SOME PROPERTIES OF JOHNSON – LINDENSTRAUSS SPACE

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Let X be a Banach space and X^* its dual. The system (x_i, f_i), $x_i \in X$, $f_i \in X^*$, $i \in I$ (where I is some set) is called a total biorthogonal system if $f_i(x_j) = \delta_{ij}$ (where δ_{ij} is the Kronecker symbol) and the set (f_i) is total on X. If, in addition, (x_f) is total on X^*, then it is an M-basis. An M-basis which can be totally ordered so that for each α there exist totally bounded projections P_α from X onto the closed linear hull of $[x_\beta: \beta < \alpha]$ parallel to $[x_\beta: \beta \geq \alpha]$ is called a projection basis. Every Banach space has an infinite-dimensional subspace with a basis (see [1]). It is natural to ask whether every Banach space X has a subspace Y with a projection basis of weight dens Y = dens X. For certain classes of Banach spaces the answer is affirmative (see [2]). We show that in the general case it is not.

LEMMA 1. Suppose that X can be decomposed into a direct sum of closed separable and reflexive subspaces. Then every subspace and every factor-space of X has the same property.

Proof. Suppose that $X = Y \oplus Z$, where Y is separable, Z is reflexive, and $X_1 \subset X$. Then X will be a WCG-space and therefore there exist a separable subspace $Y \subset Y_0 \subset X$ and a continuous projection P: $X \to Y_0$, $PX_1 \subset X_1$ (see [3, p. 140]). We denote the projection from X onto Y parallel to Z by Q. Then QP is a projection from X onto Y parallel to $\ker P \dotplus (\ker Q \cap Y_0)$. The subspace ker QP is isomorphic to X/Y and isomorphic to Z and hence is reflexive. This means that its subspace Z_0 = ker P is reflexive, and therefore so also is $Z_0 \cap X_1$. Thus $X_1 = Y_1 \oplus Z_1$, where $Y_1 = PX_1 \subset Y_0$ is separable, and $Z_1 = Z_0 \cap X_1$ is reflexive. Since, by construction, $X = Y_0 \oplus Z_0$, $X_1 = Y_1 \oplus Z_1$, $Y_1 \subset Y_0$ $Z_1 \subset Z_0$, it follows that $X/X_1 = Y_0/X_1 + Z_0/X_1$, $Y_0/(Y_1 \oplus Z_1) \cap Z_0/(Y_1 + Z_1) = 0$, $Y_0/(Y_1 \oplus Z_1)$ is isomorphic to Y_0/Y_1 and consequently is separable, and $Z_0/(Y_1 \oplus Z_1)$ is isomorphic to Z_0/Z_1 and therefore reflexive.

Let X be a space with projection basis (x_α, f_α). For each α we put $X_\alpha = [x_\beta: \beta < \alpha]$, $X^\alpha = [x_\beta, \beta \geqslant \alpha]$, $F_\alpha = [f_\beta: \beta < \alpha]$, $F^\alpha = [f_\beta: \beta \geqslant \alpha]$.

LEMMA 2. $X = X_\alpha \oplus X^\alpha$, $X^* = (X_\alpha)^\perp \oplus (X^\alpha)^\perp$, $[F_\alpha]_* = (X^\alpha)^\perp$ and $[F^\alpha]_* = (X_\alpha)^\perp$, where $[\ ]_*$ is weak * closure.

We will not define the Johnson–Lindenstrauss space (JL), rather referring the reader to [4]. We need only its properties:

1) JL is nonseparable, and there exists a countable set in JL* which is total on JL;

2) $JL^* = G \oplus H$, where G is separable and H is reflexive;

3) JL is not isomorphic to a subspace of l_∞.

THEOREM 1. JL does not have a subspace with an uncountable projection basis.

Proof. Let X be a subspace with an uncountable projection basis $(x_\alpha, f_\alpha, 1 \leq \alpha < \Omega)$. We identify the dual space with the factor-space by the annihilator: $X^* = JL^*/X^\perp$. According to Lemma 1 and property 2), the subspace $F = [f_\alpha: \alpha < \Omega] \subset X^*$ has the form $F = G_1 \oplus H_1$, where G_1 is separable and H_1 is reflexive. From Lemma 2 it follows that, for each α, $F = F_\alpha \oplus F^\alpha$. By virtue of the separability of G_1, there exists a countable ordinal γ for which the (separable) subspace $F_\gamma \supset G_1$. We show that F^γ is reflexive. Indeed, if P is the projection on F_γ parallel to F^γ, and Q is the projection on G_1 parallel to H_1, then QP is a projection on G_1 and $\ker QP \supset F^\gamma$.

The subspace ker QP is isomorphic to F/G_1, isomorphic to H_1, and consequently reflexive.

Since a reflexive subspace of the dual space is weak * closed (see [5]), it follows by Lemma 2 that $X^* = [F_\gamma]_* \oplus F^\gamma$.

The reflexive subspace $F^\gamma = (X_\gamma)^\perp$ is dual to X^γ, which is therefore also reflexive and nonseparable. There can exist no countable total set of functionals on such a subspace, which contradicts property 1).

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We recall that a subspace $F \subset X^*$ is said to be norming if $|||x||| = \sup\{|f(x)| : f \in F, \|f\| = 1\}$ is a norm which is equivalent to the original norm $\|\ \|$. The biorthogonal system (x_i, f_i) is said to be norming if the subspace $[f_i, i \in I]$ is norming. For each Banach space there exists a total biorthogonal system (see [6]), but not always a norming one.

THEOREM 2. The space JL does not have a norming biorthogonal system.

Proof. Let (x_i, f_i), $i \in I$, be a norming biorthogonal system in JL. The set I is not countable because, as was pointed out to us by I. Singer, it follows directly from property 3) that there are no separable norming subspaces in JL*. For $X = [x_i, i \in I]$, we denote the images of the functionals f_i under the factor-transformation $JL^* \to JL^*/X^\perp$ by $\hat{f}_i$. It is easy to see that the set $(x_i, \hat{f}_i)$ forms a norming M-basis of the space X. But each subspace with a norming M-basis contains a subspace of the same weight with projection basis (see [2]). But this contradicts Theorem 1.

Remark. Our definition of a projection basis is formally different from that in [2, 7]. It is not difficult to verify that they are in fact equivalent.

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c-CONVEX BANACH LATTICES

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Pisier [1] has proved the following theorem.

THEOREM. A Banach space X is superreflexive iff it is uniformly isomorphic to a convex space Y whose modulus of uniform convexity $\delta_Y(\varepsilon)$ admits a power lower bound: $\delta_Y(\varepsilon) \geq K\varepsilon^p$ $(p \geq 2)$.

We show below that if we restrict ourselves to Banach lattices, then it is possible to introduce a new numerical characteristic of a Banach space (the "modulus of complex uniform convexity") which allows us to prove an analogous result for c-convexity.

Definitions. 1. A Banach space X is superreflexive if no nonreflexive Banach space E is finitely represented in it.

2. A Banach space E is c-convex if the space c_0 is finitely represented in it.

3. A normed space E is λ-finitely represented in a normed space X if for each finite-dimensional subspace $F \subset E$ there exists a subspace of the same dimension $Y \subset X$ such that $d(F, Y) \leq \lambda$ [where $d(F, Y)$ is the Banach–Mazur distance]. A space E is finitely represented in X if it is λ-finitely represented in X for any $\lambda > 1$.

4. A Banach lattice (B.l.) is a vector lattice which is at the same time a Banach space with the following relation between order and norm:

$$0 < |x| < |y| \Rightarrow \|x\| \leqslant \|y\|.$$

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