Further,

$$\sigma_2+\sigma_3=\sum_{\substack{k\in J_1\\ k_{i-1}\leqslant k<k_i}} a_{nk}\cdot a_1y_i+\ldots+\sum_{\substack{k\in J_p\\ k_{i-1}\leqslant k<k_i}} a_{nk}\cdot a_py_i+$$
$$+\sum_{\substack{k\in J_1\\ k_i\leqslant k<k_{i+1}}} a_{nk}\cdot a_1y_{i+1}+\ldots+\sum_{\substack{k\in J_p\\ k_i\leqslant k<k_{i+1}}} a_{nk}\cdot a_py_{i+1}=$$
$$=y_i\Big[a_1\sum_{\substack{k\in J_1\\ k_{i-1}\leqslant k<k_i}} a_{nk}+\ldots+a_p\sum_{\substack{k\in J_p\\ k_{i-1}\leqslant k<k_i}} a_{nk}\Big]+$$
$$+(y_i+\beta_i)\Big[a_1\sum_{\substack{k\in J_1\\ k_i\leqslant k<k_{i+1}}} a_{nk}+\ldots+a_p\sum_{\substack{k\in J_p\\ k_i\leqslant k<k_{i+1}}} a_{nk}\Big]=$$
$$=y_i\Big[a_1\sum_{\substack{k\in J_1\\ k_{i-1}\leqslant k<k_{i+1}}} a_{nk}+\ldots+a_p\sum_{\substack{k\in J_p\\ k_{i-1}\leqslant k<k_{i+1}}} a_{nk}\Big]+$$
$$+\beta_i\Big[a_1\sum_{\substack{k\in J_1\\ k_i\leqslant k<k_{i+1}}} a_{nk}+\ldots+a_p\sum_{\substack{k\in J_p\\ k_i\leqslant k<k_{i+1}}} a_{nk}\Big].$$

Obviously, the second term tends to 0 as $i\to\infty$, since the expression in the brackets is bounded. Hence

$$\sigma_2+\sigma_3=y_i\Big[a_1\sum_{k\in J_1} a_{nk}+\ldots+a_p\sum_{k\in J_p} a_{nk}\Big]+\gamma_i-y_i\Big[a_1\sum_{\substack{k\in J_1;\ k<k_{i-1}\\ k\geqslant k_{i+1}}} a_{nk}+\ldots+a_p\sum_{\substack{k\in J_p;\ k<k_{i-1}\\ k\geqslant k_{i+1}}} a_{nk}\Big],$$

where $\gamma_i\to 0\ (i\to\infty)$.

In view of the choice of numbers k_i ($i = 1, 2, \ldots$), and the fact that $\{z_k\}$ is bounded, the third term tends to 0 as $i\to\infty$, and by (1), the first term also tends to 0 as $i\to\infty$. Hence, sequence $\{z_n\}$ is A-summable to 0.

QED.

In conclusion, I thank N. A. Davydov for advice and interest.

LITERATURE CITED

1. R. G. Cooke, Infinite Matrices and Sequence Spaces, Macmillan, New York (1950).

MEASURABILITY AND REGULARIZABILITY OF MAPPINGS INVERSE TO CONTINUOUS LINEAR OPERATORS

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The minimal σ-algebra, containing all open sets of topological space E, is called the σ-algebra of Borel sets $\mathfrak{A}(E)$ of space E. Let E and F be two topological spaces, and let A be a mapping from E into F; we call mapping A measurable if the preimage $A^{-1}(S)$ of any Borel set $S\in\mathfrak{A}(F)$ belongs to the σ-algebra $\mathfrak{A}(E)$. The set of all measurable mappings contains the set of B-measurable mappings of class α (see [1]). Among these families of mappings, the greatest interest attaches to mappings of the first class, i.e., mappings A: $E\to F$ for which the preimage of every open set $U\subset F$ is a countable union of closed sets (i.e., $A^{-1}(U)$ is a set of type F_σ).

Let E and F be metric spaces; a mapping A: $E\to F$ is regarded as Tikhonov-regularizable if there exists a one-parameter family of mappings $R_\delta\colon E\to F,\ 0\leqslant\delta<\delta_0$, such that for any $x\in E$

$$\sup\{\rho(R_\delta(y),\ A(x))\colon y\in E,\ \rho(x,y)<\delta\}\to 0$$

as $\delta\to 0$.

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It was shown in [2] that, in the case of separable metric space F, regularizability of a mapping A, acting from metric space E into F, is equivalent to mapping A being a member of the first class. In this connection, it is interesting to find necessary and sufficient conditions for measurability and membership of the first class for an unbounded linear mapping, acting from normed space E into F. A particular case of this problem is examined below: we consider mappings A^{-1}, which are inverse to bounded linear mappings A. By the classical Banach theorem on the inverse operator, the mapping A^{-1} is a bounded linear operator if A is a bounded linear operator acting from Banach space E into Banach space F. When F is an incomplete linear normed space, Banach's theorem is not valid, so that it is not possible to guarantee the continuity of mapping A^{-1}. Nevertheless, we have:

THEOREM 1. Let E be a separable Banach space, F a topological vector space, and A a continuous linear operator with zero kernel, acting from E into F. Then, the inverse operator A^{-1} is a measurable linear mapping, acting from F into E.

This theorem is a restatement of Proposition 2 of [3].

THEOREM 2. Let E be a reflexive Banach space, and F a separable locally convex space. The inverse mapping A^{-1} to the continuous linear mapping A with zero kernel, acting from E into F, is also measurable.

As a preliminary, we shall prove:

LEMMA 1. If E is a locally uniformly convex Banach space, then, given any point $x_0 \in E$, $x_0 \neq 0$, and any number $\gamma > 0$, there exist a rational number $r > \|x_0\|$ and a linear continuous functional f, such that

$$x_0 \in \{x \in E : \|x\| \leqslant r,\ f(x) > 1\} \subset S(x_0, \gamma),$$

where $S(x_0, \gamma) \equiv \{x \in E:\ \|x - x_0\| \leqslant \gamma\}$.

Proof. By the Hahn–Banach theorem, there exists a linear continuous functional g such that $g(x_0) = 1$ and $\|g\| = 1/\|x_0\|$. Since space E is locally uniformly convex, there exists $\delta > 0$ such that the diameter of the set $Q = \{x \in E: g(x) > 1 - \delta,\ \|x\| \leqslant \|x_0\|\}$ is less than $\gamma/3$ (we use the implication $UR_L \Rightarrow D_L$, see [4]). We take $\delta' > 0$ such that $\delta' \|x_0\| < \gamma/3$ and $\delta' < \delta$. Then, $(1 + \delta') Q \ni x_0$ and $(1 + \delta')Q \subset S(x_0, 2/3\gamma)$, since every point of the set Q has a displacement of less than $\gamma/3$. The set $(1 + \delta')Q$ can be written as

$$(1 + \delta') Q = \{x \in E:\ g(x) > (1 - \delta)(1 + \delta),\ \|x\| \leqslant (1 + \delta')\|x_0\|\}.$$

On choosing the rational number r from the conditions $\|x_0\| < r < (1 + \delta')\|x_0\|$ and putting $f = g/((1 - \delta)(1 + \delta'))$, we obtain the lemma. The theorem is now easily proved:

Proof of Theorem 2. We need to show that, if $X \subset E$ is a nonempty open set, then AX is a measurable set in space F. It can be assumed without loss of generality that X does not contain zero. Moreover, we shall assume that the norm in E is locally uniformly convex, since, in accordance with [5], an equivalent norm with this property exists in any reflexive space. For every point $x \in X$ we choose, in accordance with Lemma 1, a rational number $r_x > 0$, and a continuous functional f_x, such that

$$x \in \{x' \in E:\ \|x'\| \leqslant r_x,\ f_x(x') > 1\} \subset X,$$

while $f_x(x) = 1 + \varepsilon_x$, $\varepsilon_x > 0$. Since the linear continuous mapping A is injective, and space E is reflexive, the set A'F' is everywhere dense with respect to the norm in E'; hence we can choose a linear functional $v_x \in A'F'$ such that $\|f_x - v_x\| < \varepsilon_x/4r_x$; then, the functional $g_x = v_x/(1 + \varepsilon_x/2)$ belongs to the set A'F', and the following inclusions hold for it:

$$x \in \{x' \in E:\ \|x'\| \leqslant r_x,\ g_x(x') > 1\} \subset \{x' \subset E : \|x'\| \leqslant r_x,\ f_x(x') > 1\}.$$

In fact,

$$g_x(x) = v_x(x)/(1 + \varepsilon_x/2) = \frac{1}{1 + \varepsilon_x/2}[f_x(x) - (f_x - v_x)(x)] > \frac{1}{1 + \varepsilon_x/2}\left(1 + \varepsilon_x - r_x\frac{\varepsilon_x}{4r_x}\right) > 1.$$

In addition, if $\|x'\| \leqslant r_x$ and $g_x(x') > 1$, then

$$f_x(x') = v_x(x') - (v_x - f_x)(x') > v_x(x') - \frac{\varepsilon_x}{4r_x} r_x = (1 + \varepsilon_x/2) g_x(x') - \varepsilon_x/4 > 1 + \varepsilon_x/4 > 1.$$

We introduce some notation: $H_x = \{x' \in E:\ g_x(x') > 1\}$, Q is the set of positive rational numbers; and we write the set of x as $X = \bigcup_{x \in X}(S(0, r_x) \cap H_x)$. Using the fact that mapping A is injective, we obtain

$$AX = \bigcup_{x\in X}(A(S(0,r_x)) \cap A(H_x)) = \bigcup_{\xi\in Q}(A(S(0,\xi)) \cap (\bigcup_{x\in X,\, r_x=\xi} A(H_x))).$$

It remains to observe that, by hypothesis, $g \in A'F'$, and the set AH_X is open in F. Next, the linear continuous operator A: $E \to F$ is continuous from a weak into a weak topology; and since $S(0, \xi)$ is compactum in the weak topology (inasmuch as E is reflexive), then its image under a weakly continuous mapping will also be a compactum in the weak topology in F, i.e., it is weakly closed, and closed in F. In short, the set $\bigcup_{x\in X,\, r_x=\xi} A(H_x)$ is open, the set $A(S(0, \xi))$ is closed in F, their intersection is measurable in F, and the countable union of the sets is measurable in F. The measurability of set AX is proved; QED.

The essential nature of the conditions in Theorems 1 and 2 is underlined by the following:

THEOREM 3. If E is a nonseparable Banach space, adjoint to a separable space, then there is a linear bounded injective mapping $A: E \to l_2$.

Proof. If $E = E_0'$ and $(f_n)_1^\infty \subset E_0$ is a point sequence, everywhere dense in the unit sphere of space E_0, then, on associating an element $x \in E$ with a numerical sequence $(f_n(x))_1^\infty$, we obtain an isometric imbedding ψ: $E \to m$. Then, the mapping A: $E \to l_2$, associating the element $x \in E$ with the sequence $2^{-n}f_n(x)$, is a linear bounded injective mapping.

If we accept the continuum hypothesis, or the weaker hypothesis $2^{\aleph_1} > 2^{\aleph_0}$, then a B-measurable mapping will preserve separability of a space (see [1]), and hence, under these conditions, the mapping inverse to that constructed in Theorem 3 is unmeasurable.

Let us now turn to examining the regularizability of mappings A^{-1}, inverse to continuous linear injective operators, acting from Banach space E into normed space F. To this end, we consider in space E the new norm

$$\| x \|^* = \| Ax \|_F;$$

since A is a bounded linear operator, the norm $\| x \|^*$ is weaker than the initial norm of space E. By the above results, the adjoint space E_*' to a vector space E, equipped with the norm $\| \ \|^*$, is imbedded in space E' and is there an everywhere dense linear manifold in the weak topology $\sigma(E', E)$.

A subspace $M \subset E'$, everywhere dense in the weak topology $\sigma(E', E)$, is called a subspace with zero characteristic, if the norm

$$\| x \|_0 = \sup \{f(x): f \in M, \ \| f \| \leqslant 1\}$$

is not equivalent to the initial norm of space E. Otherwise, we say that the characteristic of M is nonzero (see [6]). We know [6] that a subspace has zero characteristic if and only if the closure of the unit sphere S_1 of space E in the topology $\sigma(E, M)$ is unbounded with respect to the norm of E. In the case M = A'F', this last condition is equivalent to the following assertion: the closure $\overline{S}_1$ of sphere S_1 in the norm $\| x \|^*$ is an unbounded set in the metric of space E (see [7]). If space F, into which mapping A acts, is a separable topological vector space, then, instead of the norm $\| x \|^*$ in space E we introduce the preimage $A^{-1}(\tau)$ of topology τ of space F with respect to the mapping A.

THEOREM 4. Let A be a continuous linear operator with zero kernel, acting from separable Banach space E into separable topological vector space F with topology τ. In order for the inverse operator A^{-1} to be a B-measurable mapping of the first class, it is necessary and sufficient that the closure $\overline{S}_1$ of the unit sphere S_1 in topology $A^{-1}(\tau)$ be a bounded set in the metric of space E.

The proof is obtained from the next lemma.

LEMMA 2. Let E be a separable Banach space, and τ a separable topology, matched with the structure of vector space E, which is weaker than the original topology of space E. In order for an open set of Banach space E to be a set of type F_σ in the topology τ, it is necessary and sufficient that the closure $\overline{S}_1$ of the unit sphere S_1 in topology τ be a bounded set of space E.

Proof. Sufficiency. In a separable metric space there exists a countable base, consisting of open spheres (see [1]); hence we only need to show that the open sphere S_1, center the point 0, is a set of type $F_\sigma(\tau)$. Let $\rho_n \to 1$ as $n \to \infty$ and $\rho_n < 1$; then, $S_1 = \bigcup_{n=1}^\infty S_{\rho_n}$.

Denote by $y_1^n, y_2^n, \ldots$ a countable set, everywhere dense in sphere S_{ρ_n}; it can be assumed without loss of generality that $y_m^n \in S_{\rho_n}$. Since $\overline{S}_1$ is bounded, there exists a constant c such that $c\overline{S}_1 \subset S_1$.

We choose an arbitrary number $r_n \in (0, 1-\rho_n)$ and consider the system of spheres $S(y_k^n, r_n) = \{x \in E: \|y_k^n - x\| \leqslant r_n\}$. Then,

$$S_{\rho_n} \subset \bigcup_{k=1}^{\infty} S(y_k^n, r_n), \quad S(y_k^n, r_n) \subset S_1.$$

In view of this, and our above discussion, we have

$$S_{\rho_n} \subset \bigcup_{k=1}^{\infty} \overline{S(y_k^n, r_n \cdot c)}, \quad \overline{S(y_k^n, r_n \cdot c)} \subset S(y_k^n, r_n) \subset S_1,$$

where $\overline{S(y_k^n, r_n \cdot c)}$ denotes the closure of sphere $S(y_k^n, r_n \cdot c) = \{x \in E: \|y_k^n - x\| \leqslant r_n \cdot c\}$ in topology τ. It is easily seen that

$$S_1 = \bigcup_{n=1}^{\infty} \{\bigcup_{k=1}^{\infty} S(y_k^n, r_n \cdot c)\} = \bigcup_{k,\,n=1}^{\infty} \overline{S(y_k^n, r_n \cdot c)}.$$

Hence S_1 can be expressed as the sum of a countable number of sets, closed in topology τ; consequently, $S_1 \in F_\sigma(\tau)$.

Necessity. Let $\overline{S}_1$ be unbounded. Assume that the open sphere S_1 belongs to $F_\sigma(\tau)$. Then,

$$S_1 = \bigcup_1^{\infty} X_n,$$

where X_n are closed sets in topology τ. It can easily be seen that X_n is a nowhere dense set in the original topology of space E. For, assume that, for some n, the set X_n is dense in the sphere $S(x_0, \varepsilon)$. Since τ is weaker than the original topology of space E, then the density of X_n in the metric of E implies its density in $S(x_0, \varepsilon)$ in the topology τ. The set X_n is closed in the topology τ, and hence the closure $\overline{S(x_0, \varepsilon)}$ of sphere $S(x_0, \varepsilon)$ in topology τ is contained in the set $X_n \subset S_1$; but this contradicts the unboundedness of $\overline{S(x_0, \varepsilon)}$ in the metric of space E. Consider any closed sphere $S_r \subset S_1$, $S_r = \{x \in E: \|x\| \leqslant r\}$. The sets $Y_n = X_n \cap S_r$ are nowhere dense for any integer n, whereas $S_r = \bigcup_1^{\infty} Y_n$. This last equation contradicts Baire's theorem on categories (see [1]), according to which S_r is a set of second category. This proof is similar to the proof of the lemma in [8]. Our lemma is proved.

COROLLARY 1. If, in the conditions of Theorem 4, space F is locally convex, then the operator A^{-1} is a B-measurable mapping of first class if and only if the characteristic of subspace $E'_* \subset E'$ is nonzero.

Note. In the proof of necessity in Theorem 4 (and hence in Corollary 1), no use is made of the assumption that space E is separable.

Recall that a Banach space is said to be quasireflexive if $\dim E''/E < \infty$, where E'' is the second adjoint to space E. We know (see [7]) that, in the adjoint to a quasireflexive space, the characteristic of any subspace, everywhere dense in the weak topology, is nonzero. Hence we have:

COROLLARY 2. Let E be a quasireflexive separable Banach space, and A a continuous linear injective operator, mapping E into an arbitrary normed space F. Then, the inverse operator A^{-1} is a B-measurable mapping of first class (or Tikhonov-regularizable).

Since, into any nonquasireflexive Banach space E, a weaker norm $\| \ \|_0$ can be introduced, such that the closure of the unit sphere S of space E with respect to the norm $\| \ \|_0$ is an unbounded set of space E (see [9, 10]), we have:

COROLLARY 3. Let E be a nonquasireflexive Banach space; then there exists a normed space F, and a continuous linear injective operator A: $E \to F$, such that the inverse operator A^{-1} does not belong to the first class, and hence is not Tikhonov-regularizable, as follows from [11, Theorem 1].

The following holds without the assumption that the Banach space is separable:

THEOREM 5. If E is a reflexive Banach space, and A: $E \to F$ is a linear bounded injective mapping into normed space F, then mapping A^{-1} is regularizable and hence, is a B-measurable function of the first class.

This theorem follows from Trojanski's result [5] on the isomorphism of a reflexive space and a locally uniformly convex space, and the result of [12] (see also the note to [13]).

Using Corollary 2, Theorem 5, and the decomposition of a quasireflexive Banach space into a sum of a reflexive and a separable quasireflexive space, an extension of Corollary 2 is obtained in [14]: to nonseparable spaces. Finally, note that the results proved here were announced in [15].

LITERATURE CITED

1. K. Kuratowski, Topology, Academic Press (1969).
2. V. A. Vinokurov, "On the regularizability of discontinuous mappings," Zh. Vychisl. Mat. Mat. Fiz., 11, No. 5, 1097-1112 (1971).
3. V. V. Buldygin, Yu. I. Petunin, A. N. Plichko, and I. Ya. Shneiberg, "σ-algebras of Borel sets and summation of series of independent random variables in Banach space," Ukr. Mat. Zh., 27, No. 4, 435-442 (1975).
4. M. M. Day, Normed Linear Spaces, Academic Press (1962).
5. S. Trojanski, "On locally uniformly convex and differentiable norm in certain nonseparable Banach spaces," Stud. Math., 37, 173-180 (1971).
6. J. Dixmier, "Sur un théorème de Banach," Duke Math. J., 15, 1057-1071 (1948).
7. Yu. I. Petunin, "Adjoint Banach spaces, containing subspaces of characteristic zero," Dokl. Akad. Nauk SSSR, 154, No. 3, 527-529 (1964).
8. L. G. Afanas'eva and Yu. I. Petunin, "σ-algebras generated by comparable normed topologies," Voronezhsk. Gos. Univ., Tr. Inst. Mat., 1, 3-11 (1970).
9. A. N. Plichko, "Criterion for quasireflexivity of Banach space," Dokl. Akad. Nauk Ukr. SSR, Ser. A, 5, 406-407 (1974).
10. W. Davis and J. Lindenstrauss, "On total nonnorming subspaces," Proc. Am. Math. Soc., 31, 109-111 (1972).
11. V. A. Vinokurov, "Regularizability and analytic representability," Dokl. Akad. Nauk SSSR, 220, No. 2, 269-272 (1975).
12. V. A. Vinokurov, "General properties of error of approximate solution of linear functional equations," Zh. Vychisl. Mat. Mat. Fiz., 11, No. 1, 22-28 (1971).
13. V. A. Vinokurov, "On error of approximate solution of linear problems," Zh. Vychisl. Mat. Mat. Fiz., 12, No. 3, 756-762 (1972).
14. R. V. Rakhmankulov, "Note on quasireflexive Banach spaces," Vestn. Leningr. Gos. Univ., Ser. Mat. Mekh., Astron., No. 1, 159-160 (1977).
15. V. A. Vinokurov, Yu. I. Petunin, and A. N. Plichko, "Conditions for measurability and regularizability of mappings inverse to continuous linear mappings," Dokl. Akad. Nauk SSSR, 220, No. 3, 509-511 (1975).