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ON PROJECTIVE RESOLUTIONS OF THE IDENTITY OPERATOR AND MARKUŠEVIČ BASES

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Let X be a Banach space and X^* its dual. A subspace $M \subset X^*$ is said to be λ-*norming* ($\lambda > 0$) if the Dixmier characteristic

$$\inf \sup\{f(x): f \in M, \|f\| \leq 1\}$$

equals λ, where the infimum is taken over all $x \in X$ with $\|x\| = 1$. When we are not interested in the value of λ, we speak simply of *norming subspaces*. A system x_i, f_i, $x_i \in X$, $f_i \in X^*$, $i \in J$ (J some set), is called a *Markuševič basis* (an *M-basis*) if $f_i(x_j) = \delta_{ij}$ (δ_{ij} the Kronecker symbol), the norm-closed linear space $[x_i,\ i \in J]$ coincides with X, and the subspace $[x_i\colon i \in J] = X$ is total on X. If, moreover, F is a λ-norming subspace, then the system is called a λ-*norming M-basis*.

NOTATION. ω is the first infinite ordinal, and Ω is the first uncountable ordinal; $\bar{\bar{\alpha}}$ is the cardinality of the ordinal α; dens X is the weight of X, i.e., the smallest cardinality of a dense subset of X; $S(X)$ is the unit sphere; $d(M, N)$ is the distance between the sets M and N in the metric of X; and $M^\perp$ is the corresponding annihilator.

A *projective resolution of the identity operator I* is defined to be a set of collectively bounded projections $P_\beta\colon X \to X$, $\omega \leq \beta \leq \alpha$, where α is the first ordinal for the weight of X, such that if $\omega \leq \beta, \gamma \leq \alpha$, then:

1) $P_\beta P_\gamma = P_\gamma P_\beta = P_{\min(\gamma,\beta)}$;
2) dens $P_\beta X \leq \bar{\bar{\beta}}$;
3) $P_\beta X = [P_{\gamma+1}X\colon \gamma < \beta]$;
4) $P_\alpha = I$.

For every norming M-basis we can construct a projective resolution [1], [2]. It is not known whether or not every WCG-space (a space that is the closed linear span of some weakly compact subset) has a norming M-basis. We introduce the more general concept of a countably λ-norming M-basis, for which we can construct a projective resolution. Moreover, every M-basis of a WCG-space is countably 1-norming. We show also that for every M-basis x_i, f_i we can construct an M-basis y_i, g_i with $\sup_i \|y_i\| \|g_i\| < 19$ and $\operatorname{lin}(g_i\colon i \in J) = \operatorname{lin}(f_i)$.

An M-basis x_i, f_i is said to be *countably* λ-*norming* if the subspace $G \subset X^*$ consisting of the elements g with countable support $\operatorname{supp} g = \{i \in J\colon g(x_i) \neq 0\}$ is λ-norming. Obviously, $F \subset G$. Recall that the *weak* sequential closure* of a set $M \subset X^*$ is defined to be the collection $M_{(1)}$ of all limits of sequences in M that converge weak* in X^*. By induction, the weak* sequential closure of order β is defined to be $M_{(\beta)} = \bigcup_{\gamma<\beta}(M_{(\gamma)})_{(1)}$ for any ordinal β. Since the weak* limit of a sequence of elements with countable support also has

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countable support, $F_{(\Omega)} \subset G$. For a WCG-space X and any total subspace $M \subset X^*$ we always have $M_{(\Omega)} = X$ ([3], p. 50); therefore, any M-basis of a WCG-space is countably 1-norming.

LEMMA 1. *Let x_i, f_i $(i \in J)$ be a countably 1-norming M-basis. For every infinite subset $\mathfrak{J} \subset J$ there is a subset $\mathfrak{J} \subset \mathfrak{J}' \subset J$ with the cardinality of $\mathfrak{J}$ such that $d(S(X_{\mathfrak{J}}), X^{\mathfrak{J}'}) = 1$, where $X_{\mathfrak{J}} = [x_j: j \in \mathfrak{J}]$ and $X^{\mathfrak{J}} = [x_j: j \notin \mathfrak{J}]$.*

PROOF. Since the subspace G is 1-norming, there is a subset $H \subset S(G)$ with the cardinality of $\mathfrak{J}$ such that for any $x \in X_{\mathfrak{J}}$

$$\sup\{h(x): h \in H\} = \|x\|. \tag{1}$$

Let $\mathfrak{J}' = \mathfrak{J} \cup \operatorname{supp} H$. The set $\operatorname{supp} H$ is countable for any $h \in H$; therefore, $\operatorname{card} \mathfrak{J}' = \operatorname{card} \mathfrak{J}$. The last condition of the lemma follows from (1).

THEOREM 1. *Let x_i, f_i be a countably λ-norming M-basis. Then X has a projective resolution P_β, $\omega \leqslant \beta \leqslant \alpha$, with norms $\|P_\beta\| \leqslant \lambda^{-1}$, and for any β there is a subset $J_\beta \subset J$ such that $P_\beta X = X_{J_\beta}$ and $\ker P_\beta = X^{J_\beta}$.*

PROOF. Suppose temporarily that $\lambda = 1$. We arrange the elements x_i in a transfinite sequence x_β, $\omega < \beta < \alpha$ (recall that α is the first ordinal of the weight of X). Let $P_\omega \equiv 0$. If the projections P_γ have already been constructed for $\gamma < \beta$ and the latter is a limit ordinal, then let $J_\beta = \bigcup_{\gamma<\beta} J_\gamma$. By the induction hypothesis, the norms of the projections P_γ do not exceed 1; therefore, the norm of the projection onto X_{J_β} parallel to X^{J_β} does not exceed 1. Since $\operatorname{dens} P_\gamma X \leqslant \bar{\bar{\gamma}} < \bar{\bar{\beta}}$, we have $\operatorname{dens} P_\beta X \leqslant \bar{\bar{\beta}} \times \bar{\bar{\beta}} = \bar{\bar{\beta}}$. But if β is not a limit ordinal, then we construct a sequence of subsets $J_\beta^n \subset J$ of cardinality $\leqslant \bar{\bar{\beta}}$, $n = 1, 2, \ldots$, such that

1) $x_\beta \in (x_i: i \in J_\beta^1)$, $J_{\beta-1} \subset J_\beta^1$, $J_\beta^n \subset J_\beta^{n+1}$,
2) $d(S(X_{J_\beta^n}), X^{J_\beta^{n+1}}) = 1$.

Let $J_\beta^1 = J_{\beta-1} \cup \{x_\beta\}$. Taking $\mathfrak{J} = J_\beta^1$ in Lemma 1, we find a set $J_\beta^2 = \mathfrak{J}'$ with conditions 1 and 2. If J_β^n has been constructed, then we take $\mathfrak{J} = J_\beta^n$ in Lemma 1 and construct $J_\beta^{n+1} = \mathfrak{J}'$ with conditions 1 and 2.

Let $J_\beta = \bigcup_{n=1}^\infty J_\beta^n$. Condition 2 implies that $d(S(X_{J_\beta}), X^{J_\beta}) = 1$. Therefore, the projection P_β of X onto X_{J_β} parallel to X^{J_β} exists and has norm 1. Since $\operatorname{card} J_\beta^n \leqslant \bar{\bar{\beta}}$, it follows that $\operatorname{dens} P_\beta X = \operatorname{card} J_\beta \leqslant \bar{\bar{\beta}} \times \bar{\bar{\omega}} = \bar{\bar{\beta}}$. The remaining conditions of the theorem are also easily verified. The first inclusion in 1) gives us that $J_\alpha = J$, i.e., P_α is the identity operator.

Accordingly, if $\lambda = 1$, then the resolution required in the theorem has been constructed with the norms of the projections no greater than 1. If $1 \neq \lambda > 0$, then we introduce the equivalent norm $|||x||| = \sup\{f(x): f \in (S)G\}$ on X. Since $|||x||| \leqslant \|x\| \leqslant \lambda^{-1}|||x|||$ and the subspace G is 1-norming for the space $(X, |||\cdot|||)$, we first construct a resolution P_β as required in the theorem for $(X, |||\cdot|||)$ with $\lambda = 1$, and then observe that $\|P_\beta\| \leqslant \lambda^{-1}$.

LEMMA 2. *Let x_i, f_i, $i \in J$, be an arbitrary M-basis. For every infinite subset $\mathfrak{J} \subset J$ there is a subset $\mathfrak{J} \subset \mathfrak{J}' \subset J$ with the cardinality of $\mathfrak{J}$ such that $d(S(F_{\mathfrak{J}}), F^{\mathfrak{J}'}) = 1$, where $F_{\mathfrak{J}} = (X^{\mathfrak{J}})^\perp$ and $F_{\mathfrak{J}'} = (X^{J'})^\perp$.*

PROOF. The subspace $F_{\mathfrak{J}}$ is isometric to the dual of the quotient space $X/X^{\mathfrak{J}}$, which has weight $\operatorname{card} \mathfrak{J}$. Therefore, there is a subset $\hat{Y} = S(X \setminus X^{\mathfrak{J}})$ with the cardinality of $\mathfrak{J}$ such that for any $f \in F_{\mathfrak{J}}$

$$\sup\{f(\hat{y}): \hat{y} \in \hat{Y}\} = \|f\|.$$

For every $\hat{y} \in \hat{Y}$ let $(y_n)_1^\infty$ be a sequence of representatives of the class $\hat{y}$ with $\|y_n\| \to 1$. Then the set $Y = \{y_n/\|y_n\|: n = 1, 2, \ldots, y_n \in \hat{y}, \hat{y} \in \hat{Y}\}$ has the cardinality of $\mathfrak{J}$, and for every $f \in F_{\mathfrak{J}}$

(2) $$\sup\{f(y): y \in Y\} = \|f\|.$$

Let $\mathfrak{J}' = \mathfrak{J} \cup \operatorname{supp} Y$, where $\operatorname{supp} Y = \{i \in J: \exists y \in Y, f_i(y) \neq 0\}$. Since supp y is countable for each $y \in Y$, we have card $\mathfrak{J}'$ = card $\mathfrak{J}$. The last condition of the lemma follows from (2).

LEMMA 3. *Let x_i, f_i be an M-basis for the space X. The index set can be represented in a form $J = \bigcup J_\beta$, $\omega \leqslant \beta < \alpha$, such that for any β and γ*

1) $J_\beta \subset J_\gamma$ *for* $\underline{\underline{\beta}} < \gamma$,
2) card $J_\beta = \bar{\bar{\beta}}$,
3) $J_{\beta+1} \setminus J_\beta \neq \varnothing$,
4) $d(S(F_{J_\beta}), F^{J_{\beta+1}}) = 1$.

PROOF. We arrange the elements $i \in J$ in a transfinite sequence β: $\omega \leqslant \beta < \alpha$. Let $J_\omega = \omega$. If the sets J_γ have been chosen for all $\gamma < \beta$ and β is a limit ordinal, then let $J_\beta = \bigcup_{\gamma<\beta} J_\gamma$. But if β is not a limit ordinal, then card $J_{\beta-1} < \bar{\bar{\alpha}}$ = card J. We take some element $i \in J \setminus J_{\beta-1}$ and let $\mathfrak{J} = J_{\beta-1} \cup \{i\} \cup \{\beta - 1\}$. For the set $\mathfrak{J}$ we choose a set $J_\beta = \mathfrak{J}'$ satisfying the conditions of Lemma 2. The conditions of Lemma 3 are easy to verify.

THEOREM 2. *Let X be a (nonseparable) space with an M-basis x_i, f_i, $i \in J$. Then X has an M-basis x'_i, f'_i, $i \in J$, for which*

$$\operatorname{lin}(x'_i: i \in J) = \operatorname{lin}(x_i), \quad \operatorname{lin}(f'_i: i \in J) = \operatorname{lin}(f_i),$$

and there exists a subset $\mathfrak{J} \subset J$ with the cardinality of J such that $\sup_{j\in\mathfrak{J}} \|x'_j\|, \|f'_j\| < 3$.

PROOF. Without loss of generality it can be assumed that $\|f_i\| = 1$ for all i. Let J_β be the sets in Lemma 3. For each ordinal of the form $\gamma = \delta + 3n - 2$ with δ a limit ordinal and $n > 0$ we choose an element $f'_\gamma = f_\gamma \in (f_i: i \in J_{\gamma+1} \setminus J_\gamma)$. The subspaces $F_{J_{\gamma-1}}$ and $F^{J_{\gamma+2}}$ are weak* closed, and there is a nonzero opening between them by property 4) in Lemma 3. Therefore, the sum $G_\gamma = F_{J_{\gamma-1}} + F^{J_{\gamma+2}}$ is also weak* closed. Let $g = g_1 + g_2$, $g_1 \in F_{J_{\gamma-1}}$, $g_2 \in F^{J_{\gamma+2}}$. Then $\|f'_\gamma - g\| \geqslant \|f'_\gamma - g_1\|$ by 4) in Lemma 3. Considering separately the cases $\|g_1\| > 1/2$ and $\|g_1\| \leqslant 1/2$ and using 4) in Lemma 3, we get that $\|f'_\gamma - g\| \geqslant 1/2$. Accordingly,

$$d(f'_\gamma, G_\gamma) \geqslant 1/2.$$

By the Hahn-Banach theorem, there exists an element $y_\gamma \in X$, $y_\gamma \in G_\gamma^\perp$, with norm $\|y_\gamma\| < 2 + 1/2$ and such that $f'_\gamma(y_\gamma) = 1$. Since the subspace $\operatorname{lin}(x_i: i \in J_{\gamma+2} \setminus J_{\gamma-1})$ is dense in the annihilator $G_\gamma^\perp \subset X$, there is an element $x'_\gamma \in \operatorname{lin}(x_i: i \in J_{\gamma+2} \setminus J_{\gamma-1})$ such that $f'_\gamma(x'_\gamma) = 1$ and $\|x'_\gamma\| < 3$. For the elements of the set $(x_i, f_i: i \in J_{\gamma+2} \setminus J_{\gamma-1}) \setminus \{x_\gamma, f_\gamma\}$ we let

(3) $$x'_i = x_i, \quad f'_i = f_i - f_i(x'_\gamma) f'_\gamma.$$

Since J splits into the disjoint classes $(J_{\gamma+2} \setminus J_{\gamma-1})$, when γ runs through the ordinals of the form $\delta + 3n - 2$ with δ a limit ordinal, this definition is unambiguous. We show that the system $(x'_i, f'_i) \cup (x'_\gamma, f'_\gamma)$ satisfies the conditions of the theorem.

Biorthogonality. Take any element x'_i. The index i belongs to some set $J_{\gamma+2} \setminus J_{\gamma-1}$. We go through the various possibilities. If j does not belong to $J_{\gamma+2} \setminus J_{\gamma-1}$, then by (3) we have

$f_j' \in G_\gamma$, and $f_j'(x_j') = 0$ both when $x_i' = x_\gamma'$ and when $x_i' \neq x_\gamma'$. If $j \in J_{\gamma+2} \setminus J_{\gamma-1}$ and $x_i' = x_\gamma'$, then $f_\gamma'(x_\gamma') = 1$ by the construction of x_γ', and for $f_j' \neq f_\gamma'$

$$f_j'(x_\gamma') = f_j(x_\gamma') - f_j(x_\gamma')f_\gamma'(x_\gamma') = 0.$$

But if $x_i' \neq x_\gamma'$, then by (3) we have $f_i'(x_i') = 1$ and $f_j'(x_i') = 0$ for $j \neq i$.

The equality $\mathrm{lin}(f_i') = \mathrm{lin}(f_i)$ follows immediately from the definition (3) when it is considered that $f_\gamma' = f_\gamma$.

Each element x_γ' is a linear combination of the elements $(x_i\colon i \in J_{\gamma+2} \setminus J_{\gamma-1})$, and, moreover, since $f_\gamma'(x_\gamma') = 1$, the element x_γ appears in this combination with a nonzero coefficient. Therefore, $\mathrm{lin}(x_i') = \mathrm{lin}(x_i)$.

The set $\mathfrak{J}$ of ordinals $\omega \leqslant \gamma < \alpha$ of the form $\delta + 3n - 2$ with δ a limit ordinal has cardinality $\overline{\overline{\alpha}} = \mathrm{card}\, J$, and $\|f_\gamma'\|\,\|x_\gamma'\| = \|x_\gamma'\| < 3$ for any γ. Therefore, the last condition of the theorem also holds.

COROLLARY. *If the Banach space X has an M-basis x_i, f_i, $i \in J$, then it has an M-basis y_i, g_i, $i \in J$, such that*

$$\mathrm{lin}(y_i\colon i \in J) = \mathrm{lin}(x_i\colon i \in J), \qquad \mathrm{lin}(g_i\colon i \in J) = \mathrm{lin}(f_i\colon i \in J)$$

and $\sup_i \|y_i\|\,\|g_i\| < 19$.

PROOF. For the system (x_i, f_i) we construct a system (x_i', f_i') satisfying the conditions of Theorem 2. Let J be partitioned into countably many disjoint subsets, $J = \bigcup \mathfrak{J}_\delta$, in such a way that each $\mathfrak{J}_\delta$ contains infinitely many elements of $\mathfrak{J}$. On the basis of Proposition 1 in [4] we can construct for each δ a system such that

$$\mathrm{lin}(y_\delta^n\colon n = \overline{1, \infty}) = \mathrm{lin}(x_i'\colon i \in J_\delta), \qquad \mathrm{lin}(g_\delta^n\colon n = \overline{1, \infty}) = \mathrm{lin}(f_i'\colon i \in J_\delta),$$

$$\sup_n \|y_\delta^n\|\,\|g_\delta^n\| \leqslant 3\left(1 + \sqrt{2}\right)^2 + 1.$$

The system $(y_i, g_i) = \bigcup_{n,\delta}(y_\delta^n, g_\delta^n)$ works.

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