/n$ $(q = 1, \overrightarrow{n-1})$. The existence of such elements follows from Lemma 6.

b) There exists an element $x_n''' \in E^0$ for which $\langle x_1'', x_n''' \rangle = 0, \ldots, \langle x_{n-1}'', x_n''' \rangle = 0$, $\langle x_n'', x_n''' \rangle \neq 0$. This follows from the fact that $x_n'' \notin E + L(x_1'', \ldots, x_{n-1}'')$ and E^0 is infinite-dimensional.

c) There exists a neighborhood $W_n \subset V_n$ of zero for which

$$a_n = \inf\{p_{W_n^{00}}(x'' - y'') : y'' \in H_{x_1'''\ldots x_n'''},\ p_{W_n^{00}}(y'') \geqslant 1,\quad x'' \in L(x_1'', \ldots, x_n'')\} \neq 0.$$

Indeed, we can take as W_n any neighborhood of zero whose closure in the topology $\sigma(E'', E')$ is contained in the set $V_n^{00} \cap \{x'' \in E'' : |\langle x'', x_1''' \rangle| \leq 1, \ldots, |\langle x'', x_n''' \rangle| \leq 1\}$. Lemma 6 implies that the closure of the set $(1/a_n)W_n^0 \cap M_{x_1''\dots x_n''} \cap M_{y_1''\dots y_k''}$ in the weak topology $\sigma(E', E)$ contains $W_n^0 \cap M_{y_1''\dots y_k''}$ for any set $y_1'', \dots, y_k''$ in $H_{x_1'''\dots x_n'''}$.

d) We introduce on the bounded set W_n^0 a metric ρ_n equivalent to the weak topology $\sigma(E', E)$. The previous property implies the existence of an element $x_n'^n \in (1/a_n)W_n^0 \cap M_{x_1''\dots x_n''}$ for which $\rho_n(x'^n, x_n'^n) \leq 1/n$.

Thus, it is possible to carry out the inductive step, i.e. to replace n—1 by n in the set of properties 1)-4). We will show that $M' = \bigcap_{i=1}^{\infty} M_{x_1''\dots x_i''}$ is a subspace of characteristic zero that is everywhere dense in the weak topology $\sigma(E', E)$. Property 2) implies that

$$\{x_1'^1, x_2'^1, \dots, x_n'^1, \dots;\ x_2'^2, x_3'^2, \dots, x_n'^2, \dots;\ x_n'^n, \dots\} \subset M'.$$

The sequences $x_n'^n, x_{n+1}'^n, x_{n+2}'^n \dots$ converge to elements x'^n in the weak topology $\sigma(E', E)$, so that the closure of M' in the weak topology $\sigma(E', E)$ contains the elements $x'^1, \dots, x'^n, \dots$, i.e. it coincides with E'. It follows from the construction of $\{x_n\}$ that zero is not a limit point in the strong topology $\beta(E, E')$ of the sequence $\{x_n\}$. But $\{x_n\}$ converges to zero in the topology $\beta(E, M')$. Indeed, consider any neighborhood V_i of zero in the fundamental system $\{V_n\}$ of neighborhoods of zero of E. Then

$$(V_i^0 \cap M')^0 \supset (V_i^0 \cap M_{x_i''})^0 \supset V_i^{00} + L(x_i'').$$

But $x_i'' - x_i \in V_i^{00}$, hence $x_i \in (V_i^0 \cap M')^0 \cap E$.

Thus, the topologies $\beta(E, E')$ and $\beta(E, M')$ are not equivalent. The theorem is proved.

Using Theorems 2 and 3, we can formulate the following criterion for the quasireflexivity of a Fréchet space.

THEOREM 4. For a Fréchet space E to be quasireflexive it is necessary and sufficient that the dual space E' not contain a subspace of characteristic zero that is everywhere dense in the topology $\sigma(E', E)$.

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