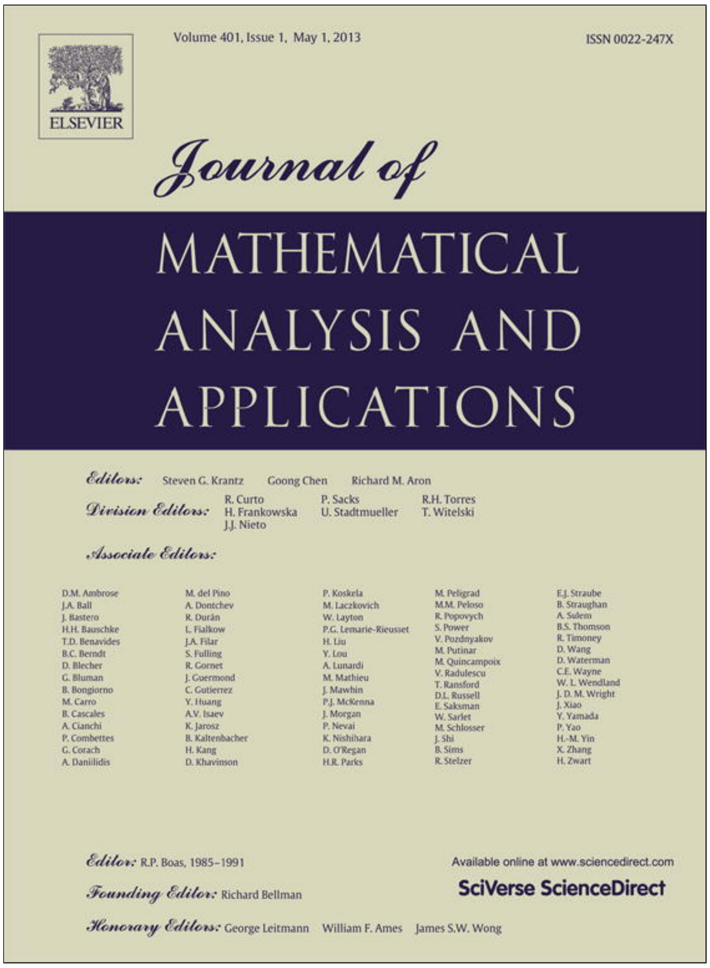
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On uniform continuity of convex bodies with respect to measures in Banach spaces

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ABSTRACT

Let μ be a probability measure on a separable Banach space X. A subset $U \subset X$ is μ-continuous if $\mu(\partial U) = 0$. In the paper the μ-continuity and uniform μ-continuity of convex bodies in X, especially of balls and half-spaces, is considered. The μ-continuity is interesting for study of the Glivenko–Cantelli theorem in Banach spaces. Answer to a question of F. Topsøe is given.

1. Introduction

We consider real *separable* Banach spaces X only, and do not mention this explicitly. By $B_r(z)$ we denote the closed ball of X with radius r and center z, by S_X the unit sphere of X, by X^* the dual of X. Given subset $U \subset X$ and $\delta > 0$, $\partial_\delta U$ stands for the δ-neighborhood of its boundary ∂U. All measures are assumed to be *probability* measures and are defined on Borel subsets of X. Let measure μ be defined on a Banach space X. Our purpose is to analyze the following concepts.

Definition 1.1. A subset U of a Banach space X is called *μ-continuous* if $\mu(\partial U) = 0$. A class $\mathcal{U}$ of subsets in X is called *μ-continuous* if each set $U \in \mathcal{U}$ is μ-continuous [13, p. 149].

We call a class $\mathcal{U}$ *uniformly μ-continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\mu(\partial_\delta U) < \varepsilon$ for all $U \in \mathcal{U}$. We call a class $\mathcal{U}$ *uniformly discontinuous* if it is not μ-uniformly continuous with respect to any measure μ.

A Banach space X is said to be *$\mathcal{U}$-ideal* if μ-continuity of $\mathcal{U}$ implies its uniform μ-continuity, for any measure μ [14, p. 283].

These notions have their origin in the theory of empirical distributions and are connected with generalizations of the Glivenko–Cantelli theorem to metric spaces [11,3,14,15,13,16,12,8]. Sometimes (see e.g. [14, p. 279]) one talks about the U-continuity of a measure μ instead of the μ-continuity of sets U. Uniform μ-continuity was repeatedly used without a name (see e.g. [11], [3, p. 2], [14, p. 282], [13, p. 151]), and is equivalent to so-called μ-uniformity, which we do not consider. Of course, every uniformly μ-continuous class is μ-continuous. Examples showing that the opposite statement is false are well known. We present some such examples below.

Of course, these concepts can be considered (and were considered) for any metric space. We restrict ourselves to Banach spaces and concentrate on the class of *convex bodies* $\mathcal{C}$ or one of the following subclasses of $\mathcal{C}$:

Half-spaces $\mathcal{H}_F$, i.e. sets of the form $H_{x^*t} = \{x \in X : x^*(x) \le t\}$, $x^* \in F$, $x^* \ne 0$, $t \in \mathbb{R}$, where F is a subset of X^*. We denote $\mathcal{H}_{X^*} = \mathcal{H}$, for short. Obviously, $\partial_\delta H_{x^*t} = \{x \in X : |x^*(x) - t| < \delta\}$.

Balls $\mathcal{B}$, and *balls with radii* ≤ 1 (small balls) $\mathcal{B}_1$. Obviously, $\partial_\delta B_r(z) = \{x \in X : r - \delta < \|x - z\| < r + \delta\}$.

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Study of balls with radii $\leq r$ can be reduced to $\mathcal{B}_1$ by introducing of a proportional norm. The $\mathcal{B}_1$-ideality is interesting also for investigation of the Vitali theorem in Banach spaces [13, p. 148]. Study of the μ-continuity for balls leads to questions on geometry of unit sphere which are interesting by itself. Necessary definitions and results in Banach space theory can be found in [2,4,6]. However we recall, for the convenience of the reader, relevant definitions.

We know only papers [11,13,16,1] concerning μ-continuity directly. Let us recall some results of the mentioned papers (mainly of [13]), simultaneously presenting relevant statements of our note.

The situation is rather clear for the class $\mathcal{H}$. Every finite-dimensional normed space X is $\mathcal{H}$-ideal (see e.g. [15, Theorem 2]). No infinite-dimensional Banach space is $\mathcal{H}$-ideal (it follows from [11]). We obtain this result from a simple and mostly known

Proposition 1.2. *Let X be an infinite-dimensional Banach space and $F \subset X^*$ be a total linear subspace. Then the class $\mathcal{H}_F$ is uniformly discontinuous.*

A relation between the μ-continuity of the classes of balls and half-spaces in c_0 was considered in [13]. Namely, let $\mathcal{H}_N$ be the class of half-spaces of the form $H_{nt} = \{(a_1, a_2, \ldots) \in c_0 : a_n \leq t\},\ n \in \mathbb{N},\ t \in \mathbb{R}$. In the space c_0, the class $\mathcal{B}$ is μ-continuous if and only if $\mathcal{H}_N$ is μ-continuous. We give an abstract version of this result and show that every Banach space X admits a measure μ for which $\mathcal{B}$ is μ-continuous, but $\mathcal{H}$ is not.

Only a few results about $\mathcal{B}$-ideal spaces were known. Topsøe [13, p. 149] writes "(practically) never Banach space is $\mathcal{B}$-ideal; just consider a measure concentrated on a line in $\mathbb{R}^2$". This idea is realized in the following statement.

Corollary 1.3. *No smooth and no rotund (infinite- or finite-dimensional) Banach space is $\mathcal{B}$-ideal.*

Further, $\mathcal{B}$ is uniformly discontinuous in the space ℓ_p of p-summable sequences, $1 \leq p < \infty$ [13, p. 155]. We generalize a part of this fact for $p > 1$.

Corollary 1.4. *The class $\mathcal{B}$ is uniformly discontinuous in every infinite-dimensional Banach space X having a smooth norm.*

The only known $\mathcal{B}$-ideal space is the space $C(S)$ of continuous functions over a compact metric set S. More precisely, using some reasoning of Topsøe [14, pp. 285–286], Aniszczyk proved [1] that $C(S)$ is $\mathcal{B}_1$-ideal and claimed that his proof implies the $\mathcal{B}$-ideality of $C(S)$.

As for $\mathcal{B}_1$-ideality, every finite-dimensional normed space is $\mathcal{B}_1$-ideal [13, p. 153]. The spaces $\ell_p,\ 1 \leq p < \infty$, are $\mathcal{B}_1$-ideal, but c_0 is not [13, p. 154]. Moreover, the class $\mathcal{B}_1$ is uniformly discontinuous for c_0 [13, p. 151]. Similarly, the space L_1 of absolutely integrable on $[0, 1]$ functions is not $\mathcal{B}_1$-ideal [16, p. 144]. The authors of [16] conjectured that $\mathcal{B}_1$ is uniformly discontinuous in L_1.

We show that every infinite-dimensional Banach space can be (equivalently) renormed in such a way that $\mathcal{B}_1$ becomes uniformly discontinuous in the new norm. However, it is not known whether every Banach space X can be renormed so that X becomes $\mathcal{B}_1$-ideal in the new norm.

Topsøe [13, p. 157] asked whether every subspace of $\ell_p,\ 1 \leq p < \infty$, is $\mathcal{B}_1$-ideal? The almost positive answer follows from the next statement (see Corollary 2.32 for details).

Corollary 1.5. *Every dual Banach space with property* $(\mathbf{m}^*)$ *is $\mathcal{B}_1$-ideal.*

See Section 2 for the definition of property $(\mathbf{m}^*)$. Here we note only that it is similar to Kalton's property $(\mathbf{M})$ from [7].

Proofs of this paper are simple and geometrically clear. We try to single out geometrical properties of Banach spaces "responsible" for various properties of μ-continuity.

2. Main results

First we make some general remarks. Intersections of the form $\bigcap_n \partial_{\delta_n} U_n$, where $U_n \in \mathcal{U}$ and $\delta_n \downarrow 0$, play a decisive role for the establishing of uniform μ-continuity. We call these intersections *Topsøe $\mathcal{U}$-sets*.

Lemma 2.1 *([14, p. 151]). A class $\mathcal{U}$ is uniformly μ-continuous if and only if $\mu(A) = 0$ for each Topsøe $\mathcal{U}$-set A.*

We will use well known

Lemma 2.2. *Let μ be a measure in a Banach space X and let $\varepsilon, \delta > 0$. Then there exists a finite-dimensional subspace $E \subset X$ so that $\mu(E_\delta) > 1 - \varepsilon$, where E_δ is the δ-neighborhood of E.*

Lemma 2.3. *Let a class $\mathcal{U}$ be shift invariant in a Banach space X, i.e. $U + x \in \mathcal{U}$ for all $U \in \mathcal{U}$ and $x \in X$. Then $\mathcal{U}$ is μ-continuous (uniformly μ-continuous) if and only if it is μ_x-continuous (resp. uniformly μ_x-continuous), where*

$$\mu_x(U) := \mu(U + x), \quad U \in \mathcal{U},\ x \in X.$$

Most classes and properties we consider are shift invariant. We often use this fact, without mentioning it explicitly. Using Lemma 2.3, we also talk about "shifting of a picture".

1. *Classes $\mathcal{C}$ and $\mathcal{H}_F$*. The class of convex sets in $\mathbb{R}^n$ is an important object of study in the theory of empirical distributions. Since each measure is almost concentrated on a compact set (the Ulam theorem) and a compact set in an infinite-dimensional Banach space coincides with its boundary, the consideration of the class $\mathcal{C}$ of convex bodies, i.e. of all convex closed sets having interior points, is natural. Formally, if there is no measure μ, respect to which $\mathcal{U}$ is μ-continuous, the space X is assumed to be $\mathcal{U}$-ideal too. So, to avoid this uninteresting situation, one wishes to be sure of the existence of at least one such measure. The following (in fact, well known) proposition guarantees the existence of the mentioned measure for convex bodies.

Recall that a subset D of a Banach space X is called *directionally porous* [2, p. 166] if there exists a $0 < \lambda < 1$ so that for every $x \in D$ there is a direction $e \in S_X$, and points $x_n = x + \varepsilon_n e$ with $\varepsilon_n \to 0$, for which D intersects no ball $B_{\lambda\varepsilon_n}(x_n)$. A *hyperplane* in a Banach space X is a shift of a closed one-codimensional subspace by some vector.

Proposition 2.4. *The class $\mathcal{C}$ is μ-continuous for every non-degenerated (i.e. non-concentrated on any hyperplane) Gaussian measure μ.*

Proof. Simple application of the Hahn–Banach theorem and the Riesz lemma shows that the boundary of any convex body is directionally porous with an arbitrary $\lambda \subset (0, 1)$. But for sets with directionally porous boundaries, the property of Proposition 2.4 holds (see e.g. [2, p. 167]). □

Remark 2.5. Simple examples show that the non-degeneracy of a (non-Gaussian) measure does not guarantee μ-continuity, even for the class $\mathcal{B}$.

Let us pass to the class $\mathcal{H}_F$. A subspace $F \subset X^*$ is said to be *total* if for every $x \in S_X$ there is a functional $x^* \in F$ such that $x^*(x) \neq 0$.

Proof of Proposition 1.2. Suppose, on the contrary, that $\mathcal{H}_F$ is uniformly μ-continuous for some measure μ. Every finite-dimensional subspace $E \subset X$ is closed in the weak topology $w(X, F)$ for a total subspace F. So, by the Hahn–Banach theorem, F is contained in a hyperplane of a form $\{x \in X : x^*(x) = 0\}$, $x^* \in F$. This hyperplane is, of course, a boundary of the half-space H_{x^*0}. Therefore, by the definition of uniform μ-continuity, $\forall\, \varepsilon > 0$ there is a $\delta > 0$ such that $\mu(E_\delta) < \varepsilon$ for each finite-dimensional subspace $E \subset X$, where E_δ is the δ-neighborhood of E. But $\mu(E_\delta) > 1 - \varepsilon$ for the subspace E from Lemma 2.2. If $\varepsilon < \frac{1}{2}$, we get a contradiction. □

For spaces ℓ_p, $1 \le p < \infty$, and $F = X^*$ Proposition 1.2 is a part of [13, Proposition 3].

Corollary 2.6. *No infinite-dimensional Banach space X is $\mathcal{H}_F$-ideal for any total linear subspace $F \subset X^*$.*

Proof. According to Proposition 2.4, every half-space is μ-continuous for each non-degenerated Gaussian measure μ on X. However, by Proposition 1.2, the class $\mathcal{H}_F$ is not uniformly μ-continuous. Therefore, X is not $\mathcal{H}_F$-ideal. □

In connection with Corollary 2.6, it is interesting to localize natural measures μ and uniformly μ-continuous subclasses $\mathcal{U} \subset \mathcal{H}$. Two such examples were presented in [11,8]; we will present one more. The following statement shows a relation between the uniform μ-continuity of balls and the uniform μ-continuity of half-spaces whose boundaries are tangent to these balls. Let us recall necessary definitions.

A hyperplane D is *tangent* to a ball B at a point x if $x \in D \cap B \subset \partial B$. A point $x \in \partial B$ is called a *smooth point* of B if for every $y \in X$ there exists $\lim_{\lambda \to 0} \lambda^{-1}(\|x + \lambda y\| - \|x\|)$ [2, p. 409]. Geometrically it means that if (for example) a hyperplane D touches a ball $B_1(z)$ ($\|z\| = 1$) at 0 and $x \in D$ then the distance $\mathrm{dist}(x, \partial B_k(kz)) \to 0$ as $k \to \infty$. A norm of a Banach space is called *smooth* if each $x \in \partial B$ is a smooth point.

Denote by $\mathcal{H}_S$ the class of half-spaces whose boundaries are tangent to balls of a Banach space X at smooth points of these balls.

Proposition 2.7. *If the class of balls $\mathcal{B}$ in a Banach space X is uniformly μ-continuous then the class of half-spaces $\mathcal{H}_S$ is uniformly μ-continuous too.*

Proof. Suppose that $\mathcal{H}_S$ is not μ-continuous. Then there exists $\varepsilon > 0$ such that $\forall\, \delta > 0$ there is a half-space $H \in \mathcal{H}_S$ (depending on δ) with $\mu(\partial_\delta H) > \varepsilon$. Since X is separable, there exists a finite collection of balls $(K_i)_1^n$ (depending on δ), with centers $(z_i)_1^n \subset \partial H$, each of radius $< \delta$, such that $\mu(\cup_1^n K_i) > \varepsilon$. Let $B_1(z)$ be a ball for which ∂H is tangent and let x be a point of tangency. Shifting, by Lemma 2.3, the whole picture on $-x$, one may assume $x = 0$. Since $x = 0$ is a smooth point of $B_1(z)$, for all i

$$\mathrm{dist}(z_i, \partial B_k(kz)) \to 0 \quad \text{as } k \to \infty.$$

Hence, for sufficiently large k,

$$\cup_1^n K_i \subset \partial_\delta B_k(kz), \quad \text{so } \mu(\partial_\delta B_k(kz)) > \varepsilon.$$

This contradicts the uniform μ-continuity of $\mathcal{B}$. □

Remark 2.8. This proof uses an idea from [12, Lemma 3], where it was proved that the uniform μ-continuity of balls implies the uniform μ-continuity of half-spaces in a Hilbert space. I would like to mention that the word "uniform" is missing in the statement of [12, Lemma 3]. Without this word the statement is false, an easy example can be constructed even in two-dimensional Euclidean space. We will return below to relations between the μ-continuity of balls and half-spaces.

From now on by $\mathcal{H}_S$ we denote the class of half-spaces of the form $H_{st} = \{x \in C(S) : x(s) \le t\}$, $s \in S$, $t \in \mathbb{R}$. By argumentation of the following remark, for $C(S)$ this "new" class $\mathcal{H}_S$ is contained in the "old one".

Remark 2.9. If the above mentioned result of Aniszczyk is true then the space $C(S)$ is $\mathcal{H}_S$-ideal.

Indeed, each half-space H_{st} touches the ball $B_t(0)$ at a smooth point—a continuous function whose modulus attains maximum at the unique point s. Since, by Aniszczyk's result [1], the space $C(S)$ is $\mathcal{B}$-ideal, Proposition 2.7 implies the $\mathcal{H}_S$-ideality of $C(S)$.

Remark 2.10. Remark 2.9 together with Corollary 2.6 show that a space X can be $\mathcal{H}_F$-ideal but not $\mathcal{H}_{\mathrm{lin}F}$-ideal.

We return to relations between the μ-continuity of balls and half-spaces. Let X be a Banach space. A point x of an arbitrary ball $B \subset X$ is called *exposed* if there is a functional $x^* \in X^*$ so that $x^*(x) > x^*(y)$ for all $y \in B$, $y \ne x$ [2, p. 108].

Proposition 2.11. *Any Banach space X admits a measure μ for which the class $\mathcal{B}$ is μ-continuous, but $\mathcal{H}$ is not.*

Proof. If X is reflexive then its unit ball contains an exposed point [2, p. 110], i.e. there exists a point $x \in S_X$ and a hyperplane D such that $D \cap S_X = \{x\}$. If X is not reflexive then, by the James theorem, there exists a functional $x^* \in X^*$ which does not attain its norm [4, p. 14]. In both cases there is a hyperplane D such that for every ball B, the intersection $D \cap B$ either is empty or contains a single point or is a convex body in D.

Take a Gaussian measure μ which is concentrated and non-generated on D. By Proposition 2.4, the class $\mathcal{B}$ is μ-continuous and, by Proposition 1.2, $\mathcal{H}$ is not. $\square$

The above mentioned result of Topsøe [13, p. 152] shows, that the converse to the previous statement is false. In c_0, even the μ-continuity of the subclass $\mathcal{H}_N \subset \mathcal{H}$ already implies the μ-continuity of $\mathcal{B}$ (see Section 1). However, under additional conditions a converse to Proposition 2.11 is valid. A Banach space X is called *rotund* if $\|x + y\| < \|x\| + \|y\|$ for all linearly independent elements $x, y \in X$.

Proposition 2.12. *Every rotund Banach space X admits a measure ν for which the class $\mathcal{H}$ is ν-continuous, but $\mathcal{B}$ is not.*

Proof. Let μ be a non-degenerated Gaussian measure on X. Given arbitrary Borel subsets $A \subset S_X$ and $U \subset X$, put

$$\nu(A) := \mu\{tx : x \in A,\ t \ge 0\} \quad \text{and} \quad \nu(U) := \nu(U \cap S_X).$$

Of course, ν is a measure on X and $\nu(S_X) = 1$. Since X is rotund, given a hyperplane D, $D \cap S_X$ is either empty or contains a single point, or is a boundary of the convex body $D \cap B_1(0)$ in D. The first and second cases are not interesting, and for the third

$$\nu(D) = \nu(D \cap S_X) = \mu\{tx : x \in D \cap S_X,\ t \ge 0\}.$$

However, the set $\{tx : x \in D \cap S_X,\ t \ge 0\}$ is a boundary of the convex body $\{tx : x \in D \cap B_1(0),\ t \ge 0\}$. According to Proposition 2.4, μ-measure of such boundary equals to zero. So, the class $\mathcal{H}$ is ν-continuous. Obviously, $\mathcal{B}$ is not ν-continuous. $\square$

Now we present the promised abstract version of Topsøe's statement on c_0. A Banach space X is called *polyhedral* [5] if a ball of every of its finite-dimensional subspace is a polyhedron.

Proposition 2.13. *For any polyhedral Banach space X there is a countable subset $F \subset S_{X^*}$ so that the class $\mathcal{B}$ is μ-continuous if and only if $\mathcal{H}_F$ is.*

Proof. Put $D_{x^*} = \{x \in X : x^*(x) = 1\}$. In view of the Fonf theorem on the structure of a sphere in a polyhedral space [5, p. 655], there is a countable subset $F \subset S_{X^*}$ so that:

(a) $S_X = \bigcup_{x^* \in F}(S_X \cap D_{x^*})$ and
(b) each set $S_X \cap D_{x^*}$, $x^* \in F$, has an interior point in D_{x^*}.

Now, let $\mu(\partial B_r(z)) > 0$ for some r and z. Then, by (a), there are $x^* \in F$ and $t \in \mathbb{R}$ such that

$$\mu\{\partial H_{x^*t} \cap \partial B_r(z)\} > 0$$

hence, $\mu(\partial H_{x^*t}) > 0$.

Conversely, let $\mu(\partial H_{x^*t}) > 0$ for some $x^* \in F$ and $t \in \mathbb{R}$. By (b), for some r and z the intersection $\partial H_{x^*t} \cap \partial B_r(z)$ has an interior point in ∂H_{x^*t}. Then

$$\mu\{\partial H_{x^*t} \cap \partial B_r(z')\} > 0$$

for a translate $B_r(z')$ of the ball $B_r(z)$. Hence, $\mu(\partial B_r(z')) > 0$. $\square$

2. *Class $\mathcal{B}$.*

Proof of Corollary 1.4. In fact, if the class $\mathcal{B}$ was uniformly μ-continuous then, according to Proposition 2.7, $\mathcal{H}$ would be uniformly μ-continuous too. This contradicts Proposition 1.2. $\square$

Corollary 1.4 implies the uniform discontinuity of $\mathcal{B}$ in the spaces ℓ_p and L_p for $1 < p < \infty$.

Proposition 2.14. *Every (infinite- or finite-dimensional) Banach space X admits a (degenerated) Gaussian measure μ for which the class $\mathcal{B}$ is not uniformly μ-continuous.*

Proof. According to the Mazur theorem [2, p. 91], each ball $B_1(z)$ of X has a smooth point x (of course, we assume $\dim X > 1$). Let D be a hyperplane in X, tangent the ball $B_1(z)$ at a point x. Take a Gaussian measure concentrated on D. Below we repeat the arguments of Proposition 2.7. Namely, given $0 < \varepsilon < 1$, for every $\delta > 0$ there exist balls $(K_i)_1^n$ of D (depending on δ), with centers $(z_i)_1^n$, each of radius $< \delta$, such that $\mu(\cup_1^n K_i) > \varepsilon$. Shifting, by Lemma 2.3, the whole picture on $-x$, one may assume $x = 0$ (then $\|z\| = 1$). Since $B_1(z)$ is smooth at point 0, the distance $\mathrm{dist}(z_i, \partial B_k(kz)) \to 0$ as $k \to \infty$, for all i. Hence, for sufficiently large k, $\cup_1^n K_i \subset \partial_\delta B_k(kz)$, so $\mu(\partial_\delta B_k(kz)) > \varepsilon$. Therefore, $\mathcal{B}$ is not uniformly μ-continuous. $\square$

Proposition 2.15. *Suppose the unit ball of a Banach space X contains a point x which is exposed and smooth simultaneously. Then X is not $\mathcal{B}$-ideal.*

Proof. Let D be a hyperplane, tangent to the unit ball of X at the point x. Take a measure μ concentrated and non-degenerated on D. By Proposition 2.14, the class $\mathcal{B}$ is not uniformly μ-continuous. Let B be an arbitrary ball in X. Since x is exposed, $B \cap D$ (provided it is nonempty and does not consist of a single point) is a convex body in D, and $\partial B \cap D = \partial(B \cap D)$. By Proposition 2.4, $\mu(\partial(B \cap D)) = 0$. Hence, μ vanished on each sphere of X, so $\mathcal{B}$ is μ-continuous. Therefore X is not $\mathcal{B}$-ideal. $\square$

Note that every Banach space X can be renormed so that in the new norm the sets of exposed and smooth points are disjoint. To verify this, one can introduce first a smooth norm in X [2, p. 89], and then the new norm as in (2.1) below.

Proof of Corollary 1.3. Consider three cases.

1. *The space X is smooth and infinite-dimensional.* By virtue of Proposition 2.4, each ball of X is μ-continuous for each non-degenerated Gaussian measure μ. On the other hand, $\mathcal{B}$ is uniformly discontinuous, by Corollary 1.4. Hence, X is not $\mathcal{B}$-ideal.
2. *X is smooth and finite-dimensional.* Then S_X has an exposed point [2, p. 110] which, as all others, is a smooth point. It remains to apply Proposition 2.15.
3. *X is rotund.* According to the Hahn–Banach theorem, every point of S_X is exposed. By the mentioned Mazur theorem, S_X contains a smooth point. We apply Proposition 2.15 once more. $\square$

The following statement shows that Corollary 1.4 is not valid without additional assumptions.

Proposition 2.16. *Suppose a Banach space has the form $X = Y \oplus L$, where Y is a closed subspace, L is a one-dimensional subspace, and*

$$\|y + l\| = \|y\| + \|l\|, \quad y \in Y,\ l \in L. \tag{2.1}$$

Let μ be a one-dimensional Gaussian measure on L. Then the class $\mathcal{B}$ is uniformly μ-continuous in X.

Proof. Take an arbitrary ball $B_r(z)$ of X. Then $B_r(z) \cap L \subset \mathrm{lin}(z, L)$, moreover, $\mathrm{lin}(z, L)$ is isometric to the two-dimensional ℓ_1^2 or to a one-dimensional space. Hence, the sphere $\partial B_r(z)$ intersects L at at most two points and the length (in the norm of X) of $\partial_\delta B \cap L$ is not greater than 4δ. Therefore, $\mu(\partial_\delta B) \to 0$ as $\delta \to 0$, uniformly on the balls. $\square$

Corollary 2.17. *Every Banach space X can be renormed so that in the new norm the class $\mathcal{B}$ becomes uniformly μ-continuous for some (degenerated) measure μ.*

Proof. Let $Y \subset X$ be a one-codimensional closed subspace, $L \subset X$ be a one-dimensional subspace and $L \cap Y = 0$. The desired norm can be introduced by (2.1). $\square$

We are unaware of publications on $\mathcal{B}$-ideal finite-dimensional spaces. In view of Corollary 1.3, a $\mathcal{B}$-ideal space cannot be smooth or rotund. On the other hand, by Aniszczyk's theorem, the n-dimensional space ℓ_∞^n is $\mathcal{B}$-ideal. One can advance, as a working hypothesis, that a finite-dimensional normed space is $\mathcal{B}$-ideal if and only if it is polyhedral.

3. *Class $\mathcal{B}_1$.* We start with negative results and single out a class of norms for which $\mathcal{B}_1$ is uniformly discontinuous.

Definition 2.18. We say that a Banach space X has a *finite universal sphere* if there exists $r > 0$ such that $\forall\, \delta > 0$ and every finite-dimensional subspace $E \subset X$ there is $z \in X$ so that the ball $K_r^E = B_r(0) \cap E$ of E belongs to $\partial_\delta(B_1(z))$.

Remark 2.19. Obviously, when we check that the condition of this definition is satisfied for X, it is sufficient to consider E from an arbitrary increasing sequence (E_n) of finite-dimensional subspaces whose union is dense in X.

Remark 2.20. Definition 2.18 can be considered as an "approximative and uniform" version of the following well known concept: A Banach space X *contains no finite-dimensional Haar (or Čebyšev) subspaces* if for every finite-dimensional subspace $E \subset X$ and for each $x \in X \setminus E$ there are at least two best approximations in E. For example, L_1 contains no finite-dimensional Haar subspaces [9, Theorem 2.5].

Proposition 2.21. *The class $\mathcal{B}_1$ is uniformly discontinuous in any Banach space X with a finite universal sphere.*

Proof. Let r be the constant from Definition 2.18 and μ be a measure on X. By Lemma 2.3, one may assume $\mu\{B_r(0)\} > \varepsilon$ for some $\varepsilon > 0$. According to Lemma 2.2, for every $\delta > 0$ there is a finite-dimensional subspace $E \subset X$ so that $\mu(E_\delta) > 1 - \frac{\varepsilon}{2}$. Then

$$\mu\{B_r(0) \setminus E_\delta\} \le \mu\{X \setminus E_\delta\} \le 1 - \left(1 - \frac{\varepsilon}{2}\right) = \frac{\varepsilon}{2},$$

so, since $K_r^E = B_r(0) \cap E$,

$$\mu\{(K_r^E)_\delta\} \ge \mu\{B_r(0)\} - \mu\{B_r(0) \setminus E_\delta\} \ge \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

By definition, for some z

$$K_r^E \subset \partial_\delta B_1(z),$$

whence

$$(K_r^E)_\delta \subset \partial_{2\delta} B_1(z).$$

Hence $\mu\{\partial_{2\delta} B_1(z)\} > \frac{\varepsilon}{2}$, i.e. $\mathcal{B}_1$ is not uniformly μ-continuous. □

Proposition 2.22. *Every infinite-dimensional Banach space X can be renormed so that the new sphere will be finite universal.*

Proof. Let (E_n) be an increasing sequence of finite-dimensional subspaces whose union is dense in X. In just the same way as in [6, p. 7], one can show the existence in X of infinite Auerbach system, i.e. a biorthogonal sequence (x_n, x_n^*), $\|x_n\| = \|x_n^*\| = 1$, with an additional condition: for all n

$$x_n^*(x) = 0 \quad \text{as soon as} \quad x \in E_n.$$

The new norm on X can be introduced by the formula

$$|||x||| = \max\left\{\|x\|, 2\sup_n |x_n^*(x)|\right\}.$$

We check that the new sphere is finite universal. Fixing n, consider the set

$$K_n = \left\{x \in B_1(0) : x_n^*(x) = \frac{1}{2}, \left\|x - \frac{1}{2}x_n\right\| < \frac{1}{2}\right\}.$$

If $m \ne n$ and $x \in K_n$ then

$$|x_m^*(x)| = \left|x_m^*\left(x - \frac{1}{2}x_n\right)\right| \le \left\|x - \frac{1}{2}x_n\right\| < \frac{1}{2},$$

so $|||x||| = 1$. Moreover, if $e \in E_n$ and $\|e\| < \frac{1}{2}$ then

$$x_n^*\left(e + \frac{1}{2}x_n\right) = \frac{1}{2}, \qquad \left\|e + \frac{1}{2}x_n\right\| \le 1 \quad \text{and} \quad \left\|\left(e + \frac{1}{2}x_n\right) - \frac{1}{2}x_n\right\| < \frac{1}{2}.$$

Therefore, $K_{1/2}^{E_n}$ belongs to the new sphere of radius 1 with center $\frac{1}{2}x_n$. Applying Remark 2.19 we get that the new sphere is finite universal with $r = \frac{1}{2}$. □

This construction of $|||\,|||$ is similar to a construction in [10].

Corollary 2.23. *If a Banach space is $\mathcal{B}_1$-ideal with respect to any equivalent norm then it is finite-dimensional. The sphere of a finite-dimensional space cannot be finite universal.*

Proof. The first part of corollary is a simple combination of Propositions 2.4, 2.21 and 2.22. The second part follows from Proposition 2.21 and mentioned Topsøe's result [13, p. 153] (see also Corollary 2.27). □

Example. Let $R = \cup_n S_n$ be an increasing sequence of metric compact sets S_n. The space $C_0(R)$ of continuous function $x(s)$ with $x(s) \to 0$ as $s \to \infty$ has finite universal sphere. In particular, the spaces c_0 and $C_0(\mathbb{R})$ have finite universal sphere.

The verification is simple. Similarly, one can easily check that every subspace of c_0 has finite universal sphere and that the space $C(S)$, with a compact metric S, has not finite universal sphere. It is not hard to prove the following

Proposition 2.24. *Suppose a Banach space Y has finite universal sphere and $X = Y \oplus Z$ with*

$$\|y + z\| = \max(\|y\|, \|z\|), \quad y \in Y,\ z \in Z.$$

Then X has finite universal sphere too.

Now we turn to positive results. First we consider a compact set of centers. The following Corollary 2.26 is more or less known (cf. with [3, Theorem 6]). We present its direct and simple proof.

Lemma 2.25. *Let X be a Banach space. Every Topsøe $\mathcal{B}$-set $A = \bigcap_n \partial_{\delta_n} B_{r_n}(z_n)$, with a convergent sequence of centers (z_n), belongs to some sphere of X.*

Proof. We show that for some sequence (n_k) the intersection $A = \bigcap_k \partial_{\delta_{n_k}} B_{r_{n_k}}(z_{n_k})$ is a sphere. Let $z_n \to z$; hence (z_n) is bounded and the sequence (r_n) is bounded too (otherwise A would be empty). Take a sequence (n_k) so that $r_{n_k} \to r$ as $k \to \infty$. Passing to a subsequence, one may assume (z_{n_k}), (r_{n_k}) and (δ_{n_k}) to be convergent very quickly. More precisely, one may assume that for all k.

$$\|z_{n_k} - z\| < 2^{-k-4}, \qquad |r_{n_k} - r| < 2^{-k-4} \quad \text{and} \quad \delta_{n_k} < 2^{-k}.$$

Now, we slightly increase δ_{n_k} (namely, take $\delta_{n_k} = 2^{-k}$). By the triangle inequality, the obtained sequence of "rings" $\partial_{\delta_{n_k}} B_{r_{n_k}}(z_{n_k})$ is decreasing and every such ring contains the sphere S with center z and radius r. Hence

$$\bigcap_k \partial_{\delta_{n_k}} B_{r_{n_k}}(z_{n_k}) = S;$$

so $A \subset S$. $\square$

Corollary 2.26. *Let X be a Banach space. Let $\mathcal{U}$ be a class consisting of balls $B_r(z)$, $z \in Z$, $r \in \mathbb{R}$, with a compact set of centers Z. Then X is $\mathcal{U}$-ideal.*

Proof. Let the class $\mathcal{U}$ be μ-continuous. Since Z is compact, by Lemma 2.25, each Topsøe $\mathcal{U}$-set belongs to the boundary ∂U of some set $U \in \mathcal{U}$. So, by Lemma 2.1, $\mathcal{U}$ is uniformly μ-continuous, hence X is $\mathcal{U}$-ideal. $\square$

Corollary 2.27. *Every finite-dimensional normed space is $\mathcal{B}_1$-ideal.*

Besides, we already know this corollary.

Definition 2.28. We say that a dual Banach space $X = Y^*$ *has property* $(\mathbf{m}^*)$ if for every weakly* null sequence $x_n \in X$ such that $\|x_n\| \to c$ as $n \to \infty$ there exists a strictly increasing function $\varphi(t) \geq t$, $t \geq 0$ (depending on c), so that for all $x \in X$

$$\lim_n \|x + x_n\| = \varphi(\|x\|).$$

This definition is inspired, on the one hand, by the proof of $\mathcal{B}_1$-ideality of ℓ_p from [13], and on the other hand, by the following well-known concept. A Banach space Y has *property* $(\mathbf{M}^*)$ of Kalton [7] if for all elements $x, y \in X = Y^*$ with $\|x\| = \|y\|$ and every weakly* null sequence (x_n)

$$\limsup_n \|x + x_n\| = \limsup_n \|y + x_n\|.$$

We suspect that there is a connection between these properties.

Remark 2.29. Every space ℓ_p, $1 \leq p < \infty$, has property $(\mathbf{m}^*)$. There are spaces, different from ℓ_p, which have property $(\mathbf{m}^*)$. Such a property has, for example, the James space J_p, $1 < p < \infty$. The property $(\mathbf{m}^*)$ is hereditary: every weakly* closed infinite-dimensional subspace of a space with property $(\mathbf{m}^*)$ has this property.

The following lemma and Corollary 1.5 generalize [13, Theorem 2].

Lemma 2.30. *Suppose a dual Banach space $X = Y^*$ has property $(\mathbf{m}^*)$. Then every Topsøe $\mathcal{B}_1$-set $A = \bigcap_n \partial_{\delta_n} B_{r_n}(z_n)$ of X belongs to a sphere of X with radius ≤ 1.*

Proof. Without loss of generality one may assume the set (z_n) to be bounded, otherwise the intersection A would be empty. Take a sequence (n_k) so that $r_{n_k} \to r \le 1$. By weak* compactness, passing to a subsequence, one may assume $z_{n_k} = z + x_{n_k}$ with weakly* null (x_{n_k}) and convergent $(\|x_{n_k}\|)$. Then for all $x \in \bigcap_k \partial_{\delta_{n_k}} B_{r_{n_k}}(z_{n_k})$

$$\|x - z - x_{n_k}\| = \|x - z_{n_k}\| \to r \quad \text{as } k \to \infty.$$

Since X has property $(\mathbf{m}^*)$,

$$\|x - z - x_{n_k}\| \to \varphi(\|x - z\|); \quad \text{as } k \to \infty.$$

Hence, for all $x \in \bigcap_k \partial_{\delta_{n_k}} B_{r_{n_k}}(z_{n_k})$

$$\varphi(\|x - z\|) = r, \quad \text{whence } \|x - z\| = \varphi^{-1}(r).$$

So, A belongs to the sphere with the center z and radius $\varphi^{-1}(r) \le r \le 1$. □

Proof of Corollary 1.5. The contrary means the existence of a measure μ with respect to which $\mathcal{B}_1$ is continuous, scalars ε, $\delta_n \to 0$ and a sequence of balls $B_{r_n}(z_n)$, for which $\mu\{\bigcap_n \partial_{\delta_n} B_{r_n}(z_n)\} > \varepsilon$. This contradicts Lemma 2.30. □

Remark 2.31. The space L_p, $1 < p < \infty$, $p \ne 2$ does not have property $(\mathbf{m}^*)$. Moreover Elena Riss has noted that for the Rademacher functions $(\mathbf{r}_n)$ and $\delta_n \downarrow 0$ the Topsøe set $\bigcap_n \partial_{\delta_n} B_1(\mathbf{r}_n)$ coincides with the set

$$E = \left\{ x \in L_p : \frac{1}{2}\|x - 1\|^p + \frac{1}{2}\|x + 1\|^p = 1 \right\}.$$

The set E, for $p \ne 2$, belongs to no sphere of L_p.

Corollary 2.32 (*Of* Corollary 1.5)**.** *Every weakly* closed subspace of* ℓ_p, $1 \le p < \infty$, *is* $\mathcal{B}_1$*-ideal.*

Let us recall, Topsøe [13, p. 157] asked whether every subspace of ℓ_p, $1 \le p < \infty$, is $\mathcal{B}_1$-ideal? Corollary 2.32 provides the positive answer for $1 < p < \infty$ and for all weakly* closed subspaces of ℓ_1.

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