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Norm approximation property

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Abstract. We introduce and study a general approximation property which takes origin in Numerical Analysis.

Let $\mathcal{F}(X)$ be the set of all finite rank bounded linear operators in a Banach space X. A Banach space X has the *approximation property* (AP) if for every $\varepsilon > 0$ and every compact subset $K \subset X$ there is $T \in \mathcal{F}(X)$ such that for all $x \in K$,

$$\|Tx - x\| \leq \varepsilon.$$

We say that a Banach space X has the *norm approximation property* (norm AP) if there is $\lambda \geq 1$ such that for every $\varepsilon > 0$ and every finite-dimensional subspace $E \subset X$ there is $T \in \mathcal{F}(X)$ with $\|T\| \leq \lambda$ and such that for all $x \in E$,

$$(1-\varepsilon)\|x\| \leq \|Tx\| \leq (1+\varepsilon)\|x\|. \tag{1}$$

Of course, one may consider the norm approximation property for different Banach spaces X and Y. This property is a kind of "finite representability" of X in Y. The norm approximation property takes origin in Numerical Analysis, see, e.g., Vainikko [6], Heinrich [1], Plichko [5].

Proposition 1. *A Banach space X has the norm AP provided there is a sequence $T_n \in \mathcal{F}(X)$ such that for all $x \in X$,*

$$\|T_n x\| \to \|x\| \ \text{ as } \ n \to \infty. \tag{2}$$

Proof. Indeed, by the Uniform Boundedness Principle, in this Proposition the operators (T_n) are automatically uniformly bounded. Moreover, (2) implies that this convergence is uniform on the unit ball of every finite-dimensional subspace $E \subset X$. □

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For separable Banach spaces the converse statement is also valid. If X has the norm AP, then there exists a sequence $T_n \in \mathcal{F}(X)$ with the property (2).

Now we present an example of a Banach space without the norm AP.

Denote by $\mathcal{L}(X,Y)$ the space of all bounded linear operators from a Banach space X to a Banach space Y. Let us recall that an operator $T \in \mathcal{L}(X,Y)$ is 2-*absolutely summing* if there is a constant C such that, for all finite subsets (x_i) in X, we have

$$\left(\sum \|Tx_i\|^2\right)^{1/2} \leq C \sup_{f \in B_{X^*}} \left(\sum |f(x_i)|^2\right)^{1/2},$$

where B_{X^*} is the dual unit ball. We denote by $\pi_2(T)$ the smallest constant C satisfying the previous inequality. This $\pi_2(T)$ is a norm on the space of all 2-absolutely summing operators and $\pi_2(T) \geq \|T\|$. Moreover, if $T \in \mathcal{L}(X,Y)$ is 2-absolutely summing, and for subspaces $X' \subset X$ and $Y' \subset Y$ we have $T(X') \subset Y'$, then the restriction $T' = T|_{X'} \in \mathcal{L}(X',Y')$ is again 2-absolutely summing and $\pi_2(T') \leq \pi_2(T)$, see, e.g., Pisier [4, p. 9].

Lemma 2. *Let $T \in \mathcal{L}(X,Y)$ and E be a subspace of X, $\dim E = n$, for which* (1) *is satisfied. Then there exists a constant $a > 0$, depending on ε only, such that*

$$\pi_2(T|_E) \geq a\sqrt{n}.$$

Proof. Let Id be the identity operator on X. As is well known, there exists an absolute constant $b > 0$ such that $\pi_2(Id|_E) \geq b\sqrt{n}$ (see, e.g., Pisier [3, p. 201] or [4, p. 145]). Since $Id|_E = (T|_E)(T|_E)^{-1}$ and $\pi_2(Id|_E) \leq \pi_2(T|_E)\|(T|_E)^{-1}\|$ (see, e.g., Pisier [4, p. 9]), the Lemma is proved. □

Pisier [3, p. 201], [4, p. 145] constructed a Banach space $\mathbf{X}$ for which there is a constant $c > 0$ such that for every $T \in \mathcal{F}(\mathbf{X})$ we have

$$\|T\| \geq c\pi_2(T). \tag{3}$$

Proposition 3. *Every Banach space X for which* (3) *is satisfied fails the norm AP.*

Proof. Indeed, let $E \subset X$, $\dim E = n$, and let an operator $T \in \mathcal{F}(X)$ satisfy condition (1). Then, by (3) and Lemma 2,

$$\|T\| \geq c\pi_2(T) \geq c\pi_2(T|_E) \geq ca\sqrt{n}.$$

Therefore, there exists no λ with $\|T\| \leq \lambda$ for the operators $T \in \mathcal{F}(X)$ satisfying (1). □

Proposition 4. *Let X be a subspace of a Banach space Y which has the norm AP and is finitely representable in X. Then X has the norm AP.*

Proof. Take a finite-dimensional subspace $E \subset X$, $\varepsilon > 0$, and let $I : X \to Y$ be the identity embedding. Since Y has the norm AP (with a constant λ), there is $T \in \mathcal{F}(Y)$ with $\|T\| \leq \lambda$ such that

$$(1-\varepsilon)\|x\| \leq \|Tx\| \leq (1+\varepsilon)\|x\|$$

for every $x \in I(E)$. Since Y is finitely representable in X, there exists an operator $S : T(Y) \to X$ with $\|S\| < 1+\varepsilon$ and $\|S^{-1}\| < 1+\varepsilon$. Put $U = STI$. Then $U \in \mathcal{F}(X)$, $\|U\| < \lambda(1+\varepsilon)$, $\|Ux\| < (1+\varepsilon)^2\|x\|$ and $\|Ux\| > \frac{1-\varepsilon}{1+\varepsilon}\|x\|$ for every $x \in E$. Hence, X has the norm AP. $\square$

Corollary 5. *Every Banach space X, which contains ℓ_∞^n uniformly, has the norm AP.*

Proof. Indeed, each Banach space X is a subspace of $Y = \ell_\infty(\Gamma)$, for a suitable set Γ, which of course has the norm AP. By the assumption of Corollary, Y is finitely representable in X and one can apply the previous Proposition. $\square$

Corollary 6. *Let $Y = (\sum_n \ell_\infty^n)_2$ and X be an arbitrary Banach space. Then the space $Z = (X \oplus Y)_{\ell_2}$ has the norm AP. So, each Banach space is isometric to a 1-complemented subspace of a Banach space with the norm AP. Each reflexive Banach space is isometric to a 1-complemented subspace of a reflexive Banach space with the norm AP.*

Corollary 7. *Every subspace X of the space ℓ_p, $1 \leq p < \infty$, or c_0 has the norm AP. So, there exists a superreflexive separable Banach space with the norm AP but without the AP.*

Proof. Indeed, each (infinite-dimensional) subspace X of ℓ_p or c_0 contains a subspace, $(1+\varepsilon)$-isometric to ℓ_p or c_0 [2, Proposition 2.a.2], and one can apply the previous Proposition. It is well known that there exists a subspace X of ℓ_p, $p \neq 2$, (or c_0) without the AP [2, p. 90]. $\square$

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