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On distribution of the norm for normal random elements in the space of continuous functions

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Dedicated to the memory of Yu.I. Petunin, our teacher and friend

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Abstract. We consider distributions of norms for normal random elements X in separable Banach spaces, in particular, in the space $C(S)$ of continuous functions on a compact space S. We prove that, under some nondegeneracy condition, the functions $\mathcal{F}_X = \{\mathbf{P}(\|X - z\| \leqslant r)\colon z \in C(S)\}$, $r \geqslant 0$, are uniformly Lipschitz and that every separable Banach space B can be ε-renormed so that the family $\mathcal{F}_X$ becomes uniformly Lipschitz in the new norm for any B-valued nondegenerate normal random element X.

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1 Introduction

Throughout the paper, X denotes a random element (r.e.) with values in a separable Banach space B. For every $z \in B$, put $X_z = X - z$. Let $F_z(r) = \mathbf{P}\{\|X_z\| \leqslant r\}$, $r \geqslant 0$, be the distribution function of $\|X_z\|$. The aim of this paper is to investigate the regularity (continuity, uniform continuity, and Lipschitz property) of these functions. A real function $f(r)$, defined on a set $T \subset \mathbb{R}$, is *uniformly continuous* on T if for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\big|f(r) - f(s)\big| < \varepsilon \quad \text{for all } r, s \in T \text{ with } |r - s| < \delta. \tag{1.1}$$

A function f is *Lipschitz* on T if there exists a constant λ such that $|f(r) - f(s)| \leqslant \lambda|r - s|$ for all $r, s \in T$. Each Lipschitz function is uniformly continuous. The converse statement is false. If a function f is differentiable on an interval T, then by the mean value theorem, its derivative is bounded on T if and only if f is Lipschitz on T. A family of functions $\mathcal{F}$ is *continuous* on T if each $f \in \mathcal{F}$ is. It is *uniformly equicontinuous* on T, provided that (1.1) holds uniformly on $f \in \mathcal{F}$. The definition of a *uniformly Lipschitz* family $\mathcal{F}$ on T is clear.

These regularity conditions appear naturally in probability. For instance, the Lipschitz property of F_z plays an important role in the approximation of sums of independent r.e. by normal distributions; see [10, Chap. 4] and [16]. The uniform equicontinuity of the family $\mathcal{F}_X = \{F_z(r)\colon z \in B\}$, $r \geqslant 0$, appears in generalizations of the Glivenko–Cantelli theorem to balls in Banach spaces; see [7] and references therein. The regularity of F_z was studied mainly for normal r.e., stable r.e., and series of random variables with vector coefficients. We restrict our attention to normal r.e.

It is reasonable to consider the regularity of F_z in a broader context. Given a subset U of a Banach space B and $\delta > 0$, we denote by $\partial_\delta U$ the δ-neighborhood of the boundary ∂U. Let a probability measure μ be defined on Borel subsets of B. A class $\mathcal{U}$ of Borel subsets in B is *μ-continuous* if $\mu(\partial U) = 0$ for each set $U \in \mathcal{U}$. This class is *uniformly μ-continuous* if $\sup_{U\in\mathcal{U}} \mu(\partial_\delta U) \to 0$ as $\delta \to 0$. We call the class $\mathcal{U}$ *uniformly μ-Lipschitz* if there exists a constant λ such that $\sup_{U\in\mathcal{U}} \mu(\partial_\delta U) \leqslant \lambda\delta$ for all $\delta > 0$. These concepts were considered, e.g., in [10, Chap. 4] and [11]; see also references therein. In particular, the class of convex sets in $\mathbb{R}^n$ is uniformly μ-Lipschitz for any nondegenerate Gaussian measure μ [10, p. 81]. Let $\mathcal{B}$ denote the class of all balls in a space B, and $\mathcal{B}_0$ denote the class of all balls with centers at zero. Let $\mu_X(A) = \mathbf{P}\{X \in A\}$ be the distribution of a r.e. X. The continuity of $F_0(r)$ on $[0, \infty)$ is equivalent to the μ_X-continuity of $\mathcal{B}_0$, and the continuity of the family $\mathcal{F}_X$ is equivalent to the μ_X-continuity of $\mathcal{B}$. The same equivalence holds for uniform equicontinuity and uniform Lipschitz property.

A closed subspace $E \subset B$ is the (linear) *support* of a probability measure μ if $\mu(E) = 1$ and $\mu(E') < 1$ for each proper closed subspace $E' \subset E$. Every probability measure μ on a separable Banach space has the support, which we denote by $\operatorname{supp}\mu$. A measure μ is called *nondegenerate* if $\operatorname{supp}\mu = B$. A r.e. X is *nondegenerate* if μ_X is. If E is a subspace of the space B with $\mu(E) = 1$, then the measure μ is said to be *concentrated* on E. A r.e. X is concentrated on E if μ_X is.

The continuity of F_z is well known for normal X and $z \in \operatorname{supp}\mu_X$; see, e.g., [4, Prop. 3.2]. If X is normal and symmetric ($\sim \mathbf{E}X = 0$), and the measure μ_X is not concentrated at a single point, then the function F_0 is uniformly continuous; see, e.g., [10, p. 83] or [11, Prop. 2.4 and Cor. 2.26]. Separability is essential here. The well-known Marcus and Shepp [6] example shows that the previous statement is false in the (nonseparable) space ℓ_∞. If a measure μ_X is degenerate, then the continuity of F_z essentially depends on geometry of B and on the subspace $\operatorname{supp}\mu_X$. Some relevant results are presented in [11]. Below, in Lemma 1, we present one more positive result. On the other hand, the following simple negative result holds.

Remark 1. If the norm of a space B is nonrotund, then there exists a (degenerate) normal r.e. X in B and $z \in B$ such that the corresponding function F_z is discontinuous.

Indeed, since the norm is nonrotund, there exist a one-dimensional subspace $L \subset B$, an element $z \in B$, and $r_0 > 0$ such that the sphere S with center z and radius r_0 intersects L by a nontrivial segment. Take a normal r.e. X with distribution concentrated on L. Then the corresponding function $F_z(r)$ is discontinuous at the point r_0.

Note also that on any (finite- or infinite-dimensional) Banach space, there exists a (degenerate) normal r.e. $X \neq \mathrm{const}$ for which the family $\mathcal{F}_X$ is not uniformly equicontinuous [11, Prop. 2.14].

Now we turn to the Lipschitz property. Convexity conditions on a norm, under which the function $F_z(r)$ has bounded on $[0, \infty)$ derivative, were studied in many papers. To the best of our knowledge, the first in this direction was Vakhania's [17, pp. 96–97] result that the derivative of $F_z(r)$ for a normal r.e. in a Hilbert space is bounded. For further progress, see [2] and [10]. Finally, Rhee and Talagrand [14] proved that $F_z(r)$ is Lipschitz on $[0, \infty)$, provided that the norm of B is uniformly convex with modulus of uniform convexity $\geqslant \varepsilon^p$ for some p.

In the opposite direction, Paulauskas [9] and Rhee and Talagrand [13] showed independently that $F_0(r)$ may be non-Lipschitz on $[0, \infty)$. Paulauskas' example concerned a c_0-valued r.e., while Rhee–Talagrand's example related to some equivalent renorming of ℓ_2. Paulauskas [9] noted that the idea how to construct a normal r.e. in $C[0, 1]$ having unbounded density belongs to Tsirelson (1977). The Paulauskas example shows that in c_0 there is a normal symmetric r.e. for which the function $F_0(r)$ is uniformly continuous but non-Lipschitz. One may ask to what extent F_0 is "non-Lipschitz"? Paulauskas and Račkauskas [10, p. 88] proved that for every continuous increasing function $\omega(r)$, $\omega(0) = 0$, there exists a normal symmetric c_0-valued r.e. X with independent components for which

$$\sup_{r>s\geqslant 0} \frac{F_0(r) - F_0(s)}{\omega(r - s)} = \infty.$$

As far as we know, the dependence of Lipschitz constant on $\|z\|$ has been considered for the first time by Paulauskas [8]. He proved that for a normal symmetric r.e. in a Hilbert space,

$$\sup_{r>0} \left|F_z'(r)\right| \leqslant c\big(1+\|z\|\big). \tag{1.2}$$

Hence, the family $\{F_z(r)\colon\ \|z\|<C\}$, $r\geqslant 0$, is uniformly Lipschitz for every $C>0$. The same concerns ℓ_p, $p>1$, but for these spaces, we have in inequality (1.2) $\|z\|^k$ instead of $\|z\|$, where $k>0$ depends on p (see [9]). On the other hand, the family $\mathcal{F}_X$ cannot be uniformly equicontinuous in any smooth infinite-dimensional Banach space (see [11, Cor. 1.4]) and, hence, cannot be uniformly Lipschitz. Until now, no Banach space was known for which $\mathcal{F}_X$ is uniformly Lipschitz for all nondegenerate normal X. Each nondegenerate normal X in the space ℓ_1 having at least one independent component has this property (see [10, p. 100]). We show, in particular, that the existence of an independent component is superfluous here.

The mentioned results were generalized to stable r.e., series of scalar random variables with vector coefficients, and seminorms in topological vector spaces; see, e.g., [15] and the references therein.

In this paper, we prove the following three theorems.

Theorem 1. (a) *Let X be a r.e. in the space $C(S)$, where S is a metric compact space, and let the class $\mathcal{C}_E$ of all convex bodies in the subspace $E=\operatorname{supp}\mu_X$ be μ-continuous. Then the family $\mathcal{F}_X$ is continuous, provided that the following condition is satisfied*:

($\mathcal{M}$) $\mathbf{P}\{X(s_0)=0\}<1$ *for every* $s_0\in S$.

(b) *If X is an arbitrary r.e. in $C(S)$ and the condition ($\mathcal{M}$) is not satisfied, then $\mathcal{F}_X$ contains a discontinuous function.*

Remark 2. The class $\mathcal{C}_B$ is μ-continuous for every nondegenerate Gaussian measure μ on B; see, e.g., [11, Prop. 2.4]. So, Theorem 1(a) implies immediately the well-known result of Ylvisaker [19] that, for normal X with $\mathbf{E}X=0$, the condition ($\mathcal{M}$) guarantees the continuity of the family $\mathcal{F}_X$.

Let W and W^0 be the Brownian process and Brownian bridge process on $[0,1]$. For $X=W$ and W^0, the Lipschitz property is especially important in the approximation of sums of independent $C[0,1]$-valued r.e. by normal distributions. Theorem 1(b) confirms the well-known fact that the function $F_z(r)$ for these r.e. is even discontinuous for some $z\in C[0,1]$.

The following question is natural: Can the above-mentioned Ylvisaker result be improved to obtain uniform equicontinuity or even uniform Lipschitz property? By an Aniszczyk statement [1], the μ-continuity of the class $\mathcal{B}$ in $C(S)$ implies its uniform μ-continuity. Unfortunately, we do not know a proof of this statement. Nevertheless, the following result is true.

Theorem 2. *Let X be a normal r.e. with $\mathbf{E}X=0$ in the space $C(S)$. If ($\mathcal{M}$) holds, then the family $\mathcal{F}_X$ is uniformly Lipschitz, i.e., there is a constant λ such that, uniformly on $z\in B$ and $r,\delta>0$,*

$$\mathbf{P}\big\{\|X_z\|\in(r,r+\delta)\big\}\leqslant\lambda\delta. \tag{1.3}$$

Therefore, for symmetric Gaussian measures, Theorems 1(b) and 2 give a sharpening of the Aniszczyk statement to the uniform Lipschitz property. Theorem 1(b) and Proposition 1 below show that the same is true for other distributions.

Theorem 3. *Each separable Banach space B can be ε-renormed so that, for every nondegenerate normal symmetric r.e. in B, the family $\mathcal{F}_X$ becomes uniformly Lipschitz in the new norm.*

Compare this “nice” renorming to Rhee’s [12] “bad” renorming: any Banach space can be ε-renormed so that, for some symmetric normal r.e. X, the corresponding function $F_0(r)$ is non-Lipschitz on $[0,\infty)$.

2 Proofs of the main results

Proof of Theorem 1. Part (a). We obtain part (a) of Theorem 1 as a corollary of the following general statement. Denote by U_{B^*} the dual unit ball of a Banach space B.

Lemma 1. *Assume that a probability measure μ on a separable Banach space B has the support E and the class $\mathcal{C}_E$ is μ-continuous. The class $\mathcal{B}$ of all balls in B is μ-continuous, provided that there exists a weakly* closed subset $\Gamma \subset U_{B^*}$ such that*:

(m) *for every $x \in B$, there is $x^* \in \Gamma$ with $|x^*(x)| = \|x\|$, and, for every $x^* \in \Gamma$, there is $x \in E$ with $x^*(x) \neq 0$.*

Proof. Take an arbitrary ball U of B with center z and radius r. There are two nontrivial possibilities:

(1) $C := U \cap E \not\subset \partial U$, where ∂U denotes the boundary in B. Then C is a convex body in E; moreover, there exists a point $x_0 \in C$ that is interior both for U in B and for C in E. Hence, any point x of ∂U is a unique intersection of a ray R starting in x_0 with ∂U. Therefore, any neighborhood of x in R contains pints from $B \setminus U$. If $R \subset E$, then, by the definition of C, the point x contains pints from $E \setminus C$ and so belongs to ∂C. So,

$$\mu(\partial U) = \mu(\partial U \cap E) = \mu(\partial C) = 0.$$

(2) $C \subset \partial U$. We show that in this case $\mu(C) = 0$. Let (x_n) be a countable dense subset of C. By the condition (m), for every n, there exists $x_n^* \in \Gamma$ such that

$$\left| x_n^* \left(\frac{1}{n} \sum_1^n x_k - z \right) \right| = r.$$

Then, for some subsequence (n_i), the values of the functionals $x_{n_i}^*$ have the same sign, say

$$x_{n_i}^* \left(\frac{1}{n_i} \sum_1^{n_i} x_k - z \right) = r.$$

Hence, $x_{n_i}^*(x_k - z) = r$ for all $k \leqslant n_i$. Since the set Γ is weakly* compact, $(x_{n_i}^*)$ contains a subsequence convergent weakly* to some functional $x^* \in \Gamma$. By construction, $x^*(C - z) \equiv r$, so $x^*(C) \equiv r + x^*(z)$. Obviously, the μ-continuity of $\mathcal{C}_E$ implies that the μ-measure of each hyperplane in E is equal to zero. Therefore, $\mu(C) = 0$, and so $\mu(\partial U) = 0$. □

Part (b). Suppose that the measure μ_X is concentrated on a subspace $\{x \in C(S)\colon\ x(s_0) = 0\}$. Pick some $0 < p < 1$. The Ulam theorem implies that there exists a convex symmetric compact set $K \subset \{x \in C(S)\colon x(s_0) = 0\}$ such that $\mu_X(K) \geqslant p$. By the equicontinuity of any compact set in $C(S)$, the function $\omega(s) = \sup_{x \in K} |x(s)|$ is continuous. Put $a = 2 \max_S \omega(s)$ and $z(s) = \omega(s) - a$. Then

$$\mathbf{P}\big\{\forall s \in S,\ \big|X(s)\big| \leqslant \omega(s)\big\} \geqslant p,$$

and, hence,

$$\mathbf{P}\big\{\forall s \in S,\ \big|X_z(s)\big| = \big|X(s) - \omega(s) + a\big| \leqslant a,\ X_z(s_0) = a\big\} \geqslant p.$$

So, $\mathbf{P}\{\|X_z\| = a\} \geqslant p$, and the corresponding function $F_z(r)$ is discontinuous at the point $r_0 = a$. □

Since $\mathbf{E}X = 0$, X may be represented in the form of an a.s. norm convergent series

$$X = \sum_{m \geqslant 1} \gamma_m x_m \tag{2.1}$$

[18, p. 262]. Here, (γ_m) are independent copies of the standard normal random variable γ, and $x_m \in C(S)$. The functions (x_m) have a few zeros in the following sense.

Lemma 2. (*A generalization of Lemma* 4 *from* [5].) *Assume that a r.e. X has the form* (2.1) *and satisfies condition* ($\mathcal{M}$). *Then there exist a collection $(S_k)_1^n$ of compact subsets of S with $\bigcup_1^n S_k = S$ and a subset $(x_{m_k})_{k=1}^n \subset (x_m)$ such that every function $x_{m_k}(s)$ is either strictly positive or strictly negative on S_k.*

Proof. By condition ($\mathcal{M}$) the sets $\{s \in S\colon x_m(s) > 1/i\}$ and $\{s \in S\colon x_m(s) < -1/i\}$, $m, i = 1, 2, \ldots,$ form an open covering of the compact set S. Choose from it a finite subcovering. Now we take the closures of the elements of this subcovering as the compact sets S_k and the corresponding x_m as x_{m_k}. □

Lemma 3. *Let $X = \gamma x + Y$, where $x \in C(S)$ with $c := \inf_S x(s) > 0$, and a r.e. Y in $C(S)$ does not depend on γ. Then, for all $z \in C(S)$ and $r, \delta > 0$,*

$$\mathbf{P}\big\{\|X_z\|_{L_\infty} \in (r, r+\delta)\big\} \leqslant \frac{\delta}{c}. \tag{2.2}$$

Proof. Let us first consider a simple (though discontinuous) function x of the form

$$x(s) = \sum_{i=1}^n a_i \chi_i(s), \tag{2.3}$$

where χ_i are the characteristic functions of Borel disjoint sets A_i with $\bigcup_1^n A_i = S$, and $a_i \geqslant c$. Put $y_i = \sup_{s\in A_i}(Y(s) - z(s))$. Then

$$P^+(\delta) := \mathbf{P}\Big\{\sup_S X_z(s) \in (r, r+\delta)\Big\} = \mathbf{P}\Big\{\max_{1\leqslant i\leqslant n}(a_i\gamma + y_i) \in (r, r+\delta)\Big\}. \tag{2.4}$$

Assume temporary Y to be nonrandom (then y_i are constants) and put

$$g(t) = \max_{1\leqslant i\leqslant n}(a_i t + y_i), \quad t \in \mathbb{R}.$$

Obviously, the function g is increasing and continuous, and its graph consists of line segments $a_i t + y_i$, $i = 1, 2, \ldots, n$. The derivative g' exists outside of finite set, and, moreover, $g'(t) \geqslant c$. By the mean value theorem the length of $g^{-1}(r, r+\delta)$ can be estimated as follows:

$$\big|g^{-1}(r, r+\delta)\big| = g^{-1}(r+\delta) - g^{-1}(r) \leqslant \frac{\delta}{c}. \tag{2.5}$$

As is well known, the standard normal distribution function is Lipschitz, more precisely,

$$\varPhi(r+\delta) - \varPhi(r) \leqslant \varPhi(\delta) - \varPhi(0) \leqslant \frac{1}{\sqrt{2\pi}}\delta < \delta.$$

This inequality, together with (2.4) and (2.5), gives

$$P^+(\delta) = \mathbf{P}\big\{\gamma \in g^{-1}(r, r+\delta)\big\} \leqslant \mathbf{P}\Big\{\gamma \in \Big(0, \frac{\delta}{c}\Big)\Big\} \leqslant \frac{\delta}{c}.$$

The right-hand side here does not depend on Y, so by the independence of γ and Y, this inequality is true for random Y as well.

Similarly, one can obtain

$$P^-(\delta) := \mathbf{P}\Big\{-\inf_S X_z(s) \in (r, r+\delta)\Big\} \leqslant \frac{\delta}{c}.$$

The last two inequalities give estimation (2.2) for functions x of the form (2.3).

Now let x be an arbitrary function satisfying the conditions of Lemma 3. Then there are functions (x_n) of the form (2.3) with $\|x_n - x\|_{L_\infty} \to 0$ as $n \to \infty$. Obviously, a.s.

$$\|\gamma x_n + Y - z\|_{L_\infty} \to \|\gamma x + Y - z\|_{L_\infty} \quad \text{as } n \to \infty.$$

Hence, inequality (2.2) is valid. □

Remark 3. Since we have a norm in inequality (2.2), the condition $\inf_S x(s) > 0$ of Lemma 3 may be replaced by $\sup_S x(s) < 0$. This does not change the conclusion of Lemma 3.

Proof of Theorem 2. Estimate (1.3) follows immediately from Lemmas 2, 3, Remark 3, and representation (2.1). Indeed, the conditions of Lemma 3 or Remark 3 are fulfilled for each set S_k from Lemma 2. Therefore,

$$\mathbf{P}\Big\{\sup_{S_k} \big|X_z(s)\big| \in (r, r+\delta)\Big\} \leqslant \frac{\delta}{c_k},$$

where $c_k = \inf_{S_k} |x_{m_k}(s)|$. Hence,

$$\mathbf{P}\big\{\|X_z\| \in (r, r+\delta)\big\} \leqslant \sum_{k=1}^{n} \mathbf{P}\Big\{\sup_{S_k} \big|X_z(s)\big| \in (r, r+\delta)\Big\} \leqslant \sum_{k=1}^{n} \frac{1}{c_k}\delta,$$

so Theorem 2 is proved. □

Remark 4. Proving Theorem 2, we established the estimate

$$\mathbf{P}\Big\{\sup_{s\in S} X_z(s) \in (r, r+\delta)\Big\} \leqslant \lambda\delta.$$

The same concerns the random variable $\inf_S X_z(s)$. This estimate is false without condition $(\mathcal{M})$, even for $z \equiv 0$. Here is a simple example: $S = [0,1]$, $X(s) = \gamma s$,

$$\mathbf{P}\Big\{\sup_{s\in[0,1]} X(s) = 0\Big\} = \frac{1}{2}.$$

Let us present an application of Theorem 2. Let $(X_i)_1^\infty$ be independent copies of a $C(S)$-valued r.e. X. Denote by

$$\mu_X^n(U) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \in U), \quad U \in \mathcal{B},$$

the empirical distribution of X, where $I(X_i \in U) = 1$ if $X_i \in U$ and $= 0$ if $X_i \notin U$. The uniform μ_X-continuity of $\mathcal{B}$ implies the Glivenko–Cantelli-type theorem: a.s.,

$$\sup_{U\in\mathcal{B}} \big|\mu_X^n(U) - \mu_X(U)\big| \to 0 \quad \text{as } n \to \infty; \tag{2.6}$$

see, e.g., [7, Thm. 1]. So, Theorem 2 gives the following:

Corollary 1. *If a normal $C(S)$-valued r.e. X with $\mathbf{E}X = 0$ satisfies condition $(\mathcal{M})$, then the Glivenko–Cantelli theorem* (2.6) *holds.*

Actually the proof of Theorem 2 gives more; namely, this theorem is valid for other distributions as well.

Proposition 1. *Let X be a symmetric $C(S)$-valued r.e. satisfying condition $(\mathcal{M})$. Assume that X has form* (2.1), *where (γ_i) are independent (not necessarily normal) random variables having uniformly bounded densities. Then the family $\mathcal{F}_X$ is uniformly Lipschitz on $[0,\infty)$.*

3 A special subset of dual unit balls

Theorem 2 can be extended to general Banach spaces as follows.

Corollary 2. *Let B be a separable Banach space, and Γ be a total weakly* compact set in B^*. Let X be a normal symmetric B-valued r.e. with $\mathbf{P}\{x^*(X) = 0\} < 1$ for every $x^* \in \Gamma$. Then there exists $\lambda > 0$ such that, for all $z \in B$ and $r, \delta > 0$,*

$$\mathbf{P}\big\{\|X_z\|_\Gamma \in (r, r+\delta)\big\} \leqslant \lambda\delta,$$

where $\|x\|_\Gamma = \sup_{x^*\in\Gamma} |x^*(x)|$.

Proof. Since the set Γ is total, i.e., for every $x \in X$, $x \neq 0$, there exists $x^* \in \Gamma$ such that $x^*(x) \neq 0$, the $\|\cdot\|_\Gamma$ indeed is a norm. Let us consider the natural embedding of B into the space $C(\Gamma)$ of weakly* continuous functions on Γ. Since B is separable, by the (trivial part of) Banach theorem (see, e.g., [3, Chap. IV, Sect. 3, Thm. 4]), Γ is weakly* metrizable. Now one may apply Theorem 2. □

DEFINITION 1. We say that the space B *has property* $(\mathcal{N})$ if there exists a weakly* compact subset $0 \notin \Gamma \subset U_{B^*}$ such that $\|x\| = \max\{|x^*(x)|\colon x^* \in \Gamma\}$ for all $x \in B$.

Corollary 3. (*Of Corollary* 2.) *Assume that a separable Banach space B has property $(\mathcal{N})$. Then, for every nondegenerate normal symmetric r.e. X in B, the corresponding family $\mathcal{F}_X$ is uniformly Lipschitz on $[0,\infty)$.*

It is interesting to know which Banach spaces have property $(\mathcal{N})$ and which do not.

1. Spaces $C(S)$, in particular, the space c, have property $(\mathcal{N})$.
2. The space ℓ_1 has property $(\mathcal{N})$. Here $\Gamma = \{x^* \in U(\ell_\infty)\colon |x^*(1)| = 1\}$.
3. Each Banach space B can be ε-renormed to have property $(\mathcal{N})$.
   Indeed, let $x_0 \in B$ with $\|x_0\| = 1$. Put $\Gamma = \{x^* \in U(B^*)\colon |x^*(x_0)| \geqslant \varepsilon\}$ and $\|x\|_0 = \max\{|x^*(x)|\colon x^* \in \Gamma\}$.
4. Each separable infinite-dimensional Banach space B can be 2-renormed so that B in the new norm does not have property $(\mathcal{N})$; see [11, Prop. 2.22].
5. An infinite-dimensional Banach space with a smooth norm does not have property $(\mathcal{N})$; see [11, Cor. 1.4].
6. The spaces $L_1[0,1]$ and $C_0(S)$, where S is an increasing sequence of metric compact sets, in particular, the space c_0, do not have property $(\mathcal{N})$; see [11, Examples 1, 2].
7. Geometrically, a Banach space B has property $(\mathcal{N})$ if and only if there are elements $(x_i)_1^n \subset B$ and positive scalars $(a_i)_1^n$ for which the convex hull
   $$\operatorname{conv}\big\{x^* \in U_{B^*}\colon x^*(x_i) \geqslant a_i\big\}_1^n = U_{B^*}.$$
8. By the Banach–Mazur theorem, every separable Banach space can be isometrically embedded into $C[0,1]$. Theorem 2 and [11, Cor. 1.4] show that for any infinite-dimensional Hilbert subspace $H \subset C[0,1]$, there is $s \in [0,1]$ such that $x(s) = 0$ for every $x \in H$.

Proof of Theorem 3. Indeed, by property 3, the space B can be ε-renormed so that B in the new norm has property $(\mathcal{N})$. By Corollary 3, for every nondegenerate normal symmetric r.e. X in B, the corresponding family $\mathcal{F}_X$ is uniformly Lipschitz on $[0,\infty)$ in the new norm. □

Remark 5. Theorem 3 is also valid for r.e. having representation (2.1), where (γ_i) are independent (not necessarily normal) random variables having uniformly bounded densities (see Proposition 1).

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