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On local convexity of nonlinear mappings between Banach spaces

Research Article

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Abstract: We find conditions for a smooth nonlinear map $f\colon U \to V$ between open subsets of Hilbert or Banach spaces to be locally convex in the sense that for some c and each positive $\varepsilon < c$ the image $f(B_\varepsilon(x))$ of each ε-ball $B_\varepsilon(x) \subset U$ is convex. We give a lower bound on c via the second order Lipschitz constant $\mathrm{Lip}_2(f)$, the Lipschitz-open constant $\mathrm{Lip}_o(f)$ of f, and the 2-convexity number $\mathrm{conv}_2(X)$ of the Banach space X.

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Introduction

The local convexity of nonlinear mappings of Banach spaces is important in many branches of applied mathematics [1, 12, 17–21], in particular, in the theory of nonlinear differential-operator equations, optimization and control

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theory, etc. Locally convex maps appear naturally in various problems of Fixed Point Theory [6–8] and Nonlinear Analysis [11, 15, 16, 22].

Let X, Y be Banach spaces. A map $f\colon U \to Y$ defined on an open subset $U \subset X$ is called *locally convex* at a point $x \in U$ if there is a positive constant $c > 0$ such that for each positive $\varepsilon \le c$ and each point $x \in U$ with $B_\varepsilon(x) \subset U$ the image $f(B_\varepsilon(x))$ is convex. Here $B_\varepsilon(x) = \{y \in X : \|x-y\| < \varepsilon\}$ stands for the open ε-ball centered at x. The local convexity of f at x can be expressed via the *local convexity radius*

$$\mathrm{lcr}_x(f) = \sup\big\{c \in [0,+\infty) \,:\, \text{for all } \varepsilon \le c \text{ and } x \in U \text{ with } B_\varepsilon(x) \subset U \text{ the set } f(B_\varepsilon(x)) \text{ is convex}\big\}.$$

It follows that f is locally convex at $x \in U$ if and only if $\mathrm{lcr}_x(f) > 0$.

A map $f\colon U \to Y$ is defined to be

- *locally convex* if f is locally convex at each point $x \in U$;
- *uniformly locally convex* if its *local convexity radius* $\mathrm{lcr}(f) = \inf\limits_{x\in U} \mathrm{lcr}_x(f)$ is not equal to zero.

For example, if a homeomorphism $f\colon U \to V$ between open subsets $U \subset X$, $V \subset Y$ with $f(0) = 0 \in U$ is norm convex in the sense that

$$\left\| f\left(\frac{x+x'}{2}\right)\right\| \le \frac{1}{2}\left(\|f(x)\| + \|f(x')\|\right) \qquad \text{for all} \quad x, x' \in f(U),$$

then the inverse map f^{-1} is locally convex at the point $y = 0$. In particular, if Y is a Banach lattice with the order $\le$ and a homeomorphism $f : U \to V$ is Jensen convex, i.e.

$$f\left(\frac{x+x'}{2}\right) \le \frac{1}{2}\left(f(x) + f(x')\right)$$

for all $x, x' \in U$, then the inverse map f^{-1} is locally convex at the point $y = 0$.

In this paper we find some conditions on a map $f\colon U \to Y$ guaranteeing that f is uniformly locally convex, and give a lower bound on the local convexity radius $\mathrm{lcr}(f)$ of f. This bound depends on the second order Lipschitz constant $\mathrm{Lip}_2(f)$ of f, the Lipschitz-open constant $\mathrm{Lip}_o(f)$ of f, and the 2-convexity number $\mathrm{conv}_2(X)$ of the Banach space X.

1. Banach spaces with modulus of convexity of power type 2

The *modulus of convexity* of a Banach space X is the function $\delta_X\colon [0,2] \to [0,1]$ assigning to each number $t \ge 0$ the real number

$$\delta_X(t) \;=\; \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \,:\, x, y \in S_X,\ \|x-y\| \ge t\right\},$$

where $S_X = \{x \in X : \|x\| = 1\}$ is the unit sphere of the Banach space X. By [14, p. 60], the modulus of convexity can be equivalently defined as

$$\delta_X(t) \;=\; \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \,:\, x, y \in B_X,\ \|x-y\| \ge t\right\},$$

where $B_X = \{x \in X : \|x\| \le 1\}$ is the closed unit ball of X.

Any Hilbert space E of dimension $\dim E > 1$ has modulus $\delta_E(t)$ of convexity

$$\frac{1}{8}\,t^2 \le \delta_E(t) = 1 - \sqrt{1 - \left(\frac{t}{2}\right)^2} \le \frac{1}{4}\,t^2.$$

By [14, p. 63] or [9], $\delta_X(t) \le \delta_{\ell_2}(t) \le t^2/4$ for each Banach space X.

Following [14, p. 63] and [4, p. 154], we say that a Banach space X has *modulus of convexity of power type p* if there is a constant $L > 0$ such that $\delta_X(t) \geq Lt^p$ for all $t \in [0,2]$. It follows from $Lt^p \leq \delta_X(t) \leq t^2/4$ that $p \geq 2$. Hilbert spaces have modulus of convexity of power type 2. Many examples of Banach spaces with modulus of convexity of power type 2 can be found in [14, § 1.e], [4, Chapter V], [2, 10, 13]. In particular, the class of Banach spaces with modulus of convexity of power type 2 includes the Banach spaces L_p for $1 < p \leq 2$, and reflexive subspaces of the Banach space L_1. By [9], a Banach space X has modulus of convexity of power type 2 if and only if for any sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ in X the convergence $2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 \to 0$ implies $\|x_n - y_n\| \to 0$.

For a Banach space X consider the constant

$$\mathrm{conv}_2(X) \;=\; \inf\left\{\frac{1-\|(x+y)/2\|}{\|x-y\|^2} \;:\; x, y \in B_X,\; x \neq y\right\} \geq 0$$

called the 2-*convexity number* of X and observe that $\mathrm{conv}_2(X) > 0$ if and only if X has modulus of convexity of power type 2. It follows from [14, p. 63] or [9] that

$$0 \leq \mathrm{conv}_2(X) \leq \mathrm{conv}_2(\ell_2) = \frac{1}{8}$$

for each Banach space X.

2. Moduli of smoothness of maps of Banach spaces

In this section we recall known information [5, § 2.7] on the moduli of smoothness $\omega_n(f,t)$ of a function $f\colon U \to Y$ defined on a subset $U \subset X$ of a Banach space X with values in a Banach space Y.

The *n-th modulus of smoothness* of f is defined as

$$\omega_n(f,t) \;=\; \sup\left\{\|\Delta_h^n(f,x)\| \;:\; h \in X,\; \|h\| \leq t,\; [x, x+nh] \subset U\right\},$$

where

$$\Delta_h^n(f,x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x+kh)$$

is the n-th difference of f. In particular,

$$\omega_1(f,t) \;=\; \sup\left\{\|f(x+h) - f(x)\| \;:\; \|h\| \leq t,\; [x, x+h] \subset U\right\} \qquad \text{and}$$
$$\omega_2(f,t) \;=\; \sup\left\{\|f(x+h) - 2f(x) + f(x-h)\| \;:\; \|h\| \leq t,\; [x-h, x+h] \subset U\right\}.$$

Here $[x,y] = \{tx + (1-t)y : t \in [0,1]\}$ stands for the segment connecting two points $x, y \in X$. The constants

$$\mathrm{Lip}_1(f) = \sup_{t>0} \frac{\omega_1(f,t)}{t} \qquad \text{and} \qquad \mathrm{Lip}_2(f) = \sup_{t>0} \frac{\omega_2(f,t)}{t^2}$$

are called the *Lipschitz constant* and the *second order Lipschitz constant* of f, respectively.

A function $f\colon U \to Y$ is called (*second order*) *Lipschitz* if its (second order) Lipschitz constant $\mathrm{Lip}_1(f)$ (resp. $\mathrm{Lip}_2(f)$) is finite. The second order Lipschitz property of a weakly Gâteaux differentiable function f can be deduced from the Lipschitz property of its derivative f'.

Let us recall [3, p. 154] that a function $f\colon U \to Y$ is *weakly Gâteaux differentiable* at a point $x \in U$ if there is a bounded linear operator $f'_x\colon X \to Y$ (called the *derivative* of f at x) such that for each $h \in X$ and each linear continuous functional $y^* \in Y^*$ we get

$$\lim_{t\to 0} \frac{y^*(f(x+th) - f(x))}{t} = y^* \circ f'_x(h).$$

If

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - f'_x(h)\|}{\|h\|} = 0,$$

then f is *Fréchet differentiable* at x. The derivative f'_x belongs to the Banach space $L(X,Y)$ of all bounded linear operators from X to Y, endowed with the operator norm $\|T\| = \sup\limits_{\|x\|\le 1} \|T(x)\|$.

Even though the following two propositions are known, we present for completeness their short proofs, useful for our further analysis.

Proposition 2.1.

Let X, Y be Banach spaces and $U \subset X$ be an open subset. A function $f\colon U \to Y$ is Lipschitz if f is weakly Gâteaux differentiable at each point of U and the derivative map $f'\colon U \to L(X,Y)$, $f'\colon x \mapsto f'_x$, is bounded. In this case $\mathrm{Lip}_1(f) \le \|f'\|_\infty = \sup\limits_{x\in U} \|f'_x\|$.

Proof. Let $L = \|f'\|_\infty$. The inequality $\mathrm{Lip}_1(f) \le L = \|f'\|_\infty$ will follow as soon as we check that

$$\|f(x+h) - f(x)\| \le L\|h\|$$

for any $x \in U$ and $h \in X$ with $[x, x+h] \subset U$. Using the Hahn–Banach Theorem, find a linear continuous functional $y^* \in Y^*$ with unit norm $\|y^*\| = 1$ such that $y^*(f(x+h) - f(x)) = \|f(x+h) - f(x)\|$. The weak Gâteaux differentiability of f implies that the function

$$g\colon [0,1] \to \mathbb{C}, \qquad g\colon\ t \mapsto y^*(f(x+th) - f(x)),$$

is differentiable and $g'(t) = y^* \circ f'_{x+th}(h)$ for each $t \in [0,1]$. Then

$$\|g'\|_\infty \le \|y^*\| \cdot \|f'_{x+th}\| \cdot \|h\| \le 1 \cdot \|f'\|_\infty \cdot \|h\| = L \cdot \|h\|$$

and

$$\|f(x+h) - f(x)\| = |g(1) - g(0)| = \left|\int_0^1 g'(t)\,dt\right| \le \int_0^1 |g'(t)|\,dt \le L\|h\| \int_0^1 dt = L\|h\|. \qquad \square$$

Proposition 2.2.

Let X, Y be Banach spaces and $U \subset X$ be an open subset. Assume that a function $f\colon U \to Y$ is weakly Gâteaux differentiable at each point of U and the derivative map $f'\colon U \to L(X,Y)$, $f'\colon x \mapsto f'_x$, is Lipschitz. Then

(1) *f is Fréchet differentiable at each point of U;*

(2) *f is second order Lipschitz with* $\mathrm{Lip}_2(f) \le \mathrm{Lip}_1(f')$.

Proof. Let $L = \mathrm{Lip}_1(f')$. The Fréchet differentiability of f at a point $x \in U$ will follow as soon as we check that

$$\|f(x+h) - f(x) - f'_x(h)\| \le \frac{1}{2} L\|h\|^2$$

for each $h \in X$ with $[x, x+h] \subset U$. Using the Hahn–Banach Theorem, choose a linear continuous functional $y^* \in Y^*$ such that $\|y^*\| = 1$ and $y^*(f(x+h) - f(x) - f'_x(h)) = \|f(x+h) - f(x) - f'_x(h)\|$. The weak Gâteaux differentiability of f implies that the function

$$g\colon [0,1] \to \mathbb{C}, \qquad g\colon\ t \mapsto y^*(f(x+th) - tf'_x(h)),$$

is differentiable. Moreover, for each $t \in [0,1]$ we get $g'(t) = y^* \circ f'_{x+th}(h) - y^* \circ f'_x(h)$ and

$$|g'(t)| = |y^*(f'_{x+th}(h) - f'_x(h))| \le \|y^*\| \cdot \|f'_{x+th}(h) - f'_x(h)\| \le \|f'_{x+th} - f'_x\| \cdot \|h\| \le \mathrm{Lip}_1(f') \cdot \|th\| \cdot \|h\| = tL\|h\|^2.$$

Then

$$\|f(x+h)-f(x)-f'_x(h)\| = |g(1)-g(0)| = \left|\int_0^1 g'(t)\,dt\right| \le \int_0^1 |g'(t)|dt \le \int_0^1 tL\|h\|^2 dt = \frac{1}{2}L\|h\|^2.$$

To see that f is second order Lipschitz, observe that for each $h \in X$ with $[x-h, x+h] \subset U$ we get

$$\|f(x+h)-2f(x)+f(x-h)\| = \big\|f(x+h)-f(x)-f'_x(h)+f(x-h)-f(x)-f'_x(-h)\big\| \le$$
$$\le \|f(x+h)-f(x)-f'_x(h)\| + \|f(x-h)-f(x)-f'_x(-h)\| \le 2\frac{1}{2}L\|h\|^2 = L\|h\|^2,$$

which implies that $\mathrm{Lip}_2(f) \le L = \mathrm{Lip}_1(f')$. □

3. Lipschitz-open maps

Let X, Y be Banach spaces. A map $f\colon U \to Y$ defined on an open subset $U \subset X$ is called *Lipschitz-open* if there is a positive constant c such that for each $x \in X$ and $\varepsilon > 0$ with $B_\varepsilon(x) \subset U$ we get $B_{c\varepsilon}(f(x)) \subset f(B_\varepsilon(x))$. Observe that a map $f\colon U \to Y$ is Lipschitz-open if and only if its *Lipschitz-open constant*

$$\mathrm{Lip}_o(f) \;=\; \sup\big\{c \in [0,\infty) : \text{for all } x \in U \text{ and } \varepsilon > 0,\ B_\varepsilon(x) \subset U \Rightarrow B_{c\varepsilon}(f(x)) \subset f(B_\varepsilon(x))\big\}$$

is strictly positive.

A map $f\colon U \to Y$ is *locally Lipschitz-open* if each point $x \in U$ has an open neighborhood $W \subset U$ such that the restriction $f{\restriction}_W\colon W \to Y$ is Lipschitz-open. Observe that a bijective map $f\colon X \to Y$ between Banach spaces is Lipschitz-open if and only if the inverse map $f^{-1}\colon Y \to X$ is Lipschitz. In this case $\mathrm{Lip}_o(f) = \mathrm{Lip}_1(f^{-1})$.

The following proposition can be derived from [3, Theorem 15.5].

Proposition 3.1.

Let X, Y be Banach spaces. A map $f\colon U \to Y$ defined on an open subspace U of X is locally Lipschitz-open if it satisfies the following two conditions:

(1) *f is weakly Gâteaux differentiable,*

(2) *the derivative $f'_x\colon X \to Y$ is surjective at each point $x \in U$, and*

(3) *the derivative $f'\colon U \to L(X, Y)$ is Lipschitz.*

4. Main results

Theorem 4.1.

Let X, Y be Banach spaces. A map $f\colon U \to Y$ defined on an open subset $U \subset X$ is uniformly locally convex if

(1) *the Banach space X has modulus of convexity of power type 2,*

(2) *f is second order Lipschitz, and*

(3) *f is Lipschitz-open.*

Moreover, in this case f has local convexity radius $\mathrm{lcr}(f) \ge 8 \cdot \mathrm{Lip}_o(f) \cdot \mathrm{conv}_2(X)/\mathrm{Lip}_2(f) > 0$.

Proof. Let $\varepsilon_0 = 8 \cdot \mathrm{Lip}_o(f) \cdot \mathrm{conv}_2(X)/\mathrm{Lip}_2(f)$. Given any point $x_0 \in U$ and a positive $\varepsilon \le \varepsilon_0$ with $B_\varepsilon(x_0) \subset U$, we need to prove that the image $f(B_\varepsilon(x_0))$ is convex. Without loss of generality, $x_0 = 0$.

Claim 1. For any points $a, b \in f(B_\varepsilon(x_0))$ we get $(a+b)/2 \in \mathrm{cl}_Y(f(B_\varepsilon(x_0)))$.

Find two points $x, y \in B_\varepsilon(x_0) = B_\varepsilon(0)$ with $a = f(x)$ and $b = f(y)$, and consider the midpoint $z = (x+y)/2$. Observe that the points $x_\varepsilon = x/\varepsilon$, $y_\varepsilon = y/\varepsilon$, and $z_\varepsilon = z/\varepsilon$ have norms ≤ 1.

The definition of the 2-convexity number $\mathrm{conv}_2(X)$ guarantees that

$$1 - \frac{1}{\varepsilon}\|z\| = 1 - \|z_\varepsilon\| \ge \mathrm{conv}_2(X)\,\|x_\varepsilon - y_\varepsilon\|^2 = \frac{1}{\varepsilon^2}\,\mathrm{conv}_2(X)\,\|x-y\|^2$$

and thus

$$\varepsilon - \|z\| \ge \frac{1}{\varepsilon}\,\mathrm{conv}_2(X)\,\|x-y\|^2.$$

Then $B_\delta(z) \subset B_\varepsilon(x_0)$, where

$$\delta = \frac{1}{\varepsilon}\,\mathrm{conv}_2(X)\,\|x-y\|^2 \ge \frac{\mathrm{Lip}_2(f)}{8\mathrm{Lip}_o(f)\cdot \mathrm{conv}_2(X)}\,\mathrm{conv}_2(X)\,\|x-y\|^2 = \frac{\mathrm{Lip}_2(f)}{8\mathrm{Lip}_o(f)}\,\|x-y\|^2$$

and hence

$$f(B_\varepsilon(x_0)) \supset f(B_\delta(z)) \supset B_{\mathrm{Lip}_o(f)\delta}(f(z)) = B_\eta(f(z)),$$

where $\eta = \mathrm{Lip}_o(f)\,\delta \ge \mathrm{Lip}_2(f)\,\|x-y\|^2/8$.

The definition of the constant $\mathrm{Lip}_2(f)$ implies that for $h = z - x$, we get

$$\left\|\frac{a+b}{2} - f(z)\right\| = \left\|\frac{f(x)+f(y)}{2} - f(z)\right\| = \frac{1}{2}\|f(z-h) - 2f(z) + f(z+h)\| \le \frac{1}{2}\,\mathrm{Lip}_2(f)\|h\|^2 = \frac{1}{8}\,\mathrm{Lip}_2(f)\,\|x-y\|^2 \le \eta$$

and hence $(a+b)/2 \in \mathrm{cl}_Y(B_\eta(f(z))) \subset \mathrm{cl}_Y(f(B_\varepsilon(x_0)))$. ■

Claim 2. For any positive numbers $\delta < \eta$ we get $\mathrm{cl}_Y(f(B_\delta(x_0))) \subset f(B_\eta(x_0))$.

Given any point $y \in \mathrm{cl}_Y(f(B_\delta(x_0))$, find a point $x \in B_\delta(x_0)$ such that $\|y - f(x)\| < (\eta - \delta)\cdot \mathrm{Lip}_o(f)$. The definition of the Lipschitz-open constant guarantees that

$$y \in B_{(\eta-\delta)\mathrm{Lip}_o(f)}(f(x)) \subset f(B_{\eta-\delta}(x)) \subset f(B_{\eta-\delta}(B_\delta(x_0))) \subset f(B_\eta(x_0)).$$

The claim is proved. ■

Claim 1 implies that for each $\delta < \varepsilon_0$ the closure $\mathrm{cl}_Y(f(B_\delta(x_0))$ is convex. Then the open set $f(B_\varepsilon(x_0))$ is convex, being the union

$$f(B_\varepsilon(x_0)) = f\left(\bigcup_{0<\delta<\varepsilon} B_\delta(x_0)\right) = \bigcup_{0<\delta<\varepsilon} \mathrm{cl}_Y(f(B_\delta(x_0))$$

of a linearly ordered chain of convex sets. □

Taking into account that each Hilbert space X has 2-convexity number $\mathrm{conv}_2(E) \ge 1/8$, and applying Theorem 4.1, we get:

Corollary 4.2.

Let Y be a Banach space and U be an open subspace of a Hilbert space X. Each Lipschitz-open second order Lipschitz map $f\colon U \to Y$ is uniformly locally convex and has local convexity radius $\mathrm{lcr}(f) \ge \mathrm{Lip}_o(f)/\mathrm{Lip}_2(f) > 0$.

Theorem 4.1 combined with Propositions 2.2 and 3.1 implies the following two corollaries.

Corollary 4.3.
Let X, Y be Banach spaces. A map $f\colon U \to Y$ defined on an open subspace $U \subset X$ is uniformly locally convex if

(1) *the Banach space X has modulus of convexity of power type 2,*

(2) *f is weakly Gâteaux differentiable and the derivative $f'\colon U \to L(X, Y)$ is Lipschitz;*

(3) *f is Lipschitz-open.*

Corollary 4.4.
Let X, Y be Banach spaces. A map $f\colon U \to Y$ defined on an open subspace $U \subset X$ is locally convex if

(1) *the Banach space X has modulus of convexity of power type 2,*

(2) *f is weakly Gâteaux differentiable and the derivative $f'\colon U \to L(X, Y)$ is Lipschitz;*

(3) *for each $x \in U$ the derivative $f'_x\colon X \to Y$ is surjective.*

5. An open problem

We do not know if the requirement on the convexity modulus of the Banach space X is essential in Theorem 4.1 and Corollaries 4.3, 4.4.

Problem 5.1.
Assume that X is a Banach space such that any Lipschitz-open second order Lipschitz map $f\colon U \to X$ defined on an open subset $U \subset X$ is locally convex. Has X the modulus of convexity of power type 2? Is X (super)reflexive?

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