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ON ORDINARY AND STANDARD "LEBESGUE MEASURES " IN SEPARABLE BANACH SPACES

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Abstract

By using results from a paper [G.R. Pantsulaia, On ordinary and standard Lebesgue measures on $\mathbb{R}^\infty$, *Bull. Pol. Acad. Sci. Math.* **57 (3-4)** (2009), 209–222] and an approach based in a paper [T. Gill, A.Kirtadze, G.Pantsulaia , A.Plichko, The existence and uniqueness of translation invariant measures in separable Banach spaces, *Functiones et Approximatio, Commentarii Mathematici*, 16 pages, to appear], a new class of translation-invariant quasi-finite Borel measures (the so called, ordinary and standard "Lebesgue Measures") in an infinite-dimensional separable Banach space X is constructed and some their properties are studied in the present paper. Also, various interesting examples of generators of two-sided (left or right) shy sets with domain in non-locally compact Polish Groups are considered.

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1. Introduction

By using an additional set-theoretical axiom asserted that *each subset of* $\mathbb{R}$ *is Lebesgue measurable* an example of an invariant measure firstly has been constructed on the powerset of a Banach space with absolutely convergent Schauder basis such that the constructed measure takes the value 1 on the standard rectangle (see [16, Th. 7.3]). A version of Lebesgue measure on every separable Banach space that has a Schauder basis firstly has been constructed in [8, Th. 12] without any additional set-theoretical assumption. More lately has

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been demonstrated that the completion $\overline{\mu}_Q$ of the Yamasaki-Kharazishvili measure μ_Q has the uniqueness property in the class of all L_1-invariant σ-finite measures in X with domain $\mathrm{dom}(\overline{\mu}_Q)$ (see comment in [7, p. 121]). The latter result has been extended by [6, Th. 4.2] to all infinite-dimensional separable Banach spaces. In [6], the problem of invariant measures for infinite-dimensional separable Banach spaces has been considered. Here has been demonstrated that their approach is distinct from that of Oxtoby [13]. The methods developed by [1] and [8], have been used for the construction of a translation invariant Borel measure in an infinite-dimensional separable Banach space with absolutely convergent Markushevich basis which gets the numerical value 1 on the standard parallelepiped defined by that basis.

In [18], new concepts of the Lebesgue measure on $\mathbb{R}^\infty$ have been proposed. Here has been demonstrated that Baker's both measures [1], [2], Mankiewicz and Preiss - Tišer generators [17] and the measure [15] are not an α-standard Lebesgue measure on $\mathbb{R}^\infty$ for $\alpha = (1, 1, \cdots)$.

Let α be an infinite parameter set and let $(\alpha_i)_{i\in I}$ be its any partition such that α_i is a non-empty finite subset for every $i \in I$. For $j \in \alpha$, let μ_j be a σ-finite Borel measure defined on a Polish metric space (E_j, ρ_j). In [20], a new concept of a standard $(\alpha_i)_{i\in I}$-product of measures $(\mu_j)_{j\in\alpha}$ has been introduced and its existence has been proved. As consequence, "a standard $(\alpha_i)_{i\in I}$-Lebesgue measure" on the Borel σ-algebra of subsets of $\mathbb{R}^\alpha$ for every infinite parameter set α has been constructed such that it is invariant under a group generated by shifts and canonical permutations. A certain interesting application of the standard Lebesgue measure m^α in $\mathbb{R}^\alpha$ for a construction of uniform measures in the Banach space of all real-valued α-sequences ℓ^α for an arbitrary parameter set α has been considered.

The purpose of the paper is to introduce a new concept of α-standard and α-ordinary "Lebesgue measures" in infinite-dimensional separable Banach spaces.

The paper is organized as follows.

In Section 2 we give some basic definitions and results from measure theory and functional analysis.

In Section 3 we present our main results.

In Section 4 study topological structures of carriers of ordinary and standard Lebesgue measures in separable Banach spaces.

In Section 5 we give some examples of generators of two-sided (left or right) shy sets in non-locally compact Polish Groups.

2. Basic definitions and results from measure theory and functional analysis

a) *Contionuous mappings.*

Lemma 2.1 [5, Theorem 2.3.6]. *Let $(Y_i)_{i\in I}$ be a family of topological spaces and $Y = \prod_{i\in I} Y_i$ be their cartesian product endowed with product topology. Let $p_j : Y \to Y_j$ be the j-projection for $j \in I$, defined by*

$$p_j((y_i)_{i\in I}) = y_j \quad (y_i)_{i\in I} \in Y.$$

If X is a topological space, then a mapping $f : X \to Y$ is continuous if and only if each composition $p_j \circ f$, $j \in I$, is continuous.

Lemma 2.2 (Lusin-Souslin [12, Theorem 15.1]). *Let X,Y be Polish spaces and $f : X \to Y$ be continuous. If $A \subseteq X$ is Borel and $f|_A$ is injective then $f(A)$ is Borel.*

b) *Markushevich basis.*

Let X be an infinite-dimensional separable Banach space. A sequence $(x_k)_{k=1}^{\infty} \subset X$ is called *minimal* if each vector x_k is not contained in the closed linear span of $(x_l)_{l\neq k}$. A sequence in X is called *fundamental* if its closed linear span coincides with X. It is easy to verify that for a fundamental minimal sequence $(x_k)_{k=1}^{\infty}$ there exists a unique sequence $(x_k^*)_{k=1}^{\infty}$ of continuous linear functionals satisfying the condition $x_k^*(x_l) = \delta_{kl}$ $(k,l \in \mathbb{N})$. This sequence is called *biorthogonal* to $(x_k)_{k=1}^{\infty}$. Thus, if (x_k) is minimal and fundamental, then to each $x \in X$ there corresponds a generalized Fourier series

$$\sum_{k=1}^{\infty} x_k^*(x) x_k.$$

The vector x is uniquely determined by this series if and only if the biorthogonal sequence $(x_k^*)_{k=1}^{\infty}$ is *total* (that is for each $x \neq 0$ there exists $k \in \mathbb{N}$ such that $x_k^*(x) \neq 0$). A fundamental minimal sequence with a total biorthogonal sequence is called the *Markushevich basis* (M-basis in short). By the Markushevich theorem, for every countably-dimensional dense subspace L of a separable Banach space X there is an M-basis $(x_k, x_k^*)_{k=1}^{\infty}$ of X such that the linear span $\mathrm{lin}(x_i)_1^{\infty} = L$. [22, p. 226]. Conversely, each Banach space with M-basis is separable. We call an M-basis *absolutely convergent* if $\sum_{k=1}^{\infty} \|x_k\| < \infty$. The following statement follows immediately from the Markushevich theorem mentioned above.

Lemma 2.3([6, Lemma 2.3]). *Every infinite-dimensional separable Banach space has absolutely convergent M-basis.*

Lemma 2.4 ([6, Lemma 2.4]). *Let (x_k, x_k^*) be an absolutely convergent M-basis in a Banach space X. Then for every bounded scalar sequence (a_k) the series $\sum_{k=1}^{\infty} a_k x_k$ is absolutely convergent to some element $x \in X$ and moreover $x_k^*(x) = a_k$ for all k.*

Let $(x_k, x_k^*)_{k=1}^{\infty}$ be an absolutely convergent M-basis of a Banach space X and define a rectangle P by:

$$P = \{x \in X : |x_k^*(x)| \leq 1/2 \ \text{ for all } \ k \in \mathbb{N}\}. \tag{2.1}$$

Obviously, P is a compact subset in X.

In the sequel, unlike N. Burbaki well known notion, under N we understand a set $\{1,2,\cdots\}$. We denote by $\mathbb{R}^{\mathbb{N}}$ the vector space of all real-valued sequences equipped with the product topology. We denote by $e_k = (0,\dots,0,1,0,\dots)$, where 1 is in the k-th position, $k = 1,2,\dots$, the unit vectors of $\mathbb{R}^{\mathbb{N}}$.

If we define the operator $T : X \to \mathbb{R}^{\mathbb{N}}$ by

$$Tx = (x_k^*(x))_{k=1}^{\infty}, \tag{2.2}$$

then T is clearly linear, injective (because the M-basis is total), continuous (by Lemma 2.1) and $Tx_k = e_k$ for all k.

Lemma 2.5([6, Lemma 2.5]). *The subspace $S := T(X) \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ and the operator $T : X \to S$ is a Borel isomorphism.*

c) *Ordinary and Standard "Lebesgue measures" in $\mathbb{R}^{\mathbb{N}}$.*

Let $(\beta_j)_{j\in\mathbb{N}} \in [0,+\infty]^N$.

We say that a number $\beta \in [0,+\infty]$ is an ordinary product of numbers $(\beta_j)_{j\in\mathbb{N}}$ if

$$\beta = \lim_{n\to\infty} \prod_{i=1}^{n} \beta_i.$$

An ordinary product of numbers $(\beta_j)_{j\in\mathbb{N}}$ is denoted by $(\mathbf{O})\prod_{i\in\mathbb{N}} \beta_i$.

A standard product of the family of numbers $(\beta_i)_{i\in\mathbb{N}}$ is denoted by $(\mathbf{S})\prod_{i\in\mathbb{N}} \beta_i$ and defined as follows:

$(\mathbf{S})\prod_{i\in\mathbb{N}} \beta_i = 0$ if $\sum_{i\in\mathbb{N}^-} \ln(\beta_i) = -\infty$, where $N^- = \{i : ln(\beta_i) < 0\}$ [1], and $(\mathbf{S})\prod_{i\in\mathbb{N}} \beta_i = e^{\sum_{i\in\mathbb{N}} \ln(\beta_i)}$ if $\sum_{i\in\mathbb{N}^-} \ln(\beta_i) \neq -\infty$.

Let $\alpha = (n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$. We set

$$F_0 = [0,n_0]\cap\mathbb{N},\ F_1 = [n_0+1, n_0+n_1]\cap\mathbb{N},\ \dots, F_k = [n_0+\cdots+n_{k-1}+1, n_0+\cdots+n_k]\cap\mathbb{N}, \dots.$$

We say that a number $\beta \in [0,+\infty]$ is an ordinary α-product of numbers $(\beta_i)_{i\in\mathbb{N}}$ if β is an ordinary product of numbers $(\prod_{i\in F_k} \beta_i)_{k\in\mathbb{N}}$. An ordinary α-product of numbers $(\beta_i)_{i\in\mathbb{N}}$ is denoted by $(\mathbf{O},\alpha)\prod_{i\in\mathbb{N}} \beta_i$.

We say that a number $\beta \in [0,+\infty]$ is a standard α-product of numbers $(\beta_i)_{i\in\mathbb{N}}$ if β is a standard product of numbers $(\prod_{i\in F_k} \beta_i)_{k\in\mathbb{N}}$. A standard α-product of numbers $(\beta_i)_{i\in\mathbb{N}}$ is denoted $(\mathbf{S},\alpha)\prod_{i\in\mathbb{N}} \beta_i$.

Let $\alpha = (n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$. Let $(\alpha)\mathcal{OR}$ be the class of all infinite-dimensional measurable α-rectangles $R = \prod_{i\in\mathbb{N}} R_i (R_i \in \mathcal{B}(\mathbb{R}^{n_i}))$ for which an ordinary product of numbers $(m^{n_i}(R_i))_{i\in\mathbb{N}}$ exists and is finite.

We say that a measure λ being the completion of a translation-invariant Borel measure is an ordinary α-Lebesgue measure on $\mathbb{R}^\infty$(or, shortly, O(α)LM) if for every $R \in (\alpha)\mathcal{OR}$ we have

$$\lambda(R) = (\mathbf{O})\prod_{k\in\mathbb{N}} m^{n_k}(R_k).$$

Let $\alpha = (n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$. Let $(\alpha)\mathcal{SR}$ be the class of all infinite-dimensional measurable α-rectangles $R = \prod_{i\in\mathbb{N}} R_i (R_i \in \mathcal{B}(\mathbb{R}^{n_i}))$ for which a standard product of numbers $(m^{n_i}(R_i))_{i\in\mathbb{N}}$ exists and is finite.

We say that a measure λ being the completion of a translation-invariant Borel measure is a standard α-Lebesgue measure on $\mathbb{R}^\infty$(or, shortly, S(α)LM) if for every $R \in (\alpha)\mathcal{SR}$ we have

$$\lambda(R) = (\mathbf{S})\prod_{k\in\mathbb{N}} m^{n_k}(R_k).$$

Lemma 2.6([18, Proposition 1]). *Note that for every $\alpha = (n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ the following strict inclusion*

$$(\alpha)\mathcal{OR} \subset (\alpha)\mathcal{SR}$$

[1] We set $ln(0) = -\infty$

holds.

Lemma 2.7 [16, Theorem 1]). *For every $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, there exists a Borel measure μ_α on $\mathbb{R}^\infty$ which is $O(\alpha)LM$.*

Lemma 2.8 [16, Theorem 2]). *For every $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, there exists a Borel measure ν_α on $\mathbb{R}^\infty$ which is $(\alpha)LM$.*

Let μ_1 and μ_2 be two measures defined on the measurable space $(\mathbb{E},\mathbb{S})$.

Following [10, p. 124], we say that the μ_1 is absolutely continuous with respect to the μ_2, in symbols $\mu_1 \ll \mu_2$, if

$$(\forall X)(X \in \mathbb{S} \ \&\ \mu_2(X) = 0 \rightarrow \mu_1(X) = 0).$$

Following [10, p. 126], two measures μ_1 and μ_2 for which both $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$ are called equivalent, in symbols $\mu_1 \equiv \mu_2$.

Lemma 2.9 [16, Theorem 3]). *For every $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, we have $\nu_\alpha \ll \mu_\alpha$ and the measures ν_α and μ_α are not equivalent.*

Let $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $n_i = n_j$ for every $i, j \in \mathbb{N}$. We set $F_i = (a_1^{(i)}, \dots, a_{n_0}^{(i)})$ for every $i \in \mathbb{N}$. Let f be any permutation of $\mathbb{N}$ such that for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $f(a_k^{(i)}) = a_k^{(j)}$ for $1 \le k \le n_0$. Then a map $A_f : \mathbb{R}^\infty \to \mathbb{R}^\infty$ defined by $A_f((z_k)_{k\in\mathbb{N}}) = (z_{f(k)})_{k\in\mathbb{N}}$ for $(z_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, is called a canonical α-permutations of $\mathbb{R}^{\mathbb{N}}$.

A group of transformations generated by all α-permutations and shifts of $\mathbb{R}^\infty$, is denoted by $\mathcal{G}_\alpha$.

Lemma 2.10 ([16, Corollary 1]). *For every $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ for which $n_i = n_j (i, j \in \mathbb{N})$, the measure ν_α is $\mathcal{G}_\alpha$-invariant.*

Lemma 2.11 ([19, Theorem 3]). *Let $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, and let $T^{n_i} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ $(i > 1)$ be a family of linear transformations with Jacobians $\Delta_i \neq 0$ and $0 < \prod_{i=1}^{\infty} \Delta_i < \infty$. Let $T^N : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be a map defined by*

$$T^{\mathbb{N}}(x) = (T^{n_1}(x_1, \dots, x_{n_1}), T^{n_2}(x_{n_1+1}, \dots, x_{n_1+n_2}), \dots),$$

where $x = (x_i)_{i\in N} \in \mathbb{R}^{\mathbb{N}}$. Then for each $E \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$, we have

$$\mu_\alpha(T^N(E)) = \Big(\prod_{i=1}^{\infty} \Delta_i\Big)\mu_\alpha(E).$$

d) *Ordinary and Standard "Lebesgue measures" in an infinite dimensional separable Banach space.*

Let X be an infinite-dimensional separable Banach space and (x_k, x_k^*) be an absolutely convergent M-basis. Let $T : X \to \mathbb{R}^{\mathbb{N}}$ be a linear operator defined by (2.2).

Let $\alpha = (n_k)_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$. We say that a translation-invariant Borel measure μ on X is an ordinary α-Lebesgue measure on X(or, shortly, O(α)LM(X)) if for every $R \in (\alpha)\mathcal{OR}$ for which $R \cap T(X) \in (\alpha)\mathcal{OR}$ we have

$$\mu(T^{-1}(R)) = \mu_\alpha(R \cap T(X)),$$

where μ_α becomes from Lemma 2.7.

We say that a translation-invariant Borel measure μ on X is a standard α-Lebesgue measure on X(or, shortly, S(α)LM(X)) if for every $R \in (\alpha)\mathcal{SR}$ for which $R \cap T(X) \in (\alpha)\mathcal{SR}$ we have

$$\mu(T^{-1}(R)) = \nu_\alpha(R \cap T(X)),$$

where ν_α becomes from Lemma 2.8.

3. Main Results

Let X be an infinite-dimensional separable Banach space and (x_k, x_k^*) be an absolutely convergent M-basis.

Theorem 3.1. *For every $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, there exists a non-zero Borel measure ψ_α in X which is O(α)LM(X).*

Proof. For $Y \in \mathcal{B}(X)$ we set $\psi_\alpha(Y) = \mu_\alpha(T(Y))$, where μ_α becomes from Lemma 2.7 and T is defined by (2.2).

Obviously, ψ_α is a non-zero Borel measure because for parallelepiped P defined by (2.1), we have

$$\psi_\alpha(P) = \mu_\alpha(T(P)) = \mu_\alpha([-\frac{1}{2}, \frac{1}{2}]^{\mathbb{N}}) = 1.$$

The latter relation follows from the evident fact that $[-\frac{1}{2}, \frac{1}{2}]^{\mathbb{N}}$ belongs to the class O(α)LM for every $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$.

Let us show that ψ_α is translation invariant measure. Indeed, for $Y \in \mathcal{B}(X)$ and $h \in X$, we have

$$\psi_\alpha(Y+h) = \mu_\alpha(T(Y+h)) = \mu_\alpha(T(Y)+T(h)) = \mu_\alpha(T(Y)) = \psi_\alpha(Y).$$

Let us show that ψ_α is O(α)LM(X). Let $R \in (\alpha)\mathcal{OR}$ for which $R \cap T(X) \in (\alpha)\mathcal{OR}$. Then we have

$$\psi_\alpha(T^{-1}(R)) = \psi_\alpha(T^{-1}(R \cap T(X))) = \mu_\alpha(T(T^{-1}(R \cap T(X))) = \mu_\alpha(R \cap T(X)).$$

□

Theorem 3.2. *For every $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, there exists a non-zero Borel measure η_α in X which is S(α)LM(X).*

Proof. For $Y \in \mathcal{B}(X)$ we set $\eta_\alpha(Y) = \nu_\alpha(T(Y))$, where ν_α becomes from Lemma 2.8 and T is defined by (2.2). Obviously, η_α is non-zero Borel measure because for parallelepiped P defined by (2.1), we have

$$\eta_\alpha(P) = \nu_\alpha(T(P)) = \nu_\alpha([-\frac{1}{2}, \frac{1}{2}]^{\mathbb{N}}) = 1.$$

The latter relation follows from the evident fact that $[-\frac{1}{2}, \frac{1}{2}]^{\mathbb{N}}$ belongs to the class S(α)LM for every $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$.

Let us show that η_α is translation invariant measure. Indeed, for $Y \in \mathcal{B}(X)$ and $h \in X$, we have

$$\eta_\alpha(Y+h) = \nu_\alpha(T(Y+h)) = \nu_\alpha(T(Y)+T(h)) = \nu_\alpha(T(Y)) = \eta_\alpha(Y).$$

Let us show that η_α is S(α)LM(X). Let $R \in (\alpha)\mathcal{SR}$ for which $R \cap T(X) \in (\alpha)\mathcal{SR}$. Then we have

$$\eta_\alpha(T^{-1}(R)) = \eta_\alpha(T^{-1}(R \cap T(X))) = \nu_\alpha(T(T^{-1}(R \cap T(X))) = \nu_\alpha(R \cap T(X)).$$

□

Theorem 3.3. *For every $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ we have $\eta_\alpha \ll \psi_\alpha$ and η_α and ψ_α are not equivalent.*

Proof. Let us show that $\eta_\alpha \ll \psi_\alpha$. Indeed, let $\psi_\alpha(Y) = 0$ for $Y \in \mathcal{B}(X)$. Then we have

$$\psi_\alpha(Y) = \mu_\alpha(T(Y)) = 0.$$

By Lemma 2.9 we know that $\nu_\alpha \ll \mu_\alpha$ for every $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ which implies that $\nu_\alpha(T(Y)) = 0$. By the definition of η_α we deduce that the value $\nu_\alpha(T(Y))$ coincides with the value $\eta_\alpha(Y)$ which implies that $\eta_\alpha(Y) = 0$.

Now we have to show that η_α and ψ_α are not equivalent. For $i \in \mathbb{N}$, we set

$$D_i = \prod_{j\in F_i} [0, e^{\frac{(-1)^i}{i\times n_i}}].$$

It is obvious that $m^{n_i}(D_i) = e^{\frac{(-1)^i}{i}}$ for $i \in \mathbb{N}$.

Note that the sequence of positive real numbers $\{e^{\frac{(-1)^i}{i\times n_i}} : i \in \mathbb{N}\}$ is bounded from above by the number $e^{\frac{1}{2}}$.

Since $(x_k, x_k^*)_{k=1}^\infty$ is an absolutely convergent M-basis of a Banach space X, by Lemma 2.4 we deduce that the series $\sum_i \sum_{j\in F_i} a_j x_j$ is absolutely convergent to some element $x \in X$ when $|a_j| \le e^{\frac{(-1)^i}{i\times n_i}}$ for $j \in F_i$. By that reason we have that $T(D) = \prod_{i\in\mathbb{N}} D_i$, where D is defined by

$$D = \{\sum_i \sum_{j\in F_i} a_j x_j : 0 \le a_j \le e^{\frac{(-1)^i}{i\times n_i}} \text{ for } j \in F_i\}.$$

On the one hand we have

$$\eta_\alpha(D) = \nu_\alpha(T(D)) = \nu_\alpha(\prod_{i\in\mathbb{N}} D_i) = 0,$$

because

$$(\mathbf{S})\prod_{i\in\mathbb{N}} m^{n_i}(D_i) = (\mathbf{S})\prod_{i\in\mathbb{N}} e^{\frac{(-1)^i}{i}} = 0.$$

On the other hand we have

$$\psi_\alpha(D) = \mu_\alpha(T(D)) = \mu_\alpha(\prod_{i\in\mathbb{N}} D_i) = 2,$$

because

$$(\mathbf{O})\prod_{i\in\mathbb{N}} m^{n_i}(D_i) = (\mathbf{O})\prod_{i\in\mathbb{N}} e^{\frac{(-1)^i}{i}} = 2.$$

The latter relations imply that η_α and ψ_α are not equivalent. □

Remark 3.1. Let X be a Banach space with an absolutely convergent M-basis (x_k, x_k^*). By using [6, Proposition 3.1]), one can prove that η_α and ψ_α are invariant inner regular semi-finite non-σ-finite Borel measures in X for every $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $\eta_\alpha(P) = \psi_\alpha(P) = 1$. By using Theorem 3.3, we give a new proof of [6, Theorem 4.4]) asserted that *the class of invariant inner regular semi-finite non-σ-finite Borel measures in X which takes the value 1 on the set P fails the uniqueness property*.

Let $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $n_i = n_j$ for every $i, j \in \mathbb{N}$. We set $F_i = (a_1^{(i)}, \dots, a_{n_0}^{(i)})$ for every $i \in \mathbb{N}$. Let f be any permutation of $\mathbb{N}$ such that for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $f(a_k^{(i)}) = a_k^{(j)}$ for $1 \le k \le n_0$. If a map $\tilde{A}_f : X \to X$ defined by $\tilde{A}_f(\sum_k x_k x_k^*) = \sum_k x_{f(k)} x_k^*$ is convergent for all $x = \sum_k x_k x_k^* \in X$, then $\tilde{A}_f$ is called a canonical α-permutations of X.

We denote by $\tilde{\mathcal{G}}_\alpha$ a group generated by shifts of X and by such canonical α-permutations $\tilde{A}_f$ of X for which the following condition

$$(\forall x)(x \in X \to \tilde{A}_f(x) = T^{-1}(A_f(T(x))))$$

holds.

Theorem 3.4. *For every $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ for which that $n_i = n_j (i, j \in \mathbb{N})$, the measure η_α is $\tilde{\mathcal{G}}_\alpha$-invariant.*

Proof. By Theorem 3.2 we know that the measure η_α is translation invariant. We have to show that η_α is invariant under action of each canonical α-permutation (of X) $\tilde{A}_f$. Indeed, by definition of the canonical α-permutation of X and by Lemma 2.10, we have

$$(\forall Y)(Y \in \mathcal{B}(X) \to \eta_\alpha(\tilde{A}_f(Y)) = \nu_\alpha(T(\tilde{A}_f(Y))) =$$

$$\nu_\alpha(A_f(T(Y))) = \nu_\alpha(T(Y)) = \eta_\alpha(Y)).$$

This ends the proof of Theorem 3.4. □

Let $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, and $L^{n_i} := \{\sum_{j\in F_i} a_j x_j : (a_j)_{j\in F_i} \in \mathbb{R}^{n_i}\}$. Let $T^{n_i} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ $(i > 1)$ be a family of linear transformations.

Let $T^N : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be a map defined by

$$T^{\mathbb{N}}(x) = (T^{n_1}(x_1, \dots, x_{n_1}), T^{n_2}(x_{n_1+1}, \dots, x_{n_1+n_2}), \dots),$$

where $x = (x_i)_{i\in N} \in \mathbb{R}^{\mathbb{N}}$.

Let $\tilde{T}^{n_i} : L^{n_i} \to L^{n_i}$ be defined by

$$\tilde{T}^{n_i}(\sum_{j\in F_i} a_j x_j) = \sum_{j\in F_i} Pr_j(T^{n_i}((a_j)_{j\in F_i}))$$

for $i \in \mathbb{N}$, where Pr_j denotes j-th projection in X defined by $Pr_j(x) = x_j^*(x) x_j$.

Let $\tilde{T}^N : X \to X$ be a map defined by

$$\tilde{T}^N(x) = \sum_i Pr_{L^{n_i}}(x),$$

where $Pr_{L^{n_i}}$ denotes a usual projection on the vector subspace L^{n_i} defined by $Pr_{L^{n_i}}(x) = \sum_{j \in F_i} x_j^*(x) x_j$.

Theorem 3.5. *Let $\alpha = (n_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, and let $T^{n_i} : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ $(i > 1)$ be a family of linear transformations with Jacobians $\Delta_i \neq 0$ and $0 < \prod_{i=1}^{\infty} \Delta_i < \infty$. If $\tilde{T}^N$ is defined for every $x \in X$ such that the condition*

$$\Big(\forall x\Big)\Big(x \in X \to \tilde{T}^N(x) = T^{-1}(T^N(T(x)))\Big)$$

holds, then for each $Y \in \mathcal{B}(X)$ the following change of variable formula for the measure ψ_α

$$\psi_\alpha(\tilde{T}^N(Y)) = \Big(\prod_{i=1}^{\infty} \Delta_i\Big) \psi_\alpha(Y)$$

is valid.

Proof. By the definition of the measure ψ_α and by the assumption of the theorem we get

$$\psi_\alpha(\tilde{T}^N(Y)) = \mu_\alpha(T(\tilde{T}^N(Y))) = \mu_\alpha(T^N(T(Y))).$$

By using Lemma 2.11, we get

$$\psi_\alpha(\tilde{T}^N(Y)) = \mu_\alpha(T^N(T(Y))) = \Big(\prod_{i=1}^{\infty} \Delta_i\Big) \mu_\alpha(T(Y)) = \Big(\prod_{i=1}^{\infty} \Delta_i\Big) \psi_\alpha(Y).$$

This ends the proof of theorem. □

4. On a certain carrier of ordinary and standard Lebesgue measures in separable Banach spaces

Remark 4.1 Let X be a separable Banach space. Following [9], a set Y is called shy if it is a subset of a Borel set Y' for which $\mu(Y' + x) = 0$ for every $x \in X$ and some Borel probability measure μ such that $\mu(K) = \mu(X)$ for some compact K. A measure μ is said to be a transverse (or a transverse measure). A complement of a shy set is called prevalence. Following [17], a Borel measure μ in X is called a generator of shy sets in X if the validity of the condition $\overline{\mu}(Y) = 0$ for $Y \subset X$ implies that Y is shy in X, where $\overline{\mu}$ denotes the usual completion of the Borel measure μ. Following [17, Corollary 2.1], every quasifinite translation invariant Borel measure in X is a generator of shy sets in X. By using the latter result, we deduce that ν_α and ψ_α are generators of shy sets in X for each $\alpha = (n_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$.

The purpose of the present section is to study the topological structure of the carrier [2] of the ordinary and standard Lebesgue measures in separable Banach spaces.

[2] Let μ be a Borel measure defined in G. A Borel subset $A \subseteq G$ is called carrier of μ if $\mu(G \setminus A) = 0$.

Theorem 4.1. *For each $\alpha = (n_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, η_α(or ψ_α) is generator of shy sets in separable Banach space X and a carrier of that generator may be choice a meagre set which is not covered by countable family of compact sets in X.*

Proof. Since η_α(or ψ_α) is quasifinite translation invariant Borel measure in $\mathbb{R}^{\mathbb{N}}$, by [17](see Corollary 2.1, p. 241) we deduce that ν_α(or ψ_α) is generator of shy sets.

Following [14](see Theorem 16.5) the real axis $\mathbb{R}$ may be written as the disjoint union of a shy set X_1 and a meagre set X_2. We set $Z_1 := X_1 \times \mathbb{R}^{\mathbb{N}\setminus\{1\}}$ and $Z_2 := X_2 \times \mathbb{R}^{\mathbb{N}\setminus\{1\}}$.

It is obvious that $\nu_\alpha(Z_1) = 0$ (or $\mu_\alpha(Z_1) = 0$) which implies that Z_2 is carrier of ν_α(or μ_α). On the one hand, Z_2 is a meagre set in $\mathbb{R}^{\mathbb{N}}$ which is not covered by a countable union of compact sets in $\mathbb{R}^{\mathbb{N}}$. Indeed, assume the contrary and $Z_2 = \cup_{k\in\mathbb{N}} F_k$, where F_k is compact in $\mathbb{R}^{\mathbb{N}}$, then by [9](see Fact 8, p. 226) we will get that F_k is a shy set in $\mathbb{R}^{\mathbb{N}}$ for each $k \in \mathbb{N}$. Since ν_α (or μ_α) is generator of shy sets we claim that Z_1 is a shy set in $\mathbb{R}^{\mathbb{N}}$. By [9](see Fact $3''$, p. 224), we deduce that $Z_1 \cup \cup_{k\in\mathbb{N}} F_k = \mathbb{R}^{\mathbb{N}}$ is a shy set in $\mathbb{R}^{\mathbb{N}}$, which is contradiction because for each quasi-finite Borel measure μ in $\mathbb{R}^{\mathbb{N}}$ we have that $\mu(\mathbb{R}^{\mathbb{N}} + t) = \mu(\mathbb{R}^{\mathbb{N}}) > 0$ for each $t \in \mathbb{R}^{\mathbb{N}}$.

Now it is obvious that $T^{-1}(Z_2)$ is carrier of η_α(or ψ_α) which satisfies all conditions in Theorem 4.1

□

Corollary 4.1. *Every infinite dimensional separable Banach space may be written as the disjoint union of a shy set and a meagre set.*

Corollary 4.2. *In every infinite dimensional separable Banach space X there is a meagre set D which is a prevalence and is not covered by a countable union of compact sets.*

Corollary 4.3. *In every infinite dimensional separable Banach space X there is a meagre set D which is not covered by a countable union of compact sets and for which the following condition holds*

$$(\forall h)(h \in X \to D \cap (D+h) \textit{is prevalence}).$$

Corollary 4.4. *In every infinite dimensional separable Banach space X there is a meagre set D such that for arbitrary $h \in X$ there is a point $y \in D$ such that an infinite arithmetic progression $y, y+h, \cdots$ belongs to D.*

Proof. Let $D := Z_2$, where Z_2 comes from Theorem 4.1. Let consider sets $\{D - ih : i \in \mathbb{N}\}$. Since $\cap_{i\in\mathbb{N}}(D - ih)$ again is prevalence, there is $y \in \cap_{i\in\mathbb{N}}(D - ih)$. Since $y \in D - ih$ for $i \in \mathbb{N}$, there are $y_i \in D (i \in \mathbb{N})$ such that $y = y_i - ih$ for $i \in \mathbb{N}$ which means that $y_i = y + ih$. Hence an infinite arithmetic progression $y, y+h, \cdots$ belongs to D. □

Corollary 4.5. *In every infinite dimensional separable Banach space X there is a meagre set D such that for arbitrary $h \in X$ a set D_h defined by*

$$D_h = \{y : y \in D \ \& \ (\forall i)(i \in \mathbb{N} \to y + ih \in D)\}$$

is prevalence in X.

Proof. Let $D := Z_2$, where Z_2 comes from Theorem 4.1. Let consider sets $\{D - ih : i \in \mathbb{N}\}$. Note that $\cap_{i\in\mathbb{N}}(D-ih)$ is prevalence. We set $B_h = \cap_{i\in\mathbb{N}}(D-ih)$. Let $y \in B_h$. Since $y \in D-ih$ for $i \in \mathbb{N}$, there is $y_i \in D(i \in \mathbb{N})$ such that $y = y_i - ih$ for $i \in \mathbb{N}$ which means that $y_i = y+ih$. Hence an infinite arithmetic progression $y, y+h, \cdots$ belongs to D. Since $B_h \subseteq D_h$ we deduce that D_h is prevalence in X. □

Corollary 4.5 admits the following reformulation.

Corollary 4.6 *In every infinite dimensional separable Banach space X there is a meagre set D such that for an arbitrary $h \in X$, a set $X \setminus D_h$ is of η_α(or ψ_α)-measure zero(and, hence shy in X), where*

$$D_h = \{y : y \in D \ \& \ (\forall i)(i \in \mathbb{N} \to y+ih \in D)\}$$

.

Corollary 4.7. *In every infinite dimensional separable Banach space X there is a meagre set D such that for an arbitrary sequence of elements $h = (h_k) \in X$ a set D_h defined by*

$$D_h = \{y : y \in D \ \& \ (\forall i)(i \in \mathbb{N} \to y + \sum_{k=1}^{i} h_k \in D)\}$$

is prevalence in X.

Proof. Let $D := Z_2$, where Z_2 comes from Theorem 4.1. Let consider sets $\{D - \sum_{k=1}^{i} h_k : i \in \mathbb{N}\}$. Note that $\cap_{i\in\mathbb{N}}(D - \sum_{k=1}^{i} h_k)$ is prevalence. We set $B_h = \cap_{k\in\mathbb{N}}(D - \sum_{k=1}^{i} h_k)$. Let $y \in B_h$. Since $y \in D - \sum_{k=1}^{i} h_k$ for $i \in \mathbb{N}$, there is $y_i \in D(i \in \mathbb{N})$ such that $y = y_i - \sum_{k=1}^{i} h_k$ for $k \in \mathbb{N}$ which means that $y_i = y + \sum_{k=1}^{i} h_k$. Hence an infinite sequence $(y + \sum_{k=1}^{i} h_k)$ belongs to D. Since $B_h \subseteq D_h$ we deduce that D_h is prevalence in X. □

5. On generators of shy sets in non-locally compact Polish groups

Let $\mathbf{G}$ be a Polish group, by which we mean a separable group with a complete metric for which the transformation (from $\mathbf{G} \times \mathbf{G}$ onto $\mathbf{G}$) which sends $(\mathbf{x}, \mathbf{y})$ into $\mathbf{x}^{-1}\mathbf{y}$ is continuous. Let $\mathcal{B}(\mathbf{G})$ denotes the σ-algebra of Borel subsets of $\mathbf{G}$.

Definition 5.1. A Borel set $X \subseteq G$ is called two-sided-shy if there exists a Borel probability measure μ over G such that $\mu(fXg) = 0$ for all $f, g \in G$. A subset of a Borel two-sided-shy set is called also two-sided-shy. The measure μ is called a two-sided transverse to the Borel set X.

Definition 5.2. A Borel set $X \subseteq G$ is called left (or right)-shy if there exists a Borel probability measure μ over G such that $\mu(fX) = 0$(or $\mu(Xf) = 0$) for all $f \in G$. A subset of a Borel left (or right)-shy set is called also left (or right)-shy.The measure μ is called a left (or right) transverse to the Borel set X.

Definition 5.3. A Borel measure μ in G is called a generator of two-sided-shy sets in G, if

$$(\forall X)(\overline{\mu}(X) = 0 \to X \in S(G)),$$

where $\overline{\mu}$ denotes a usual completion of the Borel measure μ and $S(G)$ denotes a class of all two-sided-shy sets.

Definition 5.4. A Borel measure μ in G is called a generator of left (or right) shy sets in G, if

$$(\forall X)(\overline{\mu}(X) = 0 \rightarrow X \in \mathcal{LS}(G) \text{ (or } \mathcal{RS}(G))),$$

where $\mathcal{LS}(G)$ and $\mathcal{RS}(G)$ denote classes of all left-shy and right-shy sets in G, respectively.

Definition 5.5. A Borel measure μ in G is called quasi-finite if there exists a compact set $U \subseteq G$ for which $0 < \mu(U) < \infty$.

Definition 5.6. A Borel measure μ in G is called semi-finite if for X with $\mu(X) > 0$ there exists a compact subset $F \subseteq X$ for which $0 < \mu(F) < \infty$.

Definition 5.7. A Borel measure μ in G is called left invariant if

$$(\forall X)(\forall g)(X \in B(G)\ \&\ g \in G \rightarrow \mu(gX) = \mu(X)).$$

Definition 5.8. A Borel measure μ in G is called right invariant if

$$(\forall X)(\forall g)(X \in B(G)\ \&\ g \in G \rightarrow \mu(Xg) = \mu(X)).$$

Definition 5.9. A Borel measure μ in G is called two-sided invariant if

$$(\forall X)(\forall g, f)(X \in B(G)\ \&\ g, f \in G \rightarrow \mu(gXf) = \mu(X)).$$

Definition 5.10. A Borel measure μ in G is called left quasiinvariant if

$$(\forall X)(\forall g)(X \in B(G)\ \&\ g \in G \rightarrow (\mu(gX) = 0 \Longleftrightarrow \mu(X) = 0)).$$

Definition 5.11. A Borel measure μ in G is called right quasiinvariant if

$$(\forall X)(\forall g)(X \in B(G)\ \&\ g \in G \rightarrow (\mu(Xg) = 0 \Longleftrightarrow \mu(X) = 0)).$$

Definition 5.12. A Borel measure μ in G is called two-sided quasiinvariant if

$$(\forall X)(\forall g)(\forall h)(X \in B(G)\ \&\ g, h \in G \rightarrow (\mu(gXh) = 0 \Longleftrightarrow \mu(X) = 0)).$$

Definition 5.13. A Borel measure μ in G is called locally finite if there is a neighborhood U of unity such that $0 < \mu(U) < +\infty$.

Definition 5.14. Let K be the class of measures in G. We say that a measure $\mu \in K$ has the property of uniqueness in the class K if μ and λ are equivalent for every $\lambda \in K$.

Definition 5.15. Let G be equipped with a left invariant metric. Two sets A and B are said to be congruent if there exists an element $a \in G$ such that $B = aA$.

Definition 5.16. Let G be equipped with a two sided invariant metric. Two sets A and B are said to be congruent if there exists elements $a, b \in G$ such that $B = aAb$.

Remark 5.1. Following [13](see Theorem 3, p. 220), in any complete separable metric group which is dense in itself there exists a left-invariant quasi-finite Borel measure. By the scheme due to Oxtoby[13], in [21] has been established an existence of a two-sided-invariant quasi-finite Borel measure in any complete separable metric group with two-sided invariant metric which is dense in itself and has been demonstrated that there always exist infinitely many two-sided invariant quasi-finite Borel measures and no any such a measure possesses the uniqueness property. In case the group is locally compact, the construction may in some cases generate Haar's measure. In the additive group of real numbers, that construction gives such measures which are not locally finite.

Note also that every Polish group admits a compatible left-invariant metric. Such a metric need not be complete. A compatible two-sided invariant metric on a Polish group is necessarily complete (c.f. [3]). Each abelian Polish group admit a compatible two-sided invariant metric; it is also known that all compact Polish groups have an analogous property. There are locally compact Polish groups which do not admit a compatible two-sided invariant metric (an example is $SL_2(\mathbb{R})$). For a diagram of the relationship among classes of Polish groups which admit compatible two-sided invariant metric, see [4].

In the sequel we need the following auxiliary proposition.

Lemma 5.1 *Every quasi-finite two-sided(left or right) quasiinvariant Borel measure μ defined on a Polish group G is a generator of two-sided (left or right)-shy sets in G.*

Proof. We present the proof of Lemma 5.1 for a quasi-finite two-sided quasiinvariant Borel measure μ. One can get the validity of Lemma 5.1 similarly for quasi-finite left(or right) quasiinvariant Borel measures.

Let $\overline{\mu}(S) = 0$ for $S \subseteq G$. Since $\overline{\mu}(S) = 0$, there exists a Borel set $S^{'}$ for which $S \subseteq S^{'}$ and $\mu(S^{'}) = 0$. By using a two-sided quasiinvariance of the Borel measure μ, we have

$$(\forall f,g)(f,g \in G \to \mu(fX^{'}g) = 0).$$

Since μ is quasi-finite there is a Borel set F with $0 < \mu(F) < +\infty$. We set

$$(\forall X)(X \in \mathcal{B}(G) \to \lambda(X) = \frac{\mu(X \cap F)}{\mu(F)}).$$

Let us show that λ is a two-sided transverse to the Borel set $S^{'}$. Indeed, we have:

$$(\forall f,g)(f,g \in G \to \lambda(fX^{'}g) = \frac{\mu((fX^{'}g) \cap F)}{\mu(F)} \leq \frac{\mu(fX^{'}g)}{\mu(F)} = 0).$$

The latter relation means that $S^{'}$ is a Borel two-sided–shy set. S being a subset of $S^{'}$ also is two-sided-shy set. This ends the proof of Lemma 5.1. □

Since each two-sided(left or right) invariant Borel measure same times is two-sided(left or right) quasiinvariant, we get the following corollary of the Lemma 5.1.

Corollary 5.1. *Every quasi-finite two-sided (left or right) quasiinvariant Borel measure μ defined on a Polish group G is a generator of two-sided (left or right)-shy sets in G.*

Let us construct a sequence of positive numbers $(\sigma_k)_{k\in\mathbb{N}}$ for which a function f defined by

$$f(x_1,\cdots) = \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{x_k^2}{2\sigma_k^2}}$$

is convergent for all $(x_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\infty}$.

Formally we have

$$f(x_1,\cdots)=\prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_k}e^{-\frac{x_k^2}{2\sigma_k^2}}=\prod_{k=1}^{\infty}e^{\ln(\frac{1}{\sqrt{2\pi}\sigma_k})-\frac{x_k^2}{2\sigma_k^2}}=e^{\sum_{k=1}^{\infty}\left(\ln(\frac{1}{\sqrt{2\pi}\sigma_k})-\frac{x_k^2}{2\sigma_k^2}\right)}.$$

Note that if $\sum_{k=1}^{\infty}\ln(\frac{1}{\sqrt{2\pi}\sigma_k})$ is convergent then $(\sigma_k)_{k\in N}$ must tend to $\frac{1}{\sqrt{2\pi}}$. Then

$$e^{\sum_{k=1}^{\infty}\ln(\frac{1}{\sqrt{2\pi}\sigma_k})-\frac{x_k^2}{2\sigma_k^2}}$$

will be positive for all $(x_k)_{k\in N}\in\ell_2$ and equal to zero for all $(x_k)_{k\in N}\in\mathbb{R}^{\infty}\setminus\ell_2$.

For example, setting $\sigma_k=\frac{e^{-1/2^k}}{\sqrt{2\pi}}$ for $k\in\mathbb{N}$, we get

$$f_1(x_1,x_2,\cdots)=\prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_k}e^{-\frac{x_k^2}{2\sigma_k^2}}=$$

$$\prod_{k=1}^{\infty}\frac{1}{\sqrt{2\pi}\frac{e^{-1/2^k}}{\sqrt{2\pi}}}e^{-\frac{x_k^2}{2\frac{e^{-1/2^{k-1}}}{2\pi}}}=\prod_{k=1}^{\infty}e^{1/2^k-\pi x_k^2e^{1/2^{k-1}}}.$$

Obviously, that f is convergent for all $(x_k)_{k\in\mathbb{N}}\in\mathbb{R}^{\infty}$. More precisely, f_1 is positive for all $(x_k)_{k\in N}\in\ell_2$ and equal to zero for all $(x_k)_{k\in N}\in\mathbb{R}^{\infty}\setminus\ell_2$. Setting $\sigma_k=\frac{1}{\sqrt{2\pi}}$ for $k\in\mathbb{N}$, we get more a simple example of the infinite-dimensional Gaussian density function f_2 defined by

$$f_2((x_k)_{k\in\mathbb{N}})=e^{-\pi\sum_{k=1}^{\infty}x_k^2}.$$

For each natural number $n\in N$ and for an arbitrary sequence of positive numbers $(\sigma_k)_{1\leq k\leq n}$ the following equality

$$\int_{\mathbb{R}^n}(\prod_{k=1}^{n}\frac{1}{\sqrt{2\pi}\sigma_k}e^{-\frac{x_k^2}{2\sigma_k^2}})dm^n=1 \tag{5.1}$$

holds in an finite-dimensional Euclidean vector space $\mathbb{R}^n$, where m, as usual, denotes a linear Lebesgue measure in $\mathbb{R}$.

Here naturally arises a question asking whether the formula (5.1) admits an infinite analogue.

We have the following proposition.

Theorem 5.1. *Let E be an infinite dimensional Polish topological vector space and $f:E\to\mathbb{R}$ be positive real valued measurable function. Then there does note exist a translation-invariant Borel measure μ_∞ in E for which the following formula*

$$\int_E f(z)d\mu_\infty(z)=1$$

holds.

Proof. Assume the contrary and let μ_∞ be such a measure. We define a new Borel measure μ on E by the following formula

$$(\forall A)(A \in \mathcal{B}(E) \rightarrow \mu(A) = \int_A f(z) d\mu_\infty(z)).$$

It is obvious that μ is a Borel probability measure on E. We have to show that μ is quasiinvariant with respect to the group of all translation.

Let $\mu(A) > 0$ and $h \in E$. $\mu(A) > 0$ implies that $\mu_\infty(A) > 0$ because an integral of a positive measurable function over a set of μ_∞- measure zero is zero. A translation-invariance of μ_∞ implies that $\mu_\infty(A+h) = \mu_\infty(A) > 0$. Since f is positive at all points of E we deduce that $\mu(A+h) = \int_{A+h} f(z) d\mu_\infty(z) > 0$. It is obvious that μ is a quasi-invariant Borel probability measure in E which is a contradiction[3] and thus Theorem 5.1 is proved.

□

Theorem 5.2. *For every $\alpha = (n_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, let consider ordinary and standard Lebesgue measures ν_α and μ_α, respectively. Let $(\sigma_k)_{k \in \mathbb{N}}$ be such a sequence of positive numbers that the series $\sum_{k=1}^{\infty} \ln(\frac{1}{\sqrt{2\pi}\sigma_k})$ is convergent. Let $\Phi : \mathbb{R}^\infty \rightarrow \mathbb{R}$ be defined by*

$$\Phi((x_k)_{k \in \mathbb{N}}) = \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{x_k^2}{2\sigma_k^2}}.$$

Then the following conditions

$$\int_{\mathbb{R}^\infty} \Phi(z) d\nu_\alpha(z) = \int_{\mathbb{R}^\infty} \Phi(z) d\mu_\alpha(z) = 0$$

hold.

Proof. Indeed, $A_n := \mathbb{R}^n \times \prod_{k>n} [-1/3, 1/3]$ for each $n \in \mathbb{N}$.

Obviously, $\ell_2 \subseteq \cup_{n \in \mathbb{N}} A_n$. Clearly, $\nu_\alpha(A_n) = \mu_\alpha(A_n) = 0$ for each $n \in \mathbb{N}$. Hence $\nu_\alpha(\ell_2) \leq \sum_{k \in N} \nu_\alpha(A_n) = 0$ as well $\mu_\alpha(\ell_2) \leq \sum_{k \in N} \mu_\alpha(A_n) = 0$.

We put $M := |\sum_{k=1}^{\infty} \ln(\frac{1}{\sqrt{2\pi}\sigma_k})|$.

Since Borel subsets ℓ_2 and $\mathbb{R}^\infty \setminus \ell_2$ form a partition of $\mathbb{R}^\infty$, we deduce that a function $h : \mathbb{R}^\infty \rightarrow \mathbb{R}$, defined by

$$h(z) = e^M \times \chi_{\ell_2}(z) + 0 \times \chi_{R_I^\infty \setminus \ell_2}(z),$$

where $\chi_{(\cdot)}$ denotes a characteristic function of a set in $\mathbb{R}^\infty$, is a simple function for which the following conditions

$$\int_{\mathbb{R}^\infty} h d\nu_\alpha = \int_{\mathbb{R}^\infty} h d\mu_\alpha = 0$$

hold, because $\nu_\alpha(\ell_2) = \mu_\alpha(\ell_2) = 0$.

Since

[3] It is well known that in infinite dimensional separable Polish vector space there does not exist a non-zero quasiinvariant σ-finite Borel measure.

$$0 \le \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{x_k^2}{2\sigma_k^2}} \le e^M \times \chi_{\ell_2}(z) + 0 \times \chi_{R_I^\infty \setminus \ell_2}$$

we claim that

$$0 \le \int_{\mathbb{R}^\infty} \Phi(z) d\nu_\alpha(z) \le \int_{\mathbb{R}^\infty} h d\nu_\alpha = \int_{\mathbb{R}^\infty} h d\mu_\alpha = 0.$$

□

Theorem 5.3. *Let G be a Polish group and $\Phi : G \to \mathbb{R}$ be a bounded positive real valued measurable function. Let μ_∞ be a quasi-finite two-sided(left or right)-invariant Borel measure in G. Then a measure λ_∞ defined by*

$$(\forall X)(X \in \mathcal{B}(G) \to \lambda_\infty(X) = \int_X \Phi d\mu_\infty)$$

is such a quasi-finite two-sided(left or right) quasi-invariant Borel measure in G which is equivalent to the measure μ_∞. In addition, if G is a non-locally compact Polish group then λ_∞ is non-σ-finite.

Proof. It is obvious that μ is a Borel measure in G. We have to show that μ is two-sided(left or right) quasi-invariant with respect to the group of all translation of G. Without loss of generality we can assume that μ_∞ is a quasi-finite two-sided-invariant Borel measure in G

Let $\lambda_\infty(A) > 0$ and $h \in G$. $\lambda_\infty(A) > 0$ implies that $\mu_\infty(A) > 0$ because an integral of a positive measurable function over a set of μ_∞- measure zero is zero. Hence $\mu_\infty(A) > 0$. A two-sided-invariance of μ_∞ implies that $\mu_\infty(hAg) = \mu_\infty(A) > 0$. for each $h, g \in G$.

Since Φ is positive at all points of G we deduce that $\lambda_\infty(hAg) = \int_{hAg} \Phi(z) d\mu_\infty(z) > 0$. The latter relation means that λ_∞ is a two-sided-quasiinvariant Borel probability measure in G.

Let us show that λ_∞ is quasi-finite. Indeed, since μ_∞ is quasi-finite there is a Borel subset A_0 of G such that $0 < \mu_\infty(A_0) < +\infty$. Hence

$$0 < \lambda_\infty(A_0) = \int_{A_0} \Phi d\mu_\infty < +\infty$$

because Φ is positive and bounded.

In the case when G is non-locally compact, the proof of the fact that λ_∞ is non-σ-finite follows from the proposition asserted that in a non-locally compact Polish group there does not exist a non-zero two-sided quasiinvariant σ-finite Borel measure.

The proof of the fact that λ_∞ and μ_∞ are equivalent is obvious.

□

Example 5.1. Let $(\sigma_k)_{k \in \mathbb{N}}$ be such a sequence of positive numbers that the series $\sum_{k=1}^{\infty} \ln(\frac{1}{\sqrt{2\pi}\sigma_k})$ is convergent. Let $\Phi : \ell_2 \to \mathbb{R}$ be an infinite-dimensional Gaussian density function defined by

$$\Phi((x_k)_{k \in \mathbb{N}}) = \prod_{k=1}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{x_k^2}{2\sigma_k^2}}.$$

Let μ_∞ be an arbitrary quasi-finite translation-invariant Borel measure in an infinite-dimensional separable Hilbert space ℓ_2. Then the measure λ_∞ defined by Theorem 5.3 is a quasi-finite quasi-invariant non-σ-finite Borel measure in ℓ_2. That measure is called a "Generalized Gaussian Measure" in ℓ_2 defined by an infinite-dimensional Gaussian density function Φ and a quasi-finite translation-invariant Borel measure μ_∞ and is denoted by $\Gamma_{(\Phi,\mu_\infty)}$. As we see parameters which uniquely define a "Generalized Gaussian Measure" stand a sequence of positive numbers $(\sigma_k)_{k\in\mathbb{N}}$ and a quasi-finite translation-invariant Borel measure μ_∞.

An interesting partial case of the "Generalized Gaussian Measure" in ℓ_2 can be obtained when $\sigma_k = \frac{1}{\sqrt{2}}$ for each $k \in \mathbb{N}$. In that case we have

$$(\forall Y)(Y \in \mathcal{B}(\ell_2) \to \Gamma_{(\Phi,\mu_\infty)}(Y) = \int_Y \exp^{\{-\pi||z||^2\}} d\mu_\infty(z)),$$

where $||\cdot||$ denotes a usual norm in ℓ_2.

Example 5.2. Let X be an infinite-dimensional separable Banach space, $A \in \mathbb{R}^+$ and $\Phi : X \to \mathbb{R}$ be defined as follows

$$(\forall z)(z \in X \to \Phi(z) = e^{-A||z||^2}),$$

where $||\cdot||$ denotes a usual norm in X. Let μ_∞ be an arbitrary quasi-finite translation-invariant Borel measure in X. Then the measure λ_∞ defined by Theorem 5.3 is is a quasi-finite quasi-invariant non-σ-finite Borel measure in X and is called a "Generalized Gaussian Measure" in X.

Example 5.3. Let X be an infinite-dimensional separable Banach space, $A, B \in \mathbb{R}^+$ and $\Phi : X \to \mathbb{R}$ be defined as follows

$$(\forall z)(z \in X \to \Phi(z) = \frac{A}{B + ||x||^2})$$

where $||\cdot||$ denotes a usual norm in X. Let μ_∞ be an arbitrary quasi-finite translation-invariant Borel measure in X. Then the measure λ_∞ defined by Theorem 5.3 is is a quasi-finite quasi-invariant non-σ-finite Borel measure in X and is called a "Generalized Cauchy Measure" in X.

Example 5.4. Let $\mathbb{R}^\infty$ be an infinite-dimensional topological vector space of all real-valued sequence equipped with Tychonoff topology. Let $\Phi : \mathbb{R}^\infty \to \mathbb{R}$ be defined as follows

$$(\forall (x_k)_{k\in\mathbb{N}})((x_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty \to \Phi((x_k)_{k\in\mathbb{N}}) = exp\{-\sum_{k\in\mathbb{N}} \frac{|x_k|}{2^k(1+|x_k|)}\}).$$

Let μ_∞ be an arbitrary quasi-finite translation-invariant Borel measure in $\mathbb{R}^\infty$. Then the measure λ_∞ defined by Theorem 5.3 is a quasi-finite quasi-invariant non-σ-finite Borel measure in $\mathbb{R}^\infty$.

Example 5.5. Let $\mathbb{R}^\infty$ be an infinite-dimensional topological vector space of all real-valued sequence equipped with Tychonoff topology and $A \in R^+$. Let $\Phi : \mathbb{R}^\infty \to \mathbb{R}$ be defined as follows

$$(\forall (x_k)_{k\in\mathbb{N}})((x_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty \to \Phi((x_k)_{k\in\mathbb{N}}) = exp\{-A(\sum_{k\in\mathbb{N}} \frac{|x_k|}{2^k(1+|x_k|)})^2\}).$$

Let μ_∞ be an arbitrary quasi-finite translation-invariant Borel measure in $\mathbb{R}^\infty$. Then the measure λ_∞ defined by Theorem 5.3 is a quasi-finite quasi-invariant non-σ-finite Borel measure in $\mathbb{R}^\infty$ and is called a "Generalized Gaussian Measure" in $\mathbb{R}^\infty$.

Example 5.6. Let $\mathbb{R}^\infty$ be an infinite-dimensional topological vector space of all real-valued sequence equipped with Tychonoff topology. Let $\Phi : \mathbb{R}^\infty \to \mathbb{R}$ be defined as follows

$$(\forall (x_k)_{k\in\mathbb{N}})((x_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty \to \Phi((x_k)_{k\in\mathbb{N}}) = \frac{A}{B + (\sum_{k\in\mathbb{N}} \frac{|x_k|}{2^k(1+|x_k|)})^2}),$$

where $A, B \in \mathbb{R}^+$.

Let μ_∞ be an arbitrary quasi-finite translation-invariant Borel measure in $\mathbb{R}^\infty$. Then the measure λ_∞ defined by Theorem 5.3 is a quasi-finite quasi-invariant non-σ-finite Borel measure in $\mathbb{R}^\infty$ and called a "Generalized Cauchy Measure" in $\mathbb{R}^\infty$.

Example 5.7. Let (G, ρ) be a non-locally compact Polish group and let $\Phi : G \to \mathbb{R}$ be defined as follows

$$(\forall g)(g \in G \to \Phi(g) = exp\{-\pi\rho(e,g)^2\}),$$

where e denotes a unit element of the group G. Let μ_∞ be an arbitrary quasi-finite translation-invariant Borel measure in G. Then the measure λ_∞ defined by Theorem 5.3 is a quasi-finite quasi-invariant non-σ-finite Borel measure in G and called a "Generalized Gaussian Measure" in a group G.

Example 5.8. Let (G, ρ) be a non-locally compact Polish group and let $\Phi : G \to \mathbb{R}$ be defined as follows

$$(\forall g)(g \in G \to \Phi(g) = \frac{A}{B + \rho(e,g)^2}),$$

where e denotes a unit element of the group G and $A, B \in \mathbb{R}^+$. Let μ_∞ be an arbitrary quasi-finite translation-invariant Borel measure in G. Then the measure λ_∞ defined by Theorem 5.3 is a quasi-finite quasi-invariant non-σ-finite Borel measure in G and called a "Generalized Cauchy Measure" in a group G.

Remark 5.2. Following Lemma 5.1, all measures, constructed in Examples 5.1-5.8 are generators of two-sided (left or right) shy sets.

In context of Theorem 5.3 we state the following

Problem 5.1 Let (G, ρ) be a non-locally compact Polish group and λ_∞ be a quasi-finite quasi-invariant Borel measure in G. Does there exist a quasi-finite translation-invariant Borel measure μ_∞ in G which is equivalent to the measure λ_∞?

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References

[1] R. Baker, "Lebesgue measure" on $\mathbf{R}^\infty$, *Proc. Amer. Math. Soc.*, **113** (4) (1991), 1023–1029.

[2] R. Baker, "Lebesgue measure"on $\mathbb{R}^\infty$. II. *Proc. Amer. Math. Soc.* vol. 132, no. 9, 2003, pp. 2577–2591.

[3] H. Becker and A.S. Kechris , *The Descriptive Set Theory of Polish Group Actions,* London Mathematical Society Lecture Note Series, **232**, Cambridge University Press, 1996.

[4] H. Becker, Polish group actions: Dichotomies and generalized elementary embed- dings, *J. Amer. Math. Soc.* **11(2)** (1998), 397–449.

[5] R. Engelking, *Outline of general topology*, PWN, Warsaw; North-Holland, Amsterdam, 1968.

[6] T. Gill, A.Kirtadze, G.Pantsulaia , A.Plichko, The existence and uniqueness of translation invariant measures in separable Banach spaces, *Functiones et Approximatio, Commentarii Mathematici*, 16 pages, to appear.

[7] T.L. Gill, G.R. Pantsulaia, W.W. Zachary, Constructive analysis in infinitely many variables, *Communications in Mathematical Analysis*, **13** (1), (2012), 107–141.

[8] T.L. Gill, W.W. Zachary, Lebesgue measure on a version of $\mathbb{R}^\infty$, *Real Analysis Exchange*, Summer Symposium (2009), 42–49.

[9] B.R. Hunt, T. Sauer, J.A. Yorke, Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces, *Bull. Amer. Math. Soc.* (N.S.), **27** (2), (1992), 217–238.

[10] P.R. Halmos, *Measure theory*, Princeton, Van Nostrand (1950).

[11] S. Kakutani, On equivalence of infinite product measures. *Ann. Math.*, **49** (1948), 214–224.

[12] A.S. Kechris, *Classical descriptive set theory,* Graduate Texts in Mathematics, 156. Springer-Verlag, New York, 1995. xviii+402 pp.

[13] J.C. Oxtoby, Invariant measures in groups which are not locally compact, *Trans. Amer. Math. Soc.*, **60** (1946), 215–237.

[14] J.C. Oxtoby, *Measure and Category*, Springer-Verlag, 1971, second ed. 1980.

[15] G.R. Pantsulaia, Relations between shy sets and sets of ν_p-measure zero in Solovay's Model, *Bull. Polish Acad. Sci. Math.* , **52**, No.1, (2004), 63–69.

[16] G.R. Pantsulaia, *Invariant and quasiinvariant measures in infinite-dimensional topological vector spaces.* Nova Science Publishers, Inc., New York, 2007. xii+234 pp.

[17] G.R. Pantsulaia, On generators of shy sets on Polish topological vector spaces. *New York Journal of Mathematics*, **14** (2008), 235–261.

[18] G.R. Pantsulaia, On ordinary and standard Lebesgue measures on $\mathbb{R}^\infty$, *Bull. Pol. Acad. Sci. Math.* **57 (3-4)** (2009), 209–222.

[19] G.R. Pantsulaia, Change of variable formula for "Lebesgue measures" on $\mathbb{R}^\mathbb{N}$., *J. Math. Sci. Adv. Appl.* 3, 2 (1) (2009),1–12.

[20] G.R. Pantsulaia, On ordinary and standard products of infinite family of σ -finite measures and some of their applications. *Acta Math. Sin. (Engl. Ser.)* 27 (2011), no. 3, 477–496.

[21] G.R. Pantsulaia, On existence and uniqueness of generators of shy sets in Polish groups, *J. Math. Sci. Adv. Appl. Volume no :16, Issue no: 1-2, July and August(2012) pp.61 –78.*

[22] I. Singer, *Bases in Banach spaces.* II, Springer, Berlin et al. 1981.