CONDITION OF CONJUGACY OF WCG-SPACES

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It is proved that a WCG-space E is conjugate to a Banach space if and only if its conjugate space E' contains a norm-closed total subspace M, consisting of functionals which attain supremum on the unit sphere. Moreover, M' = E in the duality established between E and E'. An example, showing that this statement is in general not true for an arbitrary Banach space, is given.

In [1] a condition of conjugacy of a separable Banach space is given in terms of the set $\mathfrak{M}$ of functionals which attain their suprema on the unit sphere. Here we will prove the validity of this condition for the wider class of WCG-spaces and give an example showing that this is, in general, not true for an arbitrary Banach space. A Banach space is called a WCG-space if it is the closed linear envelope of a weakly compact subset of itself.

LEMMA [2]. Let be a Banach space, β be a number of the interval [0, 1], and f_n be a sequence of elements of the unit ball B(E') of the conjugate space E' such that $\|f\| \geqslant \beta$ for $f \in \operatorname{conv}(f_n)_1^\infty$ (conv is the convex envelope). If $\lambda_i > 0$, $\sum_1^\infty \lambda_i = 1$, then there exist $\alpha \in [\beta, 1]$ and $g_n \in \operatorname{conv}(f_i)_n^\infty$, such that $\left\|\sum_1^\infty \lambda_n g_n\right\| = \alpha$ and

$$\left\|\sum_1^n \lambda_i g_i\right\| \leqslant \alpha\left(1 - \beta \sum_{n+1}^\infty \lambda_i\right)$$

for all n.

THEOREM. A WCG-space E is conjugate to a Banach space if and only if E' contains a norm-closed total subspace $M \subset \mathfrak{M}$. Moreover, M' = E in the duality established between E and E'.

Proof. Necessity. The proof of necessity is the same as the proof of necessity in Theorem 4 of [1].

Sufficiency. We will show that M' is isometric with E in the duality established between E and E'. Since M is total over E, it follows that $M' \supset E$. It is also easily seen that $\|x\|_{M'} \leqslant \|x\|_E$ for $x \in E$. Let us assume that $B(E) \neq B(M')$. Then, by Lemma 2 of [3, Chap. I, §3], there exists an $x_0 \in B(M')$, which does not belong to the closure $\overline{B(E)}$ of the ball B(E) in the space M'. By the Hahn–Banach theorem there exists a continuous linear functional $f_0 \in M''$, $\|f_0\|_{M''} = 1$, $0 < \beta < 1$ and ε > 0 such that

$$\sup\{f_0(x)\colon x \in \overline{B(E)}\} < \beta^2 - \varepsilon.$$

We will show that B(M) is a dense subset of B(M") in the Mackey topology τ(M", M'). Indeed, B(M) is dense in B(M") in the weak topology σ(M", M') [3, Chap. IV, Sec. 5, Proposition 5]. But M", equipped with the topology σ(M", M'), as well as with τ(M", M'), has conjugate M' [3, Chap. IV, Sec. 2, Corollary to Theorem 2]. By [3, Chap. IV, Sec. 2, Corollary 1 to Proposition 4] the closures of a convex set in the topologies σ(M", M') and τ(M", M') coincide; therefore, B(M) is dense in B(M") in the topology τ(M", M').

Since E is a WCG-space, there exists a σ(E, E')-compact convex subset U such that $\bigcup_n nU$ is dense in E. Since U is σ(E, E')-compact and $\|x\|_{M'} \leqslant \|x\|_E$ for $x \in E$, it follows that U is also σ(M', M")-compact. Let y_0 be an element of B(M") such that $f_0(y_0) \geqslant \beta$. Since B(M) is dense in B(M") in the Mackey topology τ(M", M'), i.e., in the topology of uniform convergence on σ(M', M")-compacta, we can choose a sequence $f_n \in B(M)$ such that $f_n(y_0) = f_0(y_0)$ and $f_n(x) \to f_0(x)$ as $n \to \infty$ for $x \in \bigcup_n nU$. It is easily seen that $\|f\| \geqslant \beta$ for every

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functional $f \in \operatorname{conv}(f_n)_1^\infty$. Let λ_n be a sequence of positive numbers such that $\sum_1^\infty \lambda_n = 1$. By virtue of the lemma, there exist a number α ($\beta \leqslant \alpha \leqslant 1$) and a sequence g_n such that $g_n \in \operatorname{conv}(f_i)_n^\infty$, $\left\|\sum_1^\infty \lambda_i g_i\right\| = \alpha$, and

$$\left\|\sum_1^n \lambda_i g_i\right\| \leqslant \alpha\left(1 - \beta\sum_{n+1}^\infty \lambda_i\right) \tag{1}$$

for every n.

It follows immediately from the choice of g_n that $g_n(x) \to f_0(x)$ as $n \to \infty$ for every $x \in \bigcup_n nU$. We would arrive at a contradiction if we show that the element $g = \sum_1^\infty \lambda_i g_i$, belonging to M, does not attain supremum on the unit sphere S(E). Let $x \in S(E)$. Since $\bigcup_n nU$ is dense in E, there exists a sequence x_j of elements of $\bigcup_n nU$, which converges to x in the norm of the space E. We choose j such that $\|x - x_j\| < \varepsilon$, and n such that $g_i(x_j) < \beta^2 - \varepsilon$ for i > n. Then for i > n

$$g_i(x) = g_i(x_j) + g_i(x - x_j) < \beta^2 - \varepsilon + \|g_i\|\|x - x_j\| < \beta^2 < \alpha\beta.$$

Thus,

$$\sum_1^\infty \lambda_i g_i(x) < \sum_1^n \lambda_i g_i(x) + \alpha\beta\sum_{n+1}^\infty \lambda_i \leqslant \left\|\sum_1^n \lambda_i g_i\right\| + \alpha\beta\sum_{n+1}^\infty \lambda_i.$$

Hence, it follows from inequality (1) that

$$\sum_{i=1}^\infty \lambda_i g_i(x) < \alpha\left(1 - \beta\sum_{n+1}^\infty \lambda_i\right) + \alpha\beta\sum_{n+1}^\infty \lambda_i = \alpha.$$

The theorem is proved.

In conclusion, we give an example of a norm-closed total subspace $M \subset \mathfrak{M} \subset E'$, such that $M' \neq E$ in the duality established between E and E'. Let $l_1[0, 1]$ be the space of functions x(t) such that $\sum_{t\in[0,1]} |x(t)| < \infty$ with the natural norm. Its conjugate is the space of bounded functions with the norm equal to the supremum (see [4]), and the duality is given by the equation $f(x) = \sum_{t\in[0,1]} x(t) f(t)$. It is easily seen that the subspace of continuous functions $C[0, 1] \subset l_\infty[0, 1]$ is closed, total, and consists of functionals which attain supremum. But $(C[0, 1])' \neq l_1[0, 1]$ in the duality established between $l_1[0, 1]$ and $l_\infty[0, 1]$ since, e.g., the functional $\varphi(f) = \int_0^1 f(t)\,dt$ does not coincide with any functional of the form $x(f) = \sum_{t\in[0,1]} x(t) f(t)$. We do not know any example of a space E and a closed total subspace $M \subset \mathfrak{M}$, such that $M' \neq E$ in any duality.

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