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# PROPERTY OF BOUNDED APPROXIMATION AND LINEAR FINITE-DIMENSIONAL REGULARITY


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Let E be a Banach space, $E^*$ be its dual space, and M be a linear subspace of $E^*$. One says that E has the property of bounded approximation if one can find a $\lambda$ such that for any finite-dimensional subspace $X \subset E$ and $\varepsilon > 0$ there exists a finite-dimensional subspace $R : E \to E$ with $\|R\| \le \lambda$ and $\|Rx - x\| \le \varepsilon\|x\|$ for $x \in X$ (cf. [1]). The object of our investigation will be the satisfaction of the conditions:

(A) One can find $a \ge 1$ such that for any finite-dimensional subspaces $X \subset E$ and $F \subset E^*$ there exists a linear operator $T : F \to M$ with $\langle x, f\rangle = \langle x, Tf\rangle$ for all $x \in X$, $f \in F$ and $\max\{\|T\|, \|T^{-1}\|\} < a$;

(B) E has the property of bounded approximation, and the images $R^*E^*$ of the conjugate operators belong to M.

It is easy to get from Lemma 2.4 of [1] that (B) $\Rightarrow$ (A), and from Lemma 3.1 of the same paper that if the space E has the property of bounded approximation, then (A) $\Rightarrow$ (B). From property (B) follows the normability of the subspace M [i.e., the norm $|||x||| = \sup\{f(x) : f \in M, \|f\| = 1\}$ is equivalent with the original norm $\|x\|$ of the space E]. But, as follows from [2], the normability of the subspace M does not imply (B) even if the space E has a basis (at least for complex spaces). We give an analogous example of a real space. In connection with this there is interest in clarifying the conditions under which (B) [or (A)] nevertheless holds. The most important result in this direction is the principle of local reflexivity [1], from which it follows that if $M^* = E$ in the duality established between E and M, then (A) holds. There are other results in [3]. We give one class of Banach spaces ($\mathfrak{L}_\infty$-spaces), which are important in applications, for which any normable subspace has property (B), and we indicate the connection of the results obtained with the theory of regularization in the sense of Tikhonov of ill-posed problems. In particular, we consider Fredholm integral equations of the first kind and the problem of analytic continuation of a function from a piece of the boundary to the entire domain.

Example. Let $l_1$ be the space of absolutely summable sequences. Its dual is the space $l_\infty$ of bounded functions $f(j)$ of a natural argument. We construct the set $\{f_n(j)\}_{n=2}^{\infty}$ in the following way. For $k = 1, 2, \ldots$, in the set $\{f_n(j)\} : 2^k \le n < 2^{k+1}$ we put all functions equal to zero for $j > k$ and assuming the values $\pm 1$ for $j \le k$ (there are $2^k$ different such functions). We construct a graph of subsets of the set N of natural numbers in the following way. We divide $N = N_1^1$ into two infinite sets $N_2^1$ and $N_2^2$. Each of the sets obtained we again divide into two infinite sets $N_3^1, N_3^2 \subset N_2^1$ and $N_3^3, N_3^4 \subset N_2^2$, and so on. We get a sequence of infinite subsets $(N_n^i)_{n=1,\infty}^{i=1,2^{n-1}}$, having the following properties: for any n and $i \le 2^{n-1}$,

$$N_n^i = N_{n+1}^{2i-1} \cup N_{n+1}^{2i}, \quad N_{n+1}^{2i-1} \cap N_{n+1}^{2i} = \varnothing.$$


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For $2^k \leq n < 2^{k+1}$ ($k = 1, 2, \ldots$) we set

$$g_n(j) = \begin{cases} 1 & \text{for } j \in (\{N_n^i : 1 \leqslant i \leqslant 2^{n-1},\ i \text{ odd}\} \setminus \{1, \ldots, k\}), \\ -1 & \text{for } j \in (\{N_n^i : 1 \leqslant i \leqslant 2^{n-1},\ i \text{ even}\} \setminus \{1, \ldots, k\}), \\ f_k(j) & \text{for } j \in \{1, \ldots, k\} \end{cases}$$

and we denote by M the closed (in the norm of the space $l_\infty$) linear hull of the sequence $\{g_n(j)\}_{n=2}^{\infty}$.

We shall show that for any $x \in l_1$, $\|x\| = \sup_n \langle x, g_n \rangle$. It is obvious that it is sufficient to verify this equation for sequences x having a finite number of nonzero coordinates. Let $x = (a_1, \ldots, a_k, 0, 0, \ldots)$, so one can find a $2^k \leq n < 2^{k+1}$ such that $g_n(j) = \operatorname{sign} a_j$ $(1 \leq j \leq k)$, and hence $\langle x, g_n \rangle = \|x\|$. By construction, $\|g_n\| = 1$, so the subspace M is normable.

We shall show that for any finite collection of scalars $(b_l)_2^n \left\| \sum_2^n b_l g_l \right\| = \sum_2^n |b_l|$, i.e., the subspace M is isometric with $l_1$. For this it suffices to establish the existence of a number j such that

$$g_l(j) = \operatorname{sign} b_l, \quad l = 1, \ldots, n. \tag{1}$$

In fact, we choose the set $N_2^{i_2}$ from the graph $\{N_n^i\}$ for which $g_2(j) = \operatorname{sign} b_2$ for $j \in N_3^{i_3} \setminus \{1\}$. Then $N_3^{i_3} \subset N_2^{i_2}$, for which $g_3(i) = \operatorname{sign} b_3$ for $j \in N_3^{i_3} \setminus \{1\}$ and at the m-th step $(m \leqslant n)$ $N_m^{i_m} \subset N_{m-1}^{i_{m-1}}$, for which $g_m(j) = \operatorname{sign} b_m$ for $j \in N_m^{i_m} \setminus \{1, \ldots, k_m\}$, where $k_m$ is such that $2^{k_m} \leq m < 2^{k_m+1}$. According to the construction of the functions $g_l$ such a choice is possible at each step. By the same token, for $j \in N_n^{i_n} \setminus \{1, \ldots, k_m\}$, (1) holds. But for the subspace $M \subset l_\infty$, isometric with $l_1$, condition (A), and hence also (B), is not satisfied, since $c_0 \subset l_\infty$ is not $\lambda$-finitarily representable in $l_1$ for any $\lambda$ (cf. e.g., [4, (2.4)]).

We recall that the Banach space E is called an $\mathfrak{L}_\infty$-space if there exists a $\lambda > 0$ such that for any finite-dimensional subspace $X \subset E$ there exists a finite-dimensional subspace $X \subset Y \subset E$ and a linear bijective operator $T: Y \to l_\infty^{(n)}$ with norm $\max\{\|T\| \|T^{-1}\|\} < \lambda$. Examples of $\mathfrak{L}_\infty$-spaces are the classical spaces $c_0$, $l_\infty$, $L_\infty$, $C(Q)$, where Q is a Hausdorff compactum (cf. [5]).

THEOREM 1. For an $\mathfrak{L}_\infty$-space E, normability of the subspace $M \subset E^*$ is equivalent with condition (B).

Proof. Let $X \subset E$ be a finite-dimensional subspace and Y, $\lambda$, T be objects of which one speaks in the definition of $\mathfrak{L}_\infty$-space. We denote by $j : E \to M$ the natural imbedding; since the subspace M is normable, one has $\max\{\|j\|, \|j^{-1}\|\} = \mu < \infty$. We consider the map $jTj^{-1} : jY \to l_\infty^{(n)}$. According to [6, Chap. V, Sec. 8, Theorem 3] it can be extended to an operator $S : M^* \to l_\infty^{(n)}$ preserving the norm. We set $S_1 = T^{-1}S$. Let $\varepsilon > 0$, so by Lemma 3.1 of [1] there exists a weak* continuous [i.e., $\sigma(M^*, M)$-continuous] operator $S_2 : M^* \to jY$, coinciding on the subspace jY with $S_1$ and having norm $\|S_2\| \leq (1 + \varepsilon)\|S_1\|$. We set $R = j^{-1}(S_2|_{jE})j$. We verify that (B) holds for the operator R. If $x \in X \subset Y$, then $Rx = j^{-1}(S_2|_{jE})jx = j^{-1}(S_1|_{jE})jx = j^{-1}T^{-1}Sjx = j^{-1}jT^{-1}Tj^{-1}jx = x$. Moreover, $\|R\| \leqslant \|j^{-1}\| \|S_2\| \|j\| \leqslant \mu^2(1+\varepsilon)\|S_1\| \leqslant \mu^2(1+\varepsilon)\|T^{-1}\| \|S\| \leqslant \mu^2(1+\varepsilon)\lambda \|j\| \|T\| \|j^{-1}\| \leqslant \mu^4\lambda^2(1+\varepsilon)$. We shall show, finally, that $R^*E^* \subset M$. In fact, by construction, the kernel of the restriction $S_2|_{jE}$ of the finite-dimensional operator $S_2$ has in jE finite defect and is $\sigma(E, M)$-closed. Hence, the kernel ker R is the annihilator $L^\perp$ of some finite-dimensional subspace $L \subset M$. The annihilator $(\ker R)^\perp \subset E^*$ coincides with the closure of the subspace L in the weak* topology of $\sigma(E^*, E)$. Since L is finite-dimensional, one has $R^*X^* \subset (\ker R)^\perp = L \subset M$. The theorem is proved.

We proceed to the problems of regularization. We consider a linear continuous injective map A of the Banach space X into a normed space Y. The inverse operator $A^{-1}$ is called Tikhonov regularizable if there exists a family of maps $R_\delta : Y \to X$, $\delta \in (0, \delta_0)$ such that for any $x \in X$, $\sup\{\|x - R_\delta y\| : y \in Y, \|y - Ax\| \leq \delta\} \to 0$ as $\delta \to 0$. If all the maps $R_\delta$ are linear (finite-dimensional), then the operator is considered linearly (finite-dimensionally) regularizable. On the basis of the results of [3] and the example, we conclude that there exists a real Banach space X with basis and a linear contraction $A : X \to Y$ for which $A^{-1}$ is regularizable, but not linearly regularizable. From results of [7, 8] and from Theorem 1, we get that for an operator defined in a separable $\mathfrak{L}_\infty$-space, regularizability is equivalent with linear finite-dimensional regularizability. We illustrate the last assertion on an example of integral operators.

THEOREM 2. The solution of a Fredholm integral equation of the I type $Ax = \int_a^b K(s, t)\, x(t)\, dt = y(s)$, $x(t) \in C[a, b]$, $y(s) \in L_2[a, b]$ is a linearly finite-dimensionally regularizable problem if the operator A can be extended to a continuous linear operator $\tilde{A}: L_2 \to L_2$ $\left(\text{this is possible, e.g., if } \int_a^b\int_a^b K(s,t)^2\,ds\,dt < \infty\right)$ with finite-dimensional kernel $\ker\tilde{A}$.

Proof. We denote by $\dot{I}: C \to L_2$ the natural imbedding. Then $A = \tilde{A}\dot{I}$. Since $\dim\ker\tilde{A} < \infty$, the closure in the norm of the subset $\tilde{A}^*L_2^* \subset L_2^*$ has finite defect. Moreover, it is easy to see that $\dot{I}^*L_2^* \subset C^*$ is a normable subspace. Since normability of a subspace is equivalent with normability of its closure and a complete subspace $M \subset N$ with the condition $\dim N/M < \infty$ of a normable subspace N is also normable, so $A^*L_2^* = \dot{I}^*\tilde{A}^*L_2^*$ is a normable subspace. Hence (cf. [8]) the operator $A^{-1}$ is regularizable, and consequently, linearly finite-dimensionally regularizable. The theorem is proved.

As shown in [9], the condition of finite-dimensionality of the kernel $\ker\tilde{A}$ here is essential.

Let A(D) be the Banach space of functions analytic inside the disk D and continuous up to the boundary with norm $\|x\| = \max_{z\in D}\{|x(z)|\}$, $\Gamma$ be a compact subset of the closure of D, $R: A(D) \to C(\Gamma)$ be the operator of restriction of functions from A(D) to $\Gamma$. A linear single-valued branch of the map $R^{-1}$ we call an operator of analytic extension and denote it by T.

Depending on the topological and metric properties of the set $\Gamma$ there arise various problems connected with the operator of analytic continuation. For example, if $\Gamma$ is a subset of the boundary, of linear Lebesgue measure zero, then $R(A(D)) = C(\Gamma)$, but the map $R^{-1}$ is multivalued and the question of existence of an operator T arising in connection with this reduces to the problem of isolation of a single-valued continuous linear branch of the map $R^{-1}$. This problem was solved in [10], where the results of [11] are used essentially.

Now if $\Gamma$ is a proper subset of the boundary D and $\mu\Gamma > 0$, then, as is known (cf. e.g., [12]), $R(A(D))$ is dense in $C(\Gamma)$, but does not coincide with it; the map $R^{-1}$ is single-valued, i.e., the operator of analytic continuation T exists, is defined on a dense subset of $C(\Gamma)$, and, as is easy to see, is unbounded there. In connection with this, there arises the question of the possibility of its regularization and the admissible properties of a regularizing family. The following theorem answers this question.

THEOREM 3. The operator of analytic extension from a compact subset $\Gamma$ of the boundary of the disk D, $\Gamma \neq \partial D$, $\mu\Gamma > 0$, admits a linear finite-dimensional regularization.

Our chosen method of proof of the theorem leads to the consideration of two variants: 1) the complement of $\Gamma$ is dense in $L = \partial D$, 2) $L\setminus\mathring{\Gamma}$ is not dense in L. In this note we give the proof of the second variant.

We note first that according to [7], to prove Theorem 3 it suffices to verify that the subspace $R^*(C^*\times(\Gamma)) \subset A^*(D)$ is quasibasic. We shall prove this. For the spaces dual to A(D) and $C(\Gamma)$, there are known the decompositions $A^*(D) = \mathfrak{L}^1/H_0^1 \oplus l$, $C^*(\Gamma) = \mathfrak{L}_\Gamma^1 \oplus l_\Gamma$, where $\mathfrak{L}^1(\mathfrak{L}_\Gamma^1)$ and $l(l_\Gamma)$ are the spaces of measures respectively absolutely continuous and singular with respect to the Lebesgue measure on L (on $\Gamma$), and $H_0^1$ is the subspace of $\mathfrak{L}^1$ consisting of measures on L whose Poisson integrals are analytic in D and vanish at the point $z = 0$.

It is easy to see that the imbedding $R^*: C^*(\Gamma) \to A^*(D)$ carries the summand $\mathfrak{L}_\Gamma^1$ into the subspace $\mathfrak{L}^1/H_0^1$. We shall prove that $R^*(\mathfrak{L}_\Gamma^1)$ is dense in the norm in $\mathfrak{L}^1/H_0^1$. We construct a map $\dot{I}: \mathfrak{L}_\Gamma^1 \to \mathfrak{L}^1$, $\forall x(t) \in \mathfrak{L}_\Gamma^1$

$$\dot{I}(x(t)) = \begin{cases} x(t), & t\in\Gamma, \\ 0, & t\in L\setminus\Gamma. \end{cases}$$

Obviously, $\dot{I}$ is isometric with the inclusion, and the operator $P: \mathfrak{L}^1 \to \dot{I}(\mathfrak{L}_\Gamma^1)$, defined by the equation $\forall r(t) \in \mathfrak{L}^1$

$$P(r(t)) = \begin{cases} r(t), & t\in\Gamma, \\ 0, & t\in L\setminus\Gamma, \end{cases}$$

is a continuous projector. It is also obvious that the map $R^*\dot{I}^{-1}P$ is the restriction of the subspace $\dot{I}(\mathfrak{L}^1(\Gamma))$ of the canonical map of the subspace $\mathfrak{L}^1$ onto the quotient space $\mathfrak{L}^1/H_0^1$. $\mathfrak{L}_\Gamma^1$ we identify with its $\dot{I}$-image in $\mathfrak{L}^1$ and we note that for the density of $R^*(\mathfrak{L}_\Gamma^1)$ in the space $\mathfrak{L}^1/H_0^1$ it is sufficient that the subspace $H_0^1 + \mathfrak{L}_\Gamma^1$ be dense in the norm in $\mathfrak{L}^1$. We shall prove the density of $H_0^1 + \mathfrak{L}_\Gamma^1$ in $\mathfrak{L}^1$.

The closure of the set $L\setminus\Gamma$ we denote by $\Delta$, and the subspace of functions from $\mathfrak{L}^1$, equal to zero on $L\setminus\Delta$, we denote by $\mathfrak{L}_\Delta^1$.

We note that the set of bounded functions from $H_0^1$ onto $\Delta$ is dense in $\mathfrak{L}_\Delta^1$ in norm. In fact, since $0 < \operatorname{mes}\Delta < \operatorname{mes} L$, the set of bounded functions from A(D) onto $\Delta$ is dense in $C(\Delta)$ [12, pp. 118–119]. Moreover, the set $H_0^1 \cap A(D) = A_0(D)$ is the kernel of the functional $f \in A^*(D): \forall x \in A(D),\ f(x) = \int_L x(t)\, t^{-1} dt$, unbounded in the norm $|||x||| = \sup_{t \in \Delta} |x(t)|$, so $A_0(D)$ is dense in A(D) in this norm. That is, the set of bounded functions from $A_0(D)$ onto $\Delta$ is dense in $C(\Delta)$ in norm, all the more in $\mathfrak{L}_\Delta^1$. Let us assume that $x(t) \in \mathfrak{L}^1$, and $\varepsilon > 0$. We note that $x_1 = x - Px \in \mathfrak{L}_\Delta^1$. Then one can find an element $h \in H_0^1$ such that $\| x_1 - h|_\Delta \|_{\mathfrak{L}^1_\Delta} < \varepsilon$, so we have the decomposition $x = P(x - h) + h + (x_1 - (h - Ph))$, in which $P(x-h) + h \in \mathfrak{L}_\Gamma^1 + H_0^1$, and $\| x_1 - (h - Ph) \|_{\mathfrak{L}^1} < \varepsilon$, i.e., $H_0^1 + \mathfrak{L}_\Gamma^1$ is dense in the space $\mathfrak{L}^1$. This means the subspace $R^*(\mathfrak{L}_\Gamma^1)$ is dense in $\mathfrak{L}^1/H_0^1$. Keeping in mind the functional representation of the components of $\mathfrak{L}^1/H_0^1$ [13], we note that $\mathfrak{L}^1/H_0^1$ contains a sequence of functionals, dual to the sequence $\{z^n\}_0^\infty \subset A(D)$. Consequently, since $(z^n)_{n=0}^\infty$ is an operator basis in A(D), the subspace $\mathfrak{L}^1/H_0^1 \subset A^*(D)$ is quasibasic. Whence, as proved in [7], follows the fact that $R^*(\mathfrak{L}_\Gamma^1)$ is quasibasic. The theorem is proved. We note that regularizability (but not linear finite-dimensional) of the operator of analytic continution from the curve $\Gamma$ to the entire domain was proved in [14].

Note. Property (B) was also studied in [15]. There is a nonempty intersection of this paper with the results given in [3, 7], e.g., the theorem from [15] and Theorems 2, 3 of [4] are similar. We note also that it follows from [16, Proof of Theorem IV] that if $R_n$ is a sequence of linear continuous finite-dimensional operators $R_n: E \to E$ and $\|R_n x - x\| \to 0$ for any $x \in E$, then an M-basis $(x_n, f_n)$ of the space E for which $\bigcup_1^\infty R_n E^* \subset [f_n]_1^\infty$ is a generalized summation basis. On the other hand, for any separable Banach space E and complete subspace $M \subset E^*$ there exists an M-basis $x_i, f_i$ with $f_i \in M$ [16]. Thus, from the fact that the subspace M is quasibasic follows the existence of a generalized summation basis $x_i, f_i$ with $f_i \in M$.

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