SELECTION OF SUBSPACES WITH SPECIAL PROPERTIES IN A BANACH SPACE AND SOME PROPERTIES OF QUASICOMPLEMENTS

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In this note we establish a result which has as one simple consequence the existence of subspaces with special properties in certain Banach spaces. The following proposition is well known.

LEMMA. Let X be a separable locally convex topological vector space, let Y be a finite-dimensional subspace of it, and let V be a closed, bounded, convex subset of X with $V \cap Y = \varnothing$. Then there exists a functional f in the dual space X* which separates V and Y.

THEOREM 1. Let X be a Banach space, let Y be an infinite-dimensional linear manifold in X, and let τ be a separable locally convex topology which is weaker than the norm topology. Let V_n be a sequence of convex, bounded, τ-closed subsets such that $V_n \cap Y = \varnothing$ for each n. Then Y contains an infinite-dimensional subspace Z whose τ-closure does not intersect V_n for any n.

Proof. We choose $y_1 \in Y$. Using the lemma, we separate the closed linear hull $[y_1]$ of y_1 and V_1 with the τ-continuous functional f_1. We choose y_2 from the annihilator $f_1^{\top}$. We separate $[y_1, y_2]$ and V_2 with the functional f_2. We take $y_3 \in (f_1, f_2)^{\top}$. The process of construction and the proof are now clear.

COROLLARY 1. Let X be a separable Banach space, and let $F \subset X^*$ be a linear manifold. Then there exists a subspace $G \subset F$ whose weak * closure $\overline{G}$ is contained in the weak * closure F^s of F.

Proof. As is well known, $F^s = \text{lin}\,\overline{B(F)}$, where $B(F) = \{f \in F: \|f\| \leqslant 1\}$ (see [1]). The ball $B(X^*)$ is metrizable and separable in the weak * topology with respect to the metric $\rho(f, g) = \Sigma 2^{-k} |(f - g)(x^k)|$, where x^k is a sequence which is dense in the sphere S(X). Any ball in this metric is convex. Therefore $B(X^*) \setminus \overline{B(F)}$ can be covered by a collection of bounded weak * closed convex subsets $V_n \subset B(X^*)$, $V_n \cap \overline{B(F)} = \varnothing$. Putting $X = X^*$, $\tau = \sigma(X^*, X)$, Y = F in Theorem 1, we get the existence of an infinite-dimensional subspace $G \subset F$, $\overline{G} \cap (B(X^*) \setminus \overline{B(F)}) = \varnothing$. If $f \notin F^s$, $\|f\| = 1$, then $f \notin \overline{B(F)}$ and $f \notin \overline{G}$.

COROLLARY 2 (see [3]). Let X* be separable. Every subspace $F \subset X^*$ which is closed with respect to the norm contains a weak * closed infinite-dimensional subspace.

In Theorem 1, we put $X = X^*$, $\tau = \sigma(X^*, X)$, Y = F. We cover $X^* \setminus F$ with a collection of balls V_n, $V_n \cap F = \varnothing$, and apply the theorem.

The following result is stated in [2] and proved in [3] with the help of a sequence which is a basis.

COROLLARY 3. Suppose that X** is separable. Then X and X* contain infinite-dimensional reflexive subspaces.

Proof. If we cover $X^{**} \setminus X$ with a sequence of balls, we get the first part of the statement. For the proof of the second part, we note that $X^{***} = X^* \oplus X^{\perp}$. As in Corollary 1, we can cover $C = B(X^{\perp}) \setminus 0$ with a sequence of weak * closed convex subsets of C. Applying Theorem 1, we get a subspace $F \subset X^*$, the $\sigma(X^{***}, X^{**})$-closure $\overline{F}$ of which does not intersect C. By Corollary 2, we can choose a $\sigma(X^*, X)$-closed subspace H in $\overline{F} \cap X^*$. It is not difficult to see that $\overline{H} \subset X^*$, i.e., $\overline{H}$ is reflexive.

We recall that closed subspaces X and Y of a Banach space E are said to be quasicomplementary if $X \cap Y = 0$ and $[X + Y] = E$.

THEOREM 2. Let X and Y be quasicomplementary, but not complementary, subspaces of E. Then there exists a closed subspace $Y_1 \supset Y$, $\dim Y_1/Y = \infty$, $Y_1 \cap X = 0$.

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This result has been proved in [4-6] under a variety of assumptions (reflexivity, the property of being weakly compactly generated, separability). We show that the argument in [6] carries over to the arbitrary case.

Proof. According to [6], there exists in E a countable-dimensional subspace Z such that for any $z \in Z$, $z \neq 0$, there are no bounded sequences $x_n \in X$ and $y_n \in Y$ such that $\|x_n + y_n - z\| \to 0$. The space $[Z + Y]/Y$ is separable; therefore it is possible to choose a sequence $f_i \in E^*$ for which $Y \subset (f_i)^\top$ and $(f_i)^\top \cap M = 0$, where $M = X \cap [Z + Y]$. Let

$$W_i^j = \{x \in M:\ \|x\| \leqslant j,\ f_i(x) \geqslant 1/i\}.$$

By virtue of the choice of the functionals f_i, there exist bounded sets W_i^j at a nonzero distance from Y such that $\bigcup_{i,j} W_i^j = M \setminus 0$. Indexing the W_i^j we get a sequence V_n. Furthermore, applying exactly the same argument as in [6], we can construct sequences of elements $z_n \in Z$ and closed hyperplanes $H_n \subset E$ such that for every n, a) $z_n \notin [z_j]_1^{n-1}$, b) $V_n \cap H_n$, and c) $Y + [z_i]_1^\infty \subset H_n$. The subspace $Y_1 = [Y + (z_i)_1^\infty]$ is the one sought.

We say that the Banach space X is normally imbedded in the Banach space Y if $X \subset Y$, $\|x\|_Y \leqslant \|x\|_X$ for $x \in X$, and the linear manifold X is dense in Y but does not coincide with Y. As was shown by M. I. Kadets, we have the following consequence of Theorem 2.

COROLLARY 4. Suppose that X is normally imbedded in Y. Then there exists a closed infinite-dimensional subspace $Z \subset Y$, $Z \cap X = 0$.

We get it as a consequence of Theorem 1. We choose a separable subspace $X_1 \subset X$ on which the norms $\|\ \|_X$ and $\|\ \|_Y$ are not equivalent. Let Y_1 be the closure of X_1 in Y and let $X_2 = X \cap Y_1$. Then $(X_2, \|\ \|_X)$ is normally imbedded in $(Y_1, \|\ \|_Y)$, and Y_1 is separable. We denote the closure of the unit ball in X_2 with respect to the norm in Y by C. From Lemma 1.2 of Chap. 1 of [7] it is not difficult to get (see [8]) the existence of an infinite-dimensional subspace $Z_1 \subset Y_1$, $Z_1 \cap C = 0$. We cover the set $C \setminus 0$ with a sequence of balls W_n of the space Y_1, $W_n \not\ni 0$. Then the collection $V_n = W_n \cap C$ covers $C \setminus 0$, and we can apply Theorem 1.

COROLLARY 5. Let the separable space X be normally imbedded in Y. Then $Y^* \subset X^*$, and there exists a $\sigma(X^*, X)$-closed infinite-dimensional subspace F, $F \cap Y^* = 0$ in X^*.

Proof. The ball $B(Y^*)$ is weakly $*$ compact and therefore closed in the topology $\sigma(X^*, X)$. As in the preceding corollary, there exists an infinite-dimensional subspace F_1 which does not intersect $B(Y^*)$. We cover the set $B(Y^*) \setminus 0$, which is separable and metrizable in the topology $\sigma(X^*, X)$, with a sequence of sets W_n which are convex and $\sigma(X^*, X)$-closed, and then putting $V_n = W_n \cap B(Y^*)$, we apply Theorem 1.

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