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ON THE MARCINKIEWICZ–ZYGMUND LAW OF LARGE NUMBERS IN BANACH LATTICES

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We strengthen the well-known Marcinkiewicz–Zygmund law of large numbers in the case of Banach lattices. Examples of applications to empirical distributions are presented.

1. Introduction. Main Theorem

Let $\xi, \xi_1, \xi_2, \ldots$ be independent identically distributed random variables in $\mathbb{R}$. In [1], Marcinkiewicz and Zygmund obtained the following generalization of the Kolmogorov law of large numbers: For $1 \le p < 2$, one has

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \sum_{i=1}^{n} \xi_i = 0 \quad \text{almost surely (a.s.)}$$

if $\mathbf{E}|\xi|^p < \infty$ and $\mathbf{E}\xi = 0$.

Let (X_i) be a sequence of independent copies of a random element X with values in a separable Banach space B and let

$$S_n = \sum_{i=1}^{n} X_i .$$

It is known [2, p. 259] that, for Banach spaces of the type p, $1 \le p < 2$, under the conditions

$$\mathbf{E}\| X \|^p < \infty \tag{1}$$

and $\mathbf{E}X = 0$, the Marcinkiewicz–Zygmund law of large numbers of the form

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \| S_n \| = 0 \quad \text{a.s.} \tag{2}$$

is also true.

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In what follows, B denotes a separable Banach lattice with modulus $|\cdot|$. We set

$$S_n^* = \sup_{k \le n} |S_k|, \quad n = 1, 2, \ldots$$

(here and in what follows, the relation $k \le n$ means that $1 \le k \le n$).

The following question naturally arises: Is it possible to strengthen the law of large numbers (2) in the case of Banach lattices to the equality

$$\lim_{n \to \infty} \frac{1}{n^{1/p}} \| S_n^* \| = 0 \quad \text{a.s.} \tag{3}$$

and what conditions should be imposed on the random element X for this purpose?

Let $1 \le p, q < \infty$. A Banach lattice B is called *p-convex* [3, p. 46] if there exists a constant $D^{(p)} = D^{(p)}(B)$ such that, for each n and any elements $(x_i)_1^n \subset B$, one has

$$\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\| \le D^{(p)} \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p},$$

and, similarly, it is called *q-concave* if, for a certain constant $D_{(q)} = D_{(q)}(B)$, the inverse inequality is true, i.e.,

$$\left(\sum_{i=1}^n \|x_i\|^q \right)^{1/q} \le D_{(q)} \left\| \left(\sum_{i=1}^n |x_i|^q \right)^{1/q} \right\|.$$

Theorem 1. *Let B be a p-convex $(1 \le p < 2)$ and q-concave $(q < \infty)$ Banach lattice and let X be a random element with values in B such that $\mathbf{E}X = 0$. Then condition (1) is equivalent to equality (3).*

Corollary 1. *Let X be a random element with values in the space L_p or ℓ_p for $1 \le p < 2$ and let $\mathbf{E}X = 0$. Then conditions (1) and (3) are equivalent.*

Remark 1. For general separable Banach lattices, Theorem 1 is not true. However, it was shown in [4] that the Kolmogorov-type law of large numbers

$$\lim_{n \to \infty} \frac{1}{n} \| S_n^* \| = 0 \quad \text{a.s.,}$$

provided that $\mathbf{E}\|X\| < \infty$ and $\mathbf{E}X = 0$.

Recall that a sequence (x_n) of elements of a Banach lattice B is called *o-convergent* to an element x, which is denoted by

$$x = o\text{-}\lim_{n \to \infty} x_n,$$

if there exists a sequence of nonnegative elements $v_n \in B$ such that $|x_n - x| \le v_n$ and $v_n \downarrow 0$, i.e., $v_1 \ge v_2 \ge \ldots$ and

$$\inf_{n \ge 1} v_n = 0.$$

For a random element X with values in a Banach lattice (with $\mathbf{E}X = 0$), one can consider the Marcinkiewicz–Zygmund order law of large numbers:

$$o\text{–}\lim_{n\to\infty} \frac{1}{n^{1/p}} S_n = 0 \quad \text{a.s.}$$

Remark 2. Under the conditions of Theorem 1, the Marcinkiewicz–Zygmund order law of large numbers is not true. For instance, the counterexample from [5] considered in the space ℓ_p, $1 \le p < \infty$, satisfies both inequality (1) and the following relation:

$$\left\| \sup_{n\ge1} \frac{1}{n^{1/p}} |S_n| \right\|_{\ell_p} = \infty \quad \text{a.s.}$$

2. Proof of Theorem 1

First, note that we essentially use here the proof of the Marcinkiewicz–Zygmund law of large numbers in a Banach space presented in [2, pp. 186, 187].

The implication (3) ⇒ (1) follows from the results of [2, p. 259]. Therefore, it suffices to establish the opposite implication (1) ⇒ (3).

Step 1. We present here several auxiliary lemmas.

Lemma 1 [4]**.** *Let Y be a random element with values in a finite-dimensional subspace E of a Banach lattice and let (Y_i) be its independent copies. Suppose that $1 < p \le 2$, $\mathbf{E}\|Y\|^p < \infty$, and $\mathbf{E}Y = 0$. Then*

$$\frac{1}{n^{1/p}} \left\| \sup_{k \le n} \left| \sum_{i=1}^{k} Y_i \right| \right\| \to 0 \quad \textit{a.s.} \quad \textit{as } n \to \infty.$$

Lemma 2 [6]**.** *Let B be a q-concave $(q < \infty)$ Banach ideal space and let $X = (X(t),\ t \in T)$ be a random element with values in B. Then*

$$\left(\mathbf{E}\|X\|^q\right)^{1/q} \le D_{(q)} \left\| \left(\mathbf{E}|X(t)|^q\right)^{1/q} \right\|.$$

Lemma 3 [2, p. 179]**.** *Let (X_n) and (X'_n) be independent sequences of random variables in a Banach space such that*

$$\|X_n - X'_n\| \to 0 \quad \textit{a.s.} \quad \textit{and} \quad \|X_n\| \xrightarrow{P} 0 \quad \textit{as } n \to \infty.$$

Then

$$\|X_n\| \to 0 \quad a.s.$$

The lemma presented below is similar to the known Prokhorov's result in $\mathbb{R}$ [7]. Assume that a number sequence a_n is such that $a_n \uparrow \infty$ and there exist a subsequence $(b_m) = (a_{n_m})$ and constants $C > c > 1$ such that $C \geq a_{n_{m+1}} / a_{n_m} \geq c$ for sufficiently large m. (If, e.g., $a_{n+1}/a_n \to 1$, then this subsequence exists [8, p. 330].)

Let (X_n) be a sequence of independent random elements with values in a Banach lattice B. As in the introduction, we determine S_n and S_n^* for the sequence (X_n). We set $J_m = \{n_{m-1} + 1, \ldots, n_m\}$, $m \in \mathbb{N}$, and

$$U_m = \sup_{n \in J_m} \left| S_n - S_{n_{m-1}} \right|.$$

Lemma 4. *The following relations are equivalent:*

(i) $\lim_{n\to\infty} \frac{1}{a_n} \left\| S_n^* \right\| = 0$ *a.s.;*

(ii) $\lim_{m\to\infty} \frac{1}{b_m} \|U_m\| = 0$ *a.s.;*

(iii) $\forall \delta > 0 \colon \sum_{m \geq 1} \mathbf{P}\left\{ \frac{1}{b_m} \|U_m\| > \delta \right\} < \infty.$

Proof. It suffices to prove the equivalence of relations (i) and (ii) because the equivalence of relations (ii) and (iii) follows from the Borel–Cantelli lemma.

If condition (i) is satisfied, then

$$\frac{1}{b_m}\|U_m\| \leq \frac{1}{b_m}\left\| \sup_{n \in J_m} |S_n| \right\| + \frac{\|S_{n_{m-1}}\|}{b_{m-1}} \frac{b_{m-1}}{b_m} \to 0 \quad \text{a.s.} \qquad \text{as} \quad m \to \infty.$$

Assume, on the contrary, that condition (ii) is satisfied. Then, for $n \in J_m$, we have

$$|S_n| = \left| S_n - S_{n_{m-1}} + \sum_{i=1}^{m-1} (S_{n_i} - S_{n_{i-1}}) \right| \leq \sum_{i=1}^{m} U_i .$$

Hence,

$$\frac{1}{a_n} \left\| S_n^* \right\| \leq \frac{C}{b_m} \sum_{i=1}^{m} \|U_i\|. \tag{4}$$

It follows from the properties of the sequence (b_m) that

$$\sum_{i=1}^{m} b_i \leq \frac{b_m}{1 - 1/c}. \tag{5}$$

Finally, we use the following elementary number relation (see Lemma 9 in [8, p. 327]):

$$\frac{1}{a_n} \sum_{i=1}^{n} b_i y_i \to 0$$

if

$$a_n = \sum_{i=1}^{n} b_i \uparrow \infty \quad \text{and} \quad y_n \to 0$$

as $n \to \infty$. This relation, condition (ii), and estimates (4) and (5) yield relation (i).

The lemma is proved.

Remark 3. We set

$$T_m = \left| S_{n_m} - S_{n_{m-1}} \right|$$

and consider the following conditions:

(ii′) $\lim\limits_{m \to \infty} \dfrac{1}{b_m} \| T_m \| = 0$ a.s.;

(iii′) $\forall \delta > 0: \sum\limits_{m \geq 1} \mathbf{P}\left\{ \dfrac{1}{b_m} \| T_m \| > \delta \right\} < \infty.$

If $B = \mathbb{R}$ and the random elements X_n are symmetric in the conditions of Lemma 4, then conditions (ii) and (iii) can be replaced by (ii′) and (iii′).

Step 2. First, we establish a weakened version of the implication (1) ⇒ (3), namely, we show that convergence in probability holds in (3).

It is known that the set of simple random elements is dense in $L_p(B)$ (see, e.g., Exercise 3 in [9, p. 97]). Therefore, for any $\epsilon > 0$, there exists a simple (i.e., finite-dimensional) random element Y such that

$$\left(\mathbf{E} \| X - Y \|^p \right)^{1/p} < \epsilon.$$

Since

$$\| \mathbf{E} Y \| \leq \| \mathbf{E}(Y - X) \| + \| \mathbf{E} X \| \leq \left(\mathbf{E} \| X - Y \|^p \right)^{1/p} < \epsilon,$$

using $Y - \mathbf{E}Y$ instead of Y we can assume that $\mathbf{E}Y = 0$. We set $R = X - Y$. It is clear that

$$\mathbf{E}R = 0 \quad \text{and} \quad \left(\mathbf{E}\|R\|^p\right)^{1/p} < \epsilon. \tag{6}$$

For independent copies X_n, $n \geq 1$, of X, we write $X_n = Y_n + R_n$, where Y_n are independent copies of Y and R_n are independent copies of R. We set

$$S'_n = \sup_{k \leq n} \left| \sum_{i=1}^{k} Y_i \right| \quad \text{and} \quad S''_n = \sup_{k \leq n} \left| \sum_{i=1}^{k} R_i \right|.$$

It is obvious that

$$\left\| S^*_n \right\| \leq \| S'_n \| + \| S''_n \|. \tag{7}$$

By virtue of Lemma 1, we have

$$\lim_{n \to \infty} \frac{1}{n^{1/p}} \| S'_n \| = 0 \quad \text{a.s.} \tag{8}$$

We now estimate $\| S''_n \|$ from above. Note that a separable σ-complete Banach lattice is order isometric to a certain Banach ideal space [3, p. 25] (in [3], the similar term "Köthe functional space" was used). Since a q-concave Banach lattice is σ-complete [3] (Theorem 1.a.5), we can assume, without loss of generality, that B is a separable, p-convex, q-concave Banach ideal space defined on a certain measurable space (T, Λ, μ). Let

$$X_n = X_n(t), \qquad S''_n = S''_n(t), \qquad R_n = R_n(t),$$

and let $\tilde{R}_n = \tilde{R}_n(t)$, $t \in T$, be an independent copy of R_n. Using the symmetrization procedure, we get [9, p. 222] (Lemma 3.4)

$$\mathbf{E}\| S''_n \| \leq \mathbf{E}\left\| \sup_{k \leq n} \left| \sum_{1}^{k} \left(R_i - \tilde{R}_i\right) \right| \right\|.$$

We can assume that $R_n - \tilde{R}_n = \varepsilon_n \hat{R}_n$, where ε_n are independent symmetric Bernoulli random variables and $\hat{R}_n$ are independent copies of $R - \tilde{R}$ ($\tilde{R}$ is an independent copy of R) that do not depend on (ε_n). Using the last inequality and Lemma 2, we get

$$\mathbf{E}\| S''_n \| \leq D_{(q)} \mathbf{E}\left\| \left(\hat{\mathbf{E}} \sup_{k \leq n} \left| \sum_{1}^{k} \varepsilon_i \hat{R}_i(t) \right|^q \right)^{1/q} \right\|, \tag{9}$$

where $\hat{\mathbf{E}}\varphi(\varepsilon_n \hat{R}_n)$ denotes the mathematical expectation of the random variable $\varphi(\varepsilon_n \hat{R}_n)$ for fixed values of the random variable $(\hat{R}_n)$. Further, for fixed values of $\hat{R}_n$, we successively use the Lévy moment inequality for symmetric random variables in $\mathbb{R}$ [2, p. 48]

$$\mathbf{E} \max_{k \le n} \left| \sum_1^k \xi_i \right|^q \le 2\, \mathbf{E} \left| \sum_1^n \xi_i \right|^q$$

and the well-known Kahane inequality [3] (Theorem 1.e.13)

$$\left(\hat{\mathbf{E}} \sup_{k \le n} \left| \sum_{i=1}^k \varepsilon_i \hat{R}_i(t) \right|^q \right)^{1/q} \le \left(2\, \hat{\mathbf{E}} \left| \sum_1^n \varepsilon_i \hat{R}_i(t) \right|^q \right)^{1/q} \le C_K \left(\hat{\mathbf{E}} \left| \sum_1^n \varepsilon_i \hat{R}_i(t) \right|^p \right)^{1/p} \le C_K \left(\sum_1^n \left| \hat{R}_i(t) \right|^p \right)^{1/p},$$

where $C_K = C_K(p, q)$ depends on the constant in the Kahane inequality.

Using estimate (9), the last inequality, and the p-convexity of B, we obtain (for certain absolute constants C_1 and C)

$$\mathbf{E}\| S_n'' \| \le C_1\, \mathbf{E} \left\| \left(\sum_{i=1}^n \left| \hat{R}_i \right|^p \right)^{1/p} \right\| \le C\, \mathbf{E} \left(\sum_{i=1}^n \left\| \hat{R}_i \right\|^p \right)^{1/p} \le C n^{1/p} \left(\mathbf{E} \left\| \hat{R}_i \right\|^p \right)^{1/p} \le C n^{1/p} \epsilon. \tag{10}$$

In the last inequality, we have also used the inequality from (6).

Since ϵ is arbitrary, using (7), (8), and (10) we establish that

$$\frac{1}{n^{1/p}} \left\| S_n^* \right\| \xrightarrow{P} 0. \tag{11}$$

Step 3. We now pass to the proof of the implication (1) ⇒ (3).

By virtue of Lemma 3 and relation (11), we can restrict ourselves to the case of symmetric random elements X_n. By analogy with Step 2, we represent them in the form

$$X_n = \varepsilon_n X_n' = \varepsilon_n (Y_n + R_n),$$

where the symmetric Bernoulli random variables ε_n are independent of X_n'. We set

$$S_n' = \sup_{k \le n} \left| \sum_{i=1}^k \varepsilon_i Y_i \right|, \qquad S_n'' = \sup_{k \le n} \left| \sum_{i=1}^k \varepsilon_i R_i \right|.$$

It is clear that S_n' and S_n'' satisfy inequality (7), and S_n' satisfies equality (8). Thus, to prove Theorem 1 it remains to show that

$$\lim_{n \to \infty} \frac{1}{n^{1/p}} \left\| S_n'' \right\| = 0 \quad \text{a.s.} \tag{12}$$

For each $m \in \mathbb{N}$, we denote $J_m = \left\{2^{m-1} + 1, \ldots, 2^m\right\}$. For each $j \in J_m$, we set

$$V_j = \varepsilon_j R_j \mathbb{I}\left(\| R_j \| \le 2^{m/p}\right),$$

where $\mathbb{I}(A) = 1$ if the event A takes place, and $\mathbb{I}(A) = 0$ otherwise. The random element R_j satisfies condition (6). Therefore,

$$\sum_{m \ge 1} \mathbf{P}\left\{\exists\, j \in J_m,\, V_j \ne \varepsilon_j R_j\right\} \le \sum_{m \ge 1} 2^m \mathbf{P}\left\{\| R \| > 2^{m/p}\right\} < \infty.$$

According to the Borel–Cantelli lemma, this means that, almost surely, the inequality $V_j \ne \varepsilon_j R_j$ holds finitely many times. Thus, equality (12) is true if

$$\lim_{n \to \infty} \frac{1}{n^{1/p}} \left\| \sup_{k \le n} \left| \sum_1^k V_i \right| \right\| = 0 \quad \text{a.s.}$$

Using Lemma 4 for $a_n = n^{1/p}$ and $n_m = 2^m$, we establish that the last equality is equivalent to the condition

$$\forall\, \delta > 0 : \sum_{m \ge 1} \mathbf{P}\left\{\| U_m \| > \delta\, 2^{m/p}\right\} < \infty, \tag{13}$$

where

$$U_m = \sup_{j \in J_m} \left| \sum_{i = 2^{m-1}+1}^{j} V_i \right|.$$

Using estimate (10), we get

$$\mathbf{E}\| U_m \| \le C 2^{m/p} \epsilon.$$

Taking $\epsilon = \delta/(2C)$, we establish that the condition

$$\forall\, \delta > 0 : \sum_{m \ge 1} \mathbf{P}\left\{\| U_m \| - \mathbf{E}\| U_m \| > \delta\, 2^{(m/p)-1}\right\} < \infty \tag{14}$$

is sufficient for (13).

To estimate the mth term of sum (14), we use a modification of the Yurinskii method [10] for sums of independent random elements in Banach spaces. For every $j \in J_m$, we denote

$$U_{m,\,j} \;=\; \sup_{i\,\in\, J_m} \left| \sum_{s\,\in\, J_m,\, s\,<\,i,\, s\,\neq\, j} V_s \right|,$$

$$\zeta_j \;=\; \mathbf{E}_j \,\| U_m \| - \mathbf{E}_{j-1} \| U_m \|,$$

$$\mathbf{E}_j \eta \;=\; \mathbf{E}\big(\eta \,\big|\, \mathcal{F}_j\big),$$

where $\mathcal{F}_j$ is the σ-algebra spanned by the random elements $V_i : i \in J_m,\ i \le j,$ and $\mathcal{F}_{2^{(m-1)}}$ is the trivial σ-algebra. Then the following martingale representation takes place:

$$\| U_m \| - \mathbf{E} \| U_m \| \;=\; \sum_{j\,\in\, J_m} \zeta_j\,. \tag{15}$$

Since ζ_j can be represented in the form

$$\zeta_j \;=\; \big(\mathbf{E}_j - \mathbf{E}_{j-1}\big)\big(\| U_m \| - \big\| U_{m,\,j} \big\|\big),$$

using the inequality

$$\| U_m \| - \big\| U_{m,\,j} \big\| \;\le\; \big\| V_j \big\|$$

we obtain

$$\big| \zeta_j \big| \;\le\; \big\| V_j \big\| + \mathbf{E} \big\| V_j \big\|. \tag{16}$$

However, (ζ_j) is a sequence of martingale differences. Therefore, according to estimate (16), we have

$$\mathbf{E} \left| \sum_{j \in J_m} \zeta_j \right|^2 \;\le\; \sum_{j \in J_m} \mathbf{E} \big| \zeta_j \big|^2 \;\le\; C \sum_{j \in J_m} \mathbf{E} \big\| V_j \big\|^2 \;\le\; C\, 2^m\, \mathbf{E} \big\| V_j \big\|^2.$$

Using this result and equality (15), we establish that the series in (14) can be estimated from above as follows:

$$\frac{C}{\delta^2} \sum_{m\,\ge\,1} \frac{1}{2^{2m/p}} \sum_{j \in J_m} \mathbf{E} \big\| V_j \big\|^2 \;\le\; \frac{C}{\delta^2} \sum_{m\,\ge\,1} \frac{1}{2^{m(2/p\,-\,1)}}\, \mathbf{E}\Big(\| R \|^2\, \mathbb{I}\big(\| R \| \le 2^{m/p} \big) \Big). \tag{17}$$

It is known (see the proof of Theorem 7.9 in [2, p. 187]) that, for a random variable ξ in $\mathbb{R}$, under the condition $\mathbf{E} |\, \xi \,|^p < \infty,\ 1 \le p < 2,$ one has

$$\sum_{m\,\ge\,1} \frac{1}{2^{m(2/p\,-\,1)}}\, \mathbf{E}\Big(\xi^2\, \mathbb{I}\big(|\, \xi \,|^p \le 2^m \big) \Big) \;<\; \infty.$$

Since the random variable $\| R \|$ satisfies condition (6), we conclude that series (17) converges, and, hence, series (14) also converges.

The theorem is proved.

3. Examples of Application to Empirical Distributions

1. Sampling in $\mathbb{R}$**.** For independent identically distributed random variables $\xi, \xi_1, \xi_2, \ldots$ in $\mathbb{R}$ with distribution function $F(t)$, we introduce the empirical distribution function

$$F_n^*(t) = \frac{1}{n} \sum_{i=1}^{n} I_{(-\infty, t)}(\xi_i), \qquad t \in \mathbb{R},$$

where $I_{(-\infty, t)}(\xi) = 1$ if $\xi < t$, and $I_{(-\infty, t)}(\xi) = 0$ if $\xi \geq t$.

According to the known Glivenko–Cantelli theorem, we have

$$\sup_{t \in \mathbb{R}} \left| F_n^*(t) - F(t) \right| \to 0 \quad \text{a.s.}$$

We consider the random processes

$$X_n(t) = I_{(-\infty, t)}(\xi_n) - F(t), \qquad t \in \mathbb{R}, \tag{18}$$

as random elements with values in the space $L_p(\mathbb{R})$, $1 \leq p < \infty$ (of course, this is true only under certain restrictions on the random variables ξ [4]). For the random elements X_n defined by (18) and satisfying the condition

$$\mathbf{E} \| X_n \|_{L_p(\mathbb{R})}^p < \infty, \tag{19}$$

the law of large numbers (2) in the space $L_p(\mathbb{R})$ can be rewritten in the following form:

$$\frac{1}{n} \int_{-\infty}^{\infty} n^p \left| F_n^*(t) - F(t) \right|^p dt \to 0 \quad \text{a.s.} \qquad \text{as } n \to \infty. \tag{20}$$

Theorem 1 enables us to strengthen the last relation as follows: If condition (19) is satisfied, then

$$\frac{1}{n} \int_{-\infty}^{\infty} \max_{k \leq n} k^p \left| F_k^*(t) - F(t) \right|^p dt \to 0 \quad \text{a.s.} \qquad \text{as } n \to \infty.$$

On the other hand, equalities (20) and (21) can be regarded as certain versions of the Glivenko–Cantelli theorem in the space $L_p(\mathbb{R})$.

Corollary 2. *For* $1 \le p < 2$, *the empirical distribution function* $F_n^*(t)$ *satisfies the law of large numbers (21) if and only if*

$$\mathbf{E}|\xi| < \infty. \tag{22}$$

Remark 4. It is known [4] that, under the condition $\mathbf{E}|\xi|^{1/p} < \infty$, the random element X_n belongs almost surely to $L_p(\mathbb{R})$. Therefore, condition (22) guarantees that X_n belongs to all $L_p(\mathbb{R})$ simultaneously.

Corollary 2 follows directly from Theorem 1 if the equivalence (19) ⇔ (22) is established.
This can easily be verified. Indeed,

$$\mathbf{E}\|X_n\|^p_{L_p(\mathbb{R})} = \mathbf{E}\int_{-\infty}^{\infty}\Big(|1 - F(t)|^p\,\mathbb{I}(\xi < t) + |F(t)|^p\,\mathbb{I}(\xi \ge t)\Big)\,dt$$

$$= \int_{-\infty}^{\infty}\Big(|1 - F(t)|^p\,F(t) + |F(t)|^p\,(1 - F(t))\Big)\,dt\,.$$

The last integral is bounded if and only if

$$\int_0^{\infty}(1 - F(t))\,dt + \int_{-\infty}^{0}F(t)\,dt < \infty.$$

It is known (see Lemma 2 in [11, p. 179]) that the last inequality is equivalent to condition (22).

2. Sampling in $\mathbb{R}^m$***.*** Let

$$\langle\bar{b}, \bar{c}\rangle = \sum_{i=1}^{m} b_i c_i$$

be the scalar product of elements $\bar{b} = (b_1, \ldots, b_m)$ and $\bar{c} = (c_1, \ldots, c_m)$ from $\mathbb{R}^m$, let $\|\bar{c}\| = \langle\bar{c}, \bar{c}\rangle^{1/2}$, and let $\bar{\xi}, \bar{\xi}_1, \bar{\xi}_2, \ldots$ be independent identically distributed random variables in $\mathbb{R}^m$. Out of several possible definitions of an empirical distribution function in $\mathbb{R}^m$, we choose the following:

$$F_n^*(\bar{c}, t) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty, t)}\big(\langle\bar{\xi}_i, \bar{c}\rangle\big), \qquad \langle\bar{c}, t\rangle \in D,$$

where $D = \mathbb{S}^m \times \mathbb{R}$ and $\mathbb{S}^m$ is the identity sphere of the m-dimensional Euclidean space. On D, we introduce a measure in a natural way as the product of the (normalized) spherical Lebesgue measure on $\mathbb{S}^m$ and the ordinary Lebesgue measure in $\mathbb{R}$.

We set

$$F(\overline{c},\, t) \;=\; \mathbf{P}\Big\{\big\langle \overline{\xi}_i,\, \overline{c}\big\rangle < t\Big\}$$

and consider the random functions

$$X_n(\overline{c},\, t) \;=\; I_{(-\infty,\, t)}\Big(\big\langle \overline{\xi}_n,\, \overline{c}\big\rangle\Big) \;-\; F(\overline{c},\, t)\,, \qquad (\overline{c},\, t) \in D\,,$$

as random elements X_n with values in the (separable) Banach lattice of numerical functions $L_p(D)$, $1 \le p < \infty$.

Applying Theorem 1 to the random elements X_n, we obtain the following corollary:

Corollary 3. *If*

$$\mathbf{E}\big\|\overline{\xi}\big\| \;<\; \infty\,, \tag{23}$$

then, for any $1 \le p < 2$, *one has*

$$\frac{1}{n}\int\limits_{\mathbb{S}^m} d\overline{c}\int\limits_{-\infty}^{\infty}\max_{k \le n} k^p\,\Big|F_n^*(\overline{c},\, t) - F(\overline{c},\, t)\Big|^p\, dt \;\to\; 0 \quad a.s. \quad as \;\; n \to \infty. \tag{24}$$

To prove Corollary 3, it suffices to verify condition (19). We have

$$\mathbf{E}\|X_n\|^p_{L_p(D)} \;=\; \mathbf{E}\int\limits_{\mathbb{S}^m} d\overline{c}\int\limits_{-\infty}^{\infty}\Big(|1 - F(\overline{c},\, t)|^p\, \mathbb{I}\Big(\big\langle\overline{\xi},\, \overline{c}\big\rangle < t\Big) + |F(\overline{c},\, t)|^p\, \mathbb{I}\Big(\big\langle\overline{\xi},\, \overline{c}\big\rangle \ge t\Big)\Big)\,dt$$

$$=\; \int\limits_{\mathbb{S}^m} d\overline{c}\int\limits_{-\infty}^{\infty}\Big(|1 - F(\overline{c},\, t)|^p\, F(\overline{c},\, t) + |F(\overline{c},\, t)|^p\,|1 - F(\overline{c},\, t)|\Big)\,dt$$

$$\le\; 2\int\limits_{\mathbb{S}^m} d\overline{c}\int\limits_{-\infty}^{\infty}\big(1 - F(\overline{c},\, t)\big)\,F(\overline{c},\, t)\,dt$$

$$\le\; 2\int\limits_{\mathbb{S}^m} d\overline{c}\left(\int\limits_{0}^{\infty}\big(1 - F(\overline{c},\, t)\big)\,dt \;+\; \int\limits_{-\infty}^{0} F(\overline{c},\, t)\,dt\right).$$

Estimating the last two one-dimensional integrals, we get

$$\int\limits_{0}^{\infty}|1 - F(\overline{c},\, t)|\,dt \;\le\; \int\limits_{0}^{\infty}\mathbf{P}\Big\{\Big|\big\langle\overline{\xi},\, \overline{c}\big\rangle\Big| \ge t\Big\}\,dt \;=\; \mathbf{E}\Big|\big\langle\overline{\xi},\, \overline{c}\big\rangle\Big| \;\le\; \mathbf{E}\big\|\overline{\xi}\big\|$$

and

$$\int_{-\infty}^{0} F(\bar{c}, t)\, dt = \int_{-\infty}^{0} \mathbf{P}\left\{\left\langle \xi, -\bar{c} \right\rangle > -t\right\} dt = \int_{0}^{\infty} \mathbf{P}\left\{\left\langle \xi, -\bar{c} \right\rangle > t\right\} dt \le \mathbf{E}\left\| \xi \right\|.$$

Combining the last estimates and condition (23), we obtain (19).

The corollary is proved.

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