In this note we establish a result which has as one simple consequence the existence of subspaces with special properties in certain Banach spaces. The following proposition is well known.

**Lemma.** Let $X$ be a separable locally convex topological vector space, let $Y$ be a finite-dimensional subspace of it, and let $V$ be a closed, bounded, convex subset of $X$ with $V \cap Y = \emptyset$. Then there exists a functional $f$ in the dual space $X^*$ which separates $V$ and $Y$.

**Theorem 1.** Let $X$ be a Banach space, let $Y$ be an infinite-dimensional linear manifold in $X$, and let $\tau$ be a separable locally convex topology which is weaker than the norm topology. Let $V_n$ be a sequence of convex, bounded, $\tau$-closed subsets such that $V_n \cap Y = \emptyset$ for each $n$. Then $Y$ contains an infinite-dimensional subspace $Z$ whose $\tau$-closure does not intersect $V_n$ for any $n$.

**Proof.** We choose $y_1 \in Y$. Using the lemma, we separate the closed linear hull $[y_1]$ of $y_1$ and $V_1$ with the $\tau$-continuous functional $f_1$. We choose $y_2$ from the annihilator $\text{ann}(f_1)$. We separate $[y_1, y_2]$ and $V_2$ with the functional $f_2$. We take $y_3 \in (y_1, f_2)$. The process of construction and the proof are now clear.

**Corollary 1.** Let $X$ be a separable Banach space, and let $F \subset X^*$ be a linear manifold. Then there exists a subspace $G \subset F$ whose weak * closure $\overline{G}$ is contained in the weak * closure $F^*$ of $F$.

**Proof.** As is well known, $F^* = \text{lin} B(F)$, where $B(F) = \{ f \in F: \| f \| \leq 1 \}$ (see [1]). The ball $B(X^*)$ is metrizable and separable in the weak * topology with respect to the metric $d(f, g) = 2 \sum_{k=1}^{\infty} (f - g)(x_k)^2$, where $x_k$ is a sequence which is dense in the sphere $S(X)$. Any ball in this metric is convex. Therefore $B(X^*) \setminus \overline{B(F)}$ can be covered by a collection of bounded weak * closed convex subsets $V_n \subset B(X^*)$, $V_n \cap \overline{B(F)} = \emptyset$. Putting $X = X^*$, $\tau = \sigma(X^*, X)$, $Y = F$ in Theorem 1, we get the existence of an infinite-dimensional subspace $G \subset F$, $\overline{G} \cap (B(X^*) \setminus \overline{B(F)}) = \emptyset$. If $f \in F^*$, $\| f \| = 1$, then $f \notin \overline{B(F)}$ and $f \notin \overline{\sigma(X^*, X)}$.

**Corollary 2.** Let $X^*$ be separable. Every subspace $F \subset X^*$ which is closed with respect to the norm contains a weak * closed infinite-dimensional subspace.

In Theorem 1, we put $X = X^*$, $\tau = \sigma(X^*, X)$, $Y = F$. We cover $X^* \setminus F$ with a collection of balls $V_n$. $V_n \cap F = \emptyset$, and apply the theorem.

The following result is stated in [2] and proved in [3] with the help of a sequence which is a basis.

**Corollary 3.** Suppose that $X^{**}$ is separable. Then $X$ and $X^*$ contain infinite-dimensional reflexive subspaces.

**Proof.** If we cover $X^{**} \setminus X$ with a sequence of balls, we get the first part of the statement. For the proof of the second part, we note that $X^{**} = X \oplus X^{**}$. As in Corollary 1, we can cover $C = B(X^*) \setminus 0$ with a sequence of weak * closed convex subsets of $C$. Applying Theorem 1, we get a subspace $F \subset X^*$, the $\sigma(X^{***}, X^{**})$-closure $\overline{F}$ of which does not intersect $C$. By Corollary 2, we can choose a $\sigma(X^*, X)$-closed subspace $H$ in $\overline{F} \cap X^*$. It is not difficult to see that $H \subset X^*$, i.e., $H$ is reflexive.

We recall that closed subspaces $X$ and $Y$ of a Banach space $E$ are said to be quasicomplementary if $X \cap Y = 0$ and $[X + Y] = E$.

**Theorem 2.** Let $X$ and $Y$ be quasicomplementary, but not complementary, subspaces of $E$. Then there exists a closed subspace $Y_1 \supset Y$, $\dim Y_1/Y = \infty$, $Y_1 \cap X = 0$. 

This result has been proved in [4-6] under a variety of assumptions (reflexivity, the property of being weakly compactly generated, separability). We show that the argument in [6] carries over to the arbitrary case.

Proof. According to [6], there exists in E a countable-dimensional subspace Z such that for any \( z \in Z, z \neq 0 \), there are no bounded sequences \( x_n \in X \) and \( y_n \in Y \) such that \( \| x_n + y_n - z \| \to 0 \). The space \( [Z + Y] / Y \) is separable; therefore it is possible to choose a sequence \( f_i \in E^\ast \) for which \( Y \subset \langle f_i \rangle^\perp \) and \( \langle f_i \rangle^\perp \cap M = 0 \), where \( M = X \cap [Z + Y] \). Let

\[
W_i^j = \{ x \in M: \| x \| \leq i, f_i(x) \geq 1/j \}.
\]

By virtue of the choice of the functionals \( f_i \), there exist bounded sets \( W_i^j \) at a nonzero distance from \( Y \) such that \( \bigcup_i W_i^j = M \setminus 0 \). Indexing the \( W_i^j \) we get a sequence \( V_n \). Furthermore, applying exactly the same argument as in [6], we can construct sequences of elements \( z_n \in Z \) and closed hyperplanes \( H_n \subset E \) such that for every \( n \):

1. \( z_n \notin \langle z_i \rangle^\perp \),
2. \( V_n \cap H_n \neq \emptyset \), and
3. \( Z_n = [Y + \langle z_i \rangle^\perp] \) is the one sought.

We say that the Banach space \( X \) is normally imbedded in the Banach space \( Y \) if \( X \subseteq Y \), \( \| x \|_Y \leq \| x \|_X \) for \( x \in X \), and the linear manifold \( X \) is dense in \( Y \) but does not coincide with \( Y \). As was shown by M. I. Kadets, we have the following consequence of Theorem 2.

**Corollary 4.** Suppose that \( X \) is normally imbedded in \( Y \). Then there exists a closed infinite-dimensional subspace \( Z \subset Y \), \( Z \cap X = 0 \).

We get it as a consequence of Theorem 1. We choose a separable subspace \( X_1 \subset X \) on which the norms \( \| \cdot \|_X \) and \( \| \cdot \|_Y \) are not equivalent. Let \( Y_1 \) be the closure of \( X_1 \) in \( Y \) and let \( X_2 = X \cap Y_1 \). Then \( (X_2, \| \cdot \|_X) \) is normally imbedded in \( (Y_1, \| \cdot \|_Y) \), and \( Y_1 \) is separable. We denote the closure of the unit ball in \( X_2 \) with respect to the norm in \( Y \) by \( C \). From Lemma 1.2 of Chap. 1 of [7] it is not difficult to get (see [8]) the existence of an infinite-dimensional subspace \( Z_1 \subset Y_1, Z_1 \cap C = 0 \). We cover the set \( C \setminus 0 \) with a sequence of balls \( W_n \) of the space \( Y_1, W_n \neq 0 \). Then the collection \( V_n = W_n \cap C \) covers \( C \setminus 0 \), and we can apply Theorem 1.

**Corollary 5.** Let the separable space \( X \) be normally imbedded in \( Y \). Then \( Y^* \subset X^* \), and there exists a \( \sigma(X^*, X) \)-closed infinite-dimensional subspace \( F, F \cap Y^* = 0 \) in \( X^* \).

**Proof.** The ball \( B(Y^*) \) is weakly * compact and therefore closed in the topology \( \sigma(X^*, X) \). As in the preceding corollary, there exists an infinite-dimensional subspace \( F \) which does not intersect \( B(Y^*) \). We cover the set \( B(Y^*) \setminus 0 \), which is separable and metrizable in the topology \( \sigma(X^*, X) \), with a sequence of sets \( W_n \) which are convex and \( \sigma(X^*, X) \)-closed, and then putting \( v_n = W_n \cap B(Y^*) \), we apply Theorem 1.

**Literature Cited**