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A BANACH SPACE WITHOUT A FUNDAMENTAL BIORTHOGONAL SYSTEM

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Let E be a Banach space and E^* its dual space. A system $X, F = (x_i, f_i)$, $x_i \in E$, $f_i \in E^*$, $i \in I$ (I some set), is said to be a *biorthogonal system* if $f_i(x_j) = \delta_{ij}$ (the Kronecker delta). A biorthogonal system is said to be *fundamental* if the closed linear span $[x_i: i \in I] = E$; if, moreover, the set F is total on E, then the system X, F is called an *M-basis*. An M-basis that can be well-ordered in such a way that there is a projection $P_\alpha: E \longrightarrow [x_\beta: \beta < \alpha]$ parallel to $[x_\beta: \beta \geqslant \alpha]$ for every α and these projections are collectively bounded is called a *projection* basis. A basis is said to be *monotone* if the norms of all the projections are equal to 1. It is not hard to verify that the definition of a projection basis given here is equivalent to the definitions in [1] and [2]. We shall give an example of a Banach space without a fundamental system. This answers a question in [3]. A separable Banach space has an infinite-dimensional subspace and a quotient space with an (ordinary) basis [4]. The space considered by us does not have a quotient space with a projection basis of the same cardinality.

NOTATION. ω is the first infinite ordinal number, Ω is the first uncountable ordinal, α_0 is the first ordinal with cardinality $\overline{\overline{\alpha}}_0$ equal to card I, $[M]_*$ is the weak*-closed linear span of the set M, lin M is the linear span, and $M^\perp$ is the annihilator. We define supp $M = \{i \in I: \exists x \in M, f_i(x) \neq 0\}$ for $M \subset E$, and, similarly, supp $M = \{i \in I: \exists f \in M, f(x_i) \neq 0\}$ for $M \subset E^*$.

LEMMA 1. *Suppose that the Banach space E has an uncountable fundamental system X, F. Then there exist*

a) *a partition of the set I into a well-ordered system of subsets I_α, $\omega \leqslant \alpha \leqslant \alpha_0$,*

b) *functionals $f_\alpha \in (f_i: i \in I_\alpha)$, and*

c) *subsets Y_α of the unit sphere of* lin X *with cardinality $\overline{\overline{\alpha}}$*

such that for every α:

1) card $I_\alpha \leqslant \overline{\overline{\alpha}}$;

2) $\|f\| = \sup\{f(y): y \in Y_\alpha\}$ *for* $f \in F_\alpha = [f_i: i \in I_\beta, \beta < \alpha]_*$;

3) $Y_\beta \subset Y_\alpha$ *for* $\beta < \alpha$;

4) $f_\alpha \in Y_\alpha^\perp$.

There is a proof in [5].

THEOREM 1. *If the space E has a fundamental system, then it has a quotient space with a monotone basis of cardinality* card I.

PROOF. Let $M = [f_\alpha: \omega \leqslant \alpha < \alpha_0]_*$, where the f_α are the functionals in Lemma 1. The subspace M is the dual to the quotient space $E/M^\perp$, and $M^\perp$ includes the set $[x_i: i \in I, x_i \neq x_\alpha$ for all $\alpha]$, while the quotient images $\hat{x}_\alpha$ of the elements $(x_\alpha: \omega \leqslant \alpha < \alpha_0)$ form a

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fundamental system in $E/M^{\perp}$. Thus, $\hat{x}_\alpha, f_\alpha$ form an M-basis in the space $E/M^{\perp}$. According to properties 2)–4) of Lemma 1, the projections onto $[f_\beta: \beta < \alpha]_*$ parallel to $[f_\beta: \beta \geq \alpha]_*$ in the space M have unit norm. They are adjoint to projections which also have unit norm; consequently, $(\hat{x}_\alpha, f_\alpha)$ is a monotone basis.

LEMMA 2. *For a closed subspace Y of a Banach space E there is a linear subspace $Z \subset E$ such that* card $Z =$ card E/Y *and* $Z + Y = E$.

Indeed, an algebraic complement of Y in E can be taken to be Z.

In what follows let Γ be some set of cardinality greater than c, and let X be the space of bounded sequences $x(\gamma), \gamma \in \Gamma$, such that only countably many coordinates are nonzero, with the sup norm. For a subspace $Y \subset X$ we set supp $Y = \{\gamma \in \Gamma: \exists y \in Y, y(\gamma) \neq 0\}$. If $\Gamma_0 \subset \Gamma$, then we set $X_0 = \{x \in X: x(\gamma) = 0 \text{ for } \gamma \notin \Gamma_0\}$ and $X^0 = \{x \in X: x(\gamma) = 0 \text{ for } \gamma \in \Gamma_0\}$.

LEMMA 3. card $X_0 = \max(c, \text{card } \Gamma_0)$.

The proof is by induction on the first transfinite α of cardinality card Γ_0. For countable Γ_0 the space X_0 is isometric to l_∞, whose cardinality is c. Suppose that the lemma has been proved for all $\beta < \alpha$. Then $\Gamma_0 = \bigcup_{\beta<\alpha} \Gamma_\beta$ and card $\Gamma_\beta <$ card $\Gamma_0 > \aleph_0$. By the definition of the space X, $X_0 = \bigcup_{\beta<\alpha} X_\beta$ and card $X_0 \leq \max(\text{card } \Gamma_0, c)$ card $\Gamma_0 \leq \max(c, \text{card } \Gamma_0)$. On the other hand, card $X_0 \geq c$ and card $X_0 \geq$ card Γ_0.

LEMMA 4. *For any functional $f \in X^*$ there is a countable subset $\Gamma_0 \subset \Gamma$ such that $f \in (X^0)^{\perp}$.*

PROOF. If for some $f \in X^*$ the opposite is true, then there is an uncountable set $x_\alpha \in X$, $1 < \alpha < \beta$, such that supp $x_\alpha \cap$ supp $x_\gamma = \varnothing$ for $\alpha \neq \gamma$ and $f(x_\alpha) \geq 0$. We can extract a sequence x_{α_n}, $1 \leq n < \infty$, from it such that $\sup\|x_{\alpha_n}\| < \infty$ and $\inf f(x_{\alpha_n}) > \epsilon$. But this contradicts the fact that f is bounded.

LEMMA 5. *Let Y be a closed subspace of X. Then for every subspace $Z \subset Y$ there exists a subset $\Gamma_0 \subset \Gamma$ of cardinality not exceeding* card Z *such that $Z \subset X_0$ and* $Y = (Y \cap X_0) \oplus (Y \cap X^0)$.

PROOF. We construct subsets $\Gamma_\alpha \subset \Gamma$, $1 \leq \alpha < \Omega$, such that

1) $Z \subset X_1$, $\Gamma_\alpha \subset \Gamma_\beta$ for $\alpha < \beta$, card $\Gamma_\alpha \leq$ card Z,
2) $(Y \cap X_{\alpha+1}) + (Y \cap X^\alpha) = Y$.

Let $\Gamma_1 =$ supp Z. Since card $Z \geq c$ and supp x is countable for each $x \in X$, 1) holds. If the subsets Γ_β have been constructed for all $\beta < \alpha$ and α is a limit ordinal, then let $\Gamma_\alpha = \bigcup_{\beta<\alpha} \Gamma_\beta$. The conditions 1) are easily checked for α. Suppose that α is not a limit ordinal. The mapping of $Y/(Y \cap X^{\alpha-1})$ into $Y/X^{\alpha-1}$ that assigns the quotient class $y + X^{\alpha-1}$ to the quotient class $y + (Y \cap X^{\alpha-1})$ is one-to-one, and $Y/X^{\alpha-1} \subset X/X^{\alpha-1} \cong X_{\alpha-1}$. Therefore, using Lemma 3, we get that

$$\text{card } Y/(Y \cap X^{\alpha-1}) \leq \text{card } X_{\alpha-1} \leq \max(c, \text{card } \Gamma_{\alpha-1}) \leq \text{card } Z.$$

According to Lemma 2, there is a subspace $Z' \subset Y$ with card $Z' =$ card Z and $Z' + (Y \cap X^{\alpha-1}) = Y$. Let $\Gamma_\alpha = \Gamma_{\alpha-1} \cup$ supp Z'. It is not hard to verify the conditions 1) and 2).

We set $\Gamma_0 = \bigcup_{\alpha<\Omega} \Gamma_\alpha$. Obviously, card $\Gamma_0 \leqslant$ card Z, $X_0 \supset Z$, and $(Y \cap X_0) \cap (Y \cap X^0) = 0$. We show that $(Y \cap X_0) + (Y \cap X^0) = Y$. Let $y \in Y$ and let y_0 and y^0 be its projections on X_0 and X^0. Since $(Y \cap X_{\alpha+1}) + (Y \cap X^\alpha) = Y$, we have that $y = y_\alpha + y^\alpha$, $y_\alpha \in Y \cap X_{\alpha+1}$, $y^\alpha \in Y \cap X^\alpha$. Suppose that $f \in X^*$, $f = f_0 + f^0$, $f_0 \in (X^0)^\perp$, $f_0 \in (X_0)^\perp$. Then, by Lemma 4, $f = f_{\alpha_0} \in (X^{\alpha_0})^\perp$, $\alpha_0 < \Omega$, and

$$f(y_\alpha) = (f_{\alpha_0} + f^0)(y_\alpha) = f_{\alpha_0}(y_\alpha) = f_{\alpha_0}(y) = f_{\alpha_0}(y_0) = f(y_0)$$

for $\alpha > \alpha_0$.

Thus, the sets $\{y_\alpha: \alpha < \Omega\}$ and $\{y_0\}$ cannot be strictly separated by a functional $f \in X^*$. Consequently, $y_0 \in Y \cap X_0$, $y^0 \in Y \cap X^0$, and $y \in (Y \cap X_0) + (Y \cap X^0)$.

LEMMA 6. *Let Y be a closed subspace of X such that* card $X/Y =$ card X.

Then there is a countable sequence of disjoint subsets $\Gamma_n \subset \Gamma$ of cardinality not exceeding c and such that $X_n/(X_n \cap Y) \neq 0$, card $X^n/(X^n \cap Y) > c$, *and* $Y = (Y \cap X_n) \oplus (Y \cap X^n)$.

PROOF. We choose an element $x_0 \in X$, $x_0 \notin Y$, and set $Z = \text{lin}\, x_0$. Let Γ_0 be a subset satisfying the conditions of Lemma 5. Then $X_0/(X_0 \cap Y) \neq 0$, since $x_0 \in X_0$ and $x_0 \notin Y$. The mapping $X^0/Y \to X^0/(X^0 \cap Y)$ that assigns the quotient class $x + (X^0 \cap Y)$ to the quotient class $x + Y$ is one-to-one, and card $X_0/Y \leqslant$ card $X_0 \leqslant \max(c, \text{card}\,\Gamma_0) \leqslant c$, so that card $X^0/(X^0 \cap Y) > c$. If Γ_m, $0 \leqslant m \leqslant n$, have been constructed, then we choose some element

$$x_{n+1} \in \widetilde{X}^n = \{x \in X: \ x(\gamma) = 0 \ \ \forall \gamma \in \bigcup_0^n \Gamma_m\}, \ \ x_{n+1} \notin Y.$$

Let $Z = \text{lin}\, x_{n+1}$. We choose a subset $\Gamma_{n+1} \subset \Gamma \setminus \bigcup_0^n \Gamma_m$, satisfying the conditions of Lemma 5 (here $\widetilde{X}^n$ is taken as X and $Y \cap \widetilde{X}^n$ is taken as Y). Then

$$X_{n+1} / (X_{n+1} \cap Y) = X_{n+1} \cap \widetilde{X}^n \cap Y) \neq 0,$$
$$Y = (Y \cap X_n) \oplus ((Y \cap \widetilde{X}^n \cap X_{n+1}) \oplus (Y \cap \widetilde{X}^n \cap X^{n+1}))$$
$$= (Y \cap X_{n+1}) \oplus (Y \cap X^{n+1}).$$

The mapping $X^{n+1}/Y \to X^{n+1}/(X^{n+1} \cap Y)$ is one-to-one, and card $X_{n+1}/Y \leqslant c$. Hence, card $X^{n+1}/(X^{n+1} \cap Y) > c$.

LEMMA 7. *Let Γ_n be the subset in Lemma 6. For any $\hat{x}_n \in X_n/Y$, $\|\hat{x}_n\| = 1$, and any $\epsilon > 0$ there is a representative $x_n \in \hat{x}_n$, $x_n \in X_n$, such that $\|x_n\| < 1 + \epsilon$.*

Let $\widetilde{x}_n$ be some representative in $\hat{x}_n$. The proof of the lemma follows from the relations

$$\|\hat{x}_n\| = \inf\{\|\widetilde{x}_n - y\|: \ y \in Y\} = \inf\{\|\widetilde{x}_n - y_n - y^n\|: \ y_n \in Y \cap X_n,$$
$$y^n \in Y \cap X^n\} = \inf\{\max(\|\widetilde{x}_n - y_n\|, \ \|y^n\|): \ y_n \in Y \cap X_n, \quad y^n \in Y \cap X^n\}$$
$$= \inf\{\widetilde{x}_n - y_n\|: \ y_n \in Y \cap X_n\}.$$

THEOREM 2. *A quotient space X/Y of the same cardinality as X contains a subspace isomorphic to l_∞.*

PROOF. Let Γ_n be the subsets in Lemma 6. Suppose that $\hat{x}_n \in X_n/Y$, $\|\hat{x}_n\| = 1$, $\epsilon > 0$ and that we have chosen elements $x_n \in \hat{x}_n$, $x_n \in X_n$, $\|x_n\| < 1 + \epsilon$, which exist according to Lemma 7. Since $\operatorname{supp} x_n \cap \operatorname{supp} x_m = \varnothing$ for $m \neq n$ and $1 \leqslant \|x_n\| < 1 + \epsilon$, the mapping $I: l_\infty \longrightarrow X$ that assigns to an element $a = (a_1, a_2, \ldots) \in l_\infty$ the element $Ia \in X$ whose coordinates are zero outside $\operatorname{supp}(x_n)_1^\infty$ and equal to $a_n x_n(\gamma)$ for $\gamma \in \operatorname{supp} x_n$ is an isomorphism. We show that the restriction of the quotient mapping $\varphi: X \longrightarrow X/Y$ to Il_∞ is also an isomorphism. Let $z \in Il_\infty$, $\|z\| = 1$. Then there is an n for which $z = z_n + z^n$, $z_n = \lambda x_n$, $\operatorname{supp} z^n \subset \Gamma^n$ and $\|z_n\| > 1 - \epsilon$. We have

$$\|\varphi z\| = \inf\{\|z - y\|: \ y \in Y\} = \inf\{\|z_n + z^n - y_n - y^n\|: \ y_n \in Y \cap X_n,$$
$$y^n \in Y \cap X^n\} = \inf\{\max(\|z_n - y_n\|, \ \|z^n - y^n\|): \ y_n \in Y \cap X_n,$$
$$y^n \in Y \cap X^n\} \geqslant \inf\{\|z_n - y_n\|: \ y_n \in Y \cap X_n\} = \lambda \inf\{\|x_n - y_n\|:$$
$$y_n \in Y \cap X_n\} = \lambda \|\hat{x}_n\| = \lambda.$$

Since $\|z_n\| > 1 - \epsilon$ and $\|x_n\| < 1 - \epsilon$, it follows that $\lambda > (1 - \epsilon)/(1 + \epsilon)$. Thus, φI is an isomorphism.

THEOREM 3. *The space X does not have a fundamental system.*

PROOF. Suppose the opposite. Then, by Theorem 1, there exists a quotient space X/Y of the same cardinality as X with a basis. By Theorem 2, X/Y contains a subspace isomorphic to l_∞. Consequently ([6], p. 120), X/Y does not have an equivalent locally uniformly convex norm. On the other hand, any space with a projection basis has an equivalent locally uniformly convex norm [7].

REMARK. Assuming the "diplomant's axiom", S. Shelah has shown* that there exists a nonseparable Banach space such that each biorthogonal system in it is countable.

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BIBLIOGRAPHY

1. C. Bessaga, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 15 (1967), 397.
2. K. John and V. Zizler, Comment. Math. Univ. Carolinae 15 (1974), 679.
3. W. J. Davis and W. B. Johnson, Studia Math. 45 (1973), 173.
4. Joram Lindenstrauss and Lior Tzafriri, *Classical Banach spaces*, Vol. I, Springer-Verlag, 1977.
5. A. N. Plička, Dokl. Akad. Nauk Ukrain. SSR Ser. A 1980, no. 5, 19. (Russian)
6. Joseph Diestel, *Geometry of Banach spaces—Selected topics*, Lecture Notes in Math., vol. 485, Springer-Verlag, 1975.
7. S. L. Troyanski [Stanomir Trojanski], Studia Math. 37 (1970/71), 173.

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* *Editor's note.* Shelah's paper is entitled *Uncountable constructions* (Israel J. Math., to appear), in which the diplomant's axiom is referred to as the axiom $V = L$.