FUNDAMENTAL BIORTHOGONAL SYSTEMS AND PROJECTION BASES IN BANACH SPACES

A. N. Plichko

Let X be a Banach space and X* its dual space. A system $x_i, f_i, x_i \in X, f_i \in X^*, i \in I$ (I being some set) is said to be biorthogonal if $f_i(x_j) = \delta_{ij}$ (δ is Kronecker's delta). A biorthogonal system is fundamental if the closed linear hull $[x_i: i \in I] = X$; if, furthermore, the set (f_i) is total on X, then the system x_i, f_i is called an M-basis. An M-basis which can be well-ordered in such a way that for all α there exist collectively bounded projectors $P_\alpha: X \to [x_\beta: \beta < \alpha]$ parallel to $[x_\beta: \beta \geqslant \alpha]$ is called a projection basis. If the norms of all projectors are equal to 1, a basis is said to be monotonic. It is readily seen that the definition of a projection basis stated above is equivalent to the definition in [1, 2]. We say that a biorthogonal system $x_i, f_i, i \in I$, is bounded by a number a if $\sup_{i \in I} \|x_i\| \|f_i\| \leqslant a$. We denote by dens X the weight of X, i.e., the smallest cardinality of everywhere dense subsets of the space X. The cardinal card I is called the cardinality of the biorthogonal system x_i, f_i.

THEOREM. For a nonseparable Banach space X, the following are equivalent:

1) X has a fundamental biorthogonal system;

2) X has a quotient space with a monotonic projection basis of cardinality dens X;

3) X has a quotient space with a projection basis of cardinality dens X;

4) X has a quotient space with a fundamental biorthogonal system of cardinality dens X;

5) for each $\delta > 0$, the space X has a fundamental biorthogonal system bounded by the number $4 + \delta$.

Proof. The implication $1) \Rightarrow 2)$ has been established in [3, 4]. The implications $2) \Rightarrow 3) \Rightarrow 4)$ and $5) \Rightarrow 1)$ are obvious. We will show that $2) \Rightarrow 5)$. Let E = X/Y be a quotient space with a monotonic projection basis x_α, f_α, $0 \leqslant \alpha < \alpha_0$, of cardinality dens X, P_α the corresponding projectors in the space E, and β_0 the greatest limit ordinal less than or equal to α_0. Obviously, the cardinality of the set A of all limit ordinals less than β_0 is equal to dens E. For each $\alpha \in A$, let $E_\alpha = (P_{\alpha+\omega} - P_\alpha) E$ where ω is the first infinite ordinal. We assume 0 to be a limit ordinal, so $E_0 = P_\omega E$. For a given $\varepsilon > 0$ and each $\alpha \in A$ there exists a fundamental biorthogonal system $(e_\alpha^n, \hat{g}_\alpha^n)_{n=1}^\infty$ in the separable space E_α such that $\|e_\alpha^n\| = 1$, $\|\hat{g}_\alpha^n\| < 1 + \varepsilon$, which is not an M-basis [5]. The fact that the system is not an M-basis is not directly stated in the above-mentioned article by Davis and Johnson but it follows from the proof of Theorem 1 and Lemma 1 in [5]; the sequence x_n^* in Lemma 1 of [5] should be chosen not total. A fortiori, the sequence $(e_\alpha^n)_{n=1}^\infty$ is not a basis in the space E_α, in particular, it cannot be equivalent to the standard basis of the space l_1. Extend each functional $\hat{g}_\alpha^n$ to a functional g_α^n defined on the entire space E by the formula

$$g_\alpha^n(e) = \hat{g}_\alpha^n((P_{\alpha+\omega} - P_\alpha) e), \quad e \in E.$$

Since $\hat{g}_\alpha^n((P_{\alpha+\omega} - P_\alpha) e_\beta^m)$ is equal to one only when $\alpha = \beta$ and n = m and is equal to zero otherwise, the obtained system $e_\alpha^n, g_\alpha^n, \alpha \in A, n = 1, \infty$, is biorthogonal. Furthermore, for ordinals $\beta_0 \leqslant \beta < \alpha_0$

$$f_\beta(e_\alpha^n) = g_\alpha^n(x_\beta) = 0.$$

Since $\|P_\alpha\| = 1$ for all α, we have

$$\|g_\alpha^n\| = \sup\{g_\alpha^n(e): e \in E, \|e\| \leqslant 1\} =$$

$$= \sup\{\hat{g}_\alpha^n((P_{\alpha+\omega} - P_\alpha) e): e \in E, \|e\| \leqslant 1\} \leqslant \|\hat{g}_n^\alpha\| \|P_{\alpha+\omega} - P_\alpha\| \leqslant 2(1 + \varepsilon).$$

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Since the system x_α, f_α forms a monotonic projection basis, it is bounded, more precisely, $\|x_\alpha\| \|f_\alpha\| \leqslant 2$ (p. 585 in [6]). To make the construction suitable for applying the theorem below, we adjoin finitely many elements $x_\beta, f_\beta, \beta_0 \leqslant \beta_0 < \alpha_0$ to e_0^n, g_0^n, i.e., we set

$$(e_0^n, g_0^n)_{n=1}^\infty = (e_0^n, g_0^n)_{n=1}^\infty \cup (x_\beta, f_\beta : \beta_0 \leqslant \beta < \alpha_0).$$

Obviously, $[e_\alpha^n]_{\alpha\in A}^{n=1,\infty} = [E_\alpha]_{\alpha\in A} = E$. Thus, there exists a fundamental biorthogonal system e_α^n, g_α^n, $\alpha \in A$, $n = 1, \infty$, in the quotient space E such that

a) $\|e_\alpha^n\| \|g_\alpha^n\| < 2(1+\varepsilon)$,

b) card A = dens E,

c) for each $\alpha \in A$ the sequence $(e_\alpha^n)_{n=1}^\infty$ is not equivalent to the standard basis of the space l_1.

Thus, all conditions of the Lifting Theorem (p. 862 in [6]) are satisfied and, by its conclusion, the space X has a fundamental biorthogonal system bounded by the number $4(1+\varepsilon)+\varepsilon$, i.e., $4+\delta$, for a sufficiently small ε.

Thus, $2) \Rightarrow 5)$. To close the ring of implications, it suffices to establish that $4) \Rightarrow 5)$. Suppose that a space X has a quotient space X/Y with a fundamental biorthogonal system of cardinality dens X. Since $1) \Rightarrow 2)$, the space X/Y has a quotient space (X/Y)/Z with a monotonic projection basis of cardinality dens X. Then the spaces (X/Y)/Z and $X/\varphi^{-1}Z$ are isometric where $\varphi: X \to X/Y$ is the quotient map.

Thus, the space X has a quotient space $X/\varphi^{-1}Z$ with a monotonic projection basis of cardinality dens X. As we have shown, $2) \Rightarrow 5)$, so $4) \Rightarrow 5)$. The theorem is proved.

Remarks. It is known that conditions 1)-5) are almost satisfied for each separable Banach space; more precisely, a monotonic projection basis in condition 2) should be replaced by a δ-monotonic basis, and the constant $4+\delta$ in condition 5) may be replaced by $1+\delta$. For nonseparable spaces they are not always fulfilled [4]. The implication $1) \Rightarrow 5)$ has been stated in [3]; the proof given there relies on the proof of Theorem 2 in [5]. But the proof of this theorem has a gap. Its correct proof (and the implication $1) \Rightarrow 5)$ without the estimate $4+\delta$) has been given by Godun (p. 862 in [6]).

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