$$= O(1) z^{1/p} \sum_{k=[z]-1}^{\infty} \frac{|\Delta R_k|}{k+2} + O(1) z^{1/p} \sum_{k=[z]-1}^{\infty} R_k \left| \frac{P_k}{P_{k+1}} \frac{1}{k+1} - \frac{1}{k+2} \right|$$

$$= O(1) \frac{z^{1/p}}{[z]+1} \sum_{k=[z]-1}^{\infty} |\Delta R_k| + O(1) z^{1/p} \sum_{k=[z]-1}^{\infty} \frac{1}{(k+1)(k+2)} = O(1).$$

Finally,

$$\Sigma_{223} = O(1) z^{1/p} \sum_{n=[z]+1}^{\infty} \frac{p_{n-1}}{n^{1/p} P_{n-1}} = O(1) z^{1/p} \sum_{n=[z]+1}^{\infty} R_n \frac{1}{n^{1+1/p}} = O(1).$$

Consequently, the condition for the $|WN, p_n|_p$-method is satisfied. It remains to apply Theorem A.

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M-BASES IN SEPARABLE AND REFLEXIVE BANACH SPACES

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Let E be a normed linear space, E' its dual. A set $\{e_\alpha, f_\alpha\}_{\alpha \in I}$ is called a total biorthogonal system if $f_\alpha(e_\beta) = \delta_{\alpha\beta}$ and $\{f_\alpha\}$ is total in E'. If, in addition, $\{e_\alpha\}$ is total in E, $\{e_\alpha, f_\alpha\}$ is called an M-basis (Markushevich basis). We say that an M-basis is bounded if $\sup_\alpha \|f_\alpha\| \|e_\alpha\| < \infty$, ε-normed if $\sup_\alpha \|f_\alpha\| \|e_\alpha\| \leqslant 1 + \varepsilon$, and normed if $\varepsilon = 0$.

The following two problems are posed in [1, p. 205]: 1) does there exist a normed M-basis, in every separable Banach space, and 2) does there exist a bounded M-basis in such a space.

We prove a proposition which is weaker than the first problem, but stronger than the second, along with other closely related assertions. In this paper we use the following notation: [M] is the closure of the linear span of the set M, $M^\perp = \{f \in E' : f(M) = 0\}$ if $M \subset E$ and $M^\perp = \{x \in E : x(M) = 0\}$ if $M \subset E'$; d(M, N) is the distance between the sets M and N.

Separable Spaces. The following statement was announced in [2].

THEOREM 1. For every separable normed linear space E and $\varepsilon > 0$, there exists an ε-normed M-basis.

Proof. Construction. Let $\{n_i\}$ be a strictly increasing sequence of natural numbers and write $P_i = \left\{n : \sum_{s=1}^{n_{i-1}} n_s < n \leqslant \sum_{s=1}^{n_i} n_s\right\}$. On the basis of Lemma 1 of [3], we choose a biorthogonal system $\{x_n, g_n\}$ in which $\|x_n\| = \|g_n\| = 1$ and

a) $d(x, [x_n : n \in P_s, s > i]) > \|x\|/2$ for $x \in [x_n : n \in P_s, s \le i]$, and hence $d(x, [x_n : n \notin P_i]) > \|x\|/8$ for $x \in [x_n : n \in P_i]$;

b) $^1/_2 \|x\|_{l_2} \le \|x\| \le {}^3/_2 \|x\|_{l_2}$, where $x = \sum_{n \in P_i} \alpha_n x_n$, $\|x\|_{l_2} = \sqrt{\sum_{n \in P_i} (\alpha_n)^2}$;

c) $[[x_n]_1^\infty + \{g_n\}^\perp] = E$, $[x_n]_1^\infty \cap \{g_n\}^\perp = 0$.

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According to Theorem 3 in [4], $\{x_n, g_n\}$ can be extended to an M-basis using sets $\{y_j\} \subset \{g_n\}^{\perp}$ and $\{h_j\} \subset ([x_n]_1^{\infty})^{\perp}$ with $\|y_j\| = 1$.

Let $d(y_j, h_j^{\perp}) = a_j$. We choose a double sequence $\{n_j^k\}_{j,k=1}^{\infty}$, $\{n_j^k\} \subset \{n_i\}$ in such a way that for all k and j

$$\frac{6}{\sqrt{n_i^k}} \leqslant \varepsilon; \quad \frac{(n_j^k - 1)\,\varepsilon}{8} - \frac{2^9}{\varepsilon} > \begin{cases} n_j^{k-1} & \text{for } k \neq 1, \\ \sqrt{\dfrac{n_j^1}{a_j}} & \text{for } k = 1. \end{cases} \tag{1}$$

We denote by P_j^k that P_i for which $n_j^k = n_i$. We put

$$e_n = \begin{cases} x_n & \text{for } n \notin \bigcup_{jk} P_j^k, \\ x_n + y_j/\sqrt{n_j^1} & \text{for } n \in P_j^1, \\ x_n + \Big(\sum\limits_{s \in P_j^{k-1}} x_s\Big)/n_j^{k-1} & \text{for } n \in P_j^k, \quad k \neq 1, \end{cases} \tag{2}$$

and $D_n = [\{e_m\}_1^{\infty} \setminus e_n]$. Choose $\{f_n\}$ such that $f_m(e_n) = \delta_{mn}$ (it will be shown below that $e_n \notin D_n$, and hence such a choice is possible).

Totality of $\{e_n\}$. Applying (2), condition b), and (1) in succession, we obtain

$$\left\| \sum_{k=1} (-1)^k \Big(\sum_{n \in P_j^k} e_n\Big) \Big/ n_j^k + \frac{1}{\sqrt{n_j^1}} y_j \right\| = \left\| \frac{1}{n_j^t} \Big(\sum_{n \in P_j^t} x_n\Big) \right\| \leqslant \frac{3}{2\sqrt{n_j^t}} \underset{t \to \infty}{\to} 0.$$

Thus $\{y_j\} \subset [e_n]_1^{\infty}$; by (2) and condition c), $\{x_n\} \subset [e_n]_1^{\infty}$ and $[e_n]_1^{\infty} = E$.

Totality of $\{f_n\}$. By the construction of $\{f_n\}$ and the density of $\{e_n\}$ in the space E, the totality of $\{f_n\}$ is equivalent to the equality $\bigcap_m [e_n]_m^{\infty} = 0$.

But $\bigcap_m [e_n]_m^{\infty} \subset (\{g_n\} \bigcup \{h_j\})^{\perp} = 0$ and hence $\{f_n\}^{\perp} = 0$.

Inequality $\sup_n \|f_n\| \|e_n\| \leqslant 1 + \varepsilon$. We first show that

$$d(e_n, D_n) \geqslant 1 - \varepsilon/2. \tag{3}$$

Choose any element z in the linear span of the set $\{e_m\}_1^{\infty} \setminus e_n$:

$$z = \sum_{i=1} z_i, \quad z_i \in [e_s : s \in P_i]. \tag{4}$$

If $n \notin \bigcup_{jk} P_j^k$, then (3) follows at once from the construction; assume that $n \in P_j^k$. By (2), condition b), and (1), $\|e_n - z\| \geq \|x_n - z\| - \varepsilon/4$. Assume that

$$\|x_n - z\| < 1 - \varepsilon/4. \tag{5}$$

Then by (2) there exists a term $z_j^{k+1} = \sum\limits_{s \in P_j^{k+1}} \alpha_s e_s$ in the sum (4) for which

$$b_j^k = \Big(\sum_{s \in P_j^{k+1}} \alpha_s\Big) \Big/ n_j^k > \varepsilon/4. \tag{6}$$

1. We show that there exists a term $z_j^k = \sum\limits_{s \in P^k} \alpha_s e_s$ in sum (4) for which

$$\Big|\sum_{s \in P_j^k} \alpha_s\Big| > \begin{cases} n_j^{k-1} & \text{for } k \neq 1, \\ \sqrt{n_j^1}/a_j & \text{for } k = 1. \end{cases} \tag{7}$$

Indeed, applying (5), condition a), and condition b) successively, we get:

$$1-\varepsilon/4 > \|x_n - z\| = \left\|\left(x_n - \sum_{s\in P_j^k}(b_j^k+\alpha_s)x_s\right)\right.$$

$$\left.\left(z - b_j^k\sum_{s\in P_j^k}x_s - \sum_{s\in P_j^k}\alpha_s x_s\right)\right\| \geqslant \frac{1}{8}\left\|x_n - \sum_{s\in P_j^k}(b_j^k+\alpha_s)x_s\right\| \geqslant \frac{1}{16}\left\|x_n - \sum_{s\in P_j^k}(b_j^k+\alpha_s)x_s\right\|_{l_2} \geqslant \frac{1}{16}\sqrt{\sum_{\substack{s\in P_j^k\\ s\neq n}}(b_j^k+\alpha_s)^2}.$$

Hence $(1-\varepsilon/4)^2 \geqslant 2^{-8}\sum_{s\in P_j^k, s\neq n}(b_j^k+\alpha_s)^2$. Making some algebraic transformations, we have

$$-2b_j^k\sum_{s\in P_j^k, s\neq n}\alpha_s \geqslant (n_j^k-1)(b_j^k)^2 - 2^8(1-\varepsilon/4)^2.$$

Dividing the last inequality by $2|b_j^k|$ and using (6) and (1), we obtain

$$\left|\sum_{\substack{s\in P_j^k\\ s\neq n}}\alpha_s\right| > \begin{cases} n_j^{k-1} & \text{for } k\neq 1,\\ \sqrt{n_j^1}/a_j & \text{for } k=1.\end{cases} \tag{8}$$

It follows from the choice of z that $\alpha_n = 0$ and hence (7) is proved. For $k \neq 1$ we go to Part 2 below, for $k = 1$, to Part 3 below.

2. Thus,

$$|b_j^{k-1}| \geqslant 1, \tag{9}$$

where $b_j^{k-1} = \left(\sum_{s\in P_j^k}\alpha_s\right)\Big/ n_j^{k-1}$; put $u_j^{k-1} = b_j^{k-1}\sum_{s\in P_j^{k-1}}x_s$. We use once more arguments similar to those of 1. We show that there exists a term $z_j^{k-1} = \sum_{s\in P_j^{k-1}}\alpha_s e_s$ in the sum (4) for which

$$\left|\sum_{s\in P_j^{k-1}}\alpha_s\right| > \begin{cases} n_j^{k-2} & \text{for } k\neq 2,\\ \sqrt{n_j^1}/a_j & \text{for } k=2.\end{cases} \tag{10}$$

Indeed, applying (5) and conditions a) and b) in succession, we obtain

$$1-\varepsilon/4 \geqslant \|x_n - z\| = \left\|x_n - u_j^{k-1} - \sum_{s\in P_j^{k-1}}\alpha_s x_s - \left(z - u_j^{k-1} - \sum_{s\in P_j^{k-1}}\alpha_s x_s\right)\right\|$$

$$\geqslant \frac{1}{8}\left\|u_j^{k-1} + \sum_{s\in P_j^{k-1}}\alpha_s x_s\right\| \geqslant \frac{1}{16}\left\|u_j^{k-1} + \sum_{s\in P_j^{k-1}}\alpha_s x_s\right\|_{l_1} = \frac{1}{16}\sqrt{\sum_{s\in P_j^{k-1}}(b_j^{k-1}+\alpha_s)^2}.$$

Hence $(1-\varepsilon/4)^2 \geqslant 2^{-8}\sum_{s\in P_j^{k-1}}(b_j^{k-1}+\alpha_s)^2$. Making some algebraic transformations, we have

$$-2b_j^{k-1}\sum_{s\in P_j^{k-1}}\alpha_s \geqslant n_j^{k-1}(b_j^{k-1})^2 - 2^8(1-\varepsilon/4)^2.$$

Dividing this inequality by $2|b_j^{k-1}|$ and bearing in mind (9) and (1), we get (10). For $k \neq 2$, we go back to the beginning of Part 2, replacing k by $k-1$; for $k = 2$, we go to Part 3 below.

3. After finitely many steps, we come to the conclusion that sum (4) contains a term $z_j^1 = \sum_{s\in P_j^1}\alpha_s x_s + y$, $y = \left(\sum_{s\in P_j^1}\alpha_s\right)y_j/\sqrt{n_j^1}$ and $\left|\sum_{s\in P_j^1}\alpha_s\right| \geqslant \sqrt{n_j^1}/a_j$. Since $x_n - (z-y) \in h_j^\perp$, we have $\|x_n - (z-y) - y\| \geq a_j\|y\| \geq 1$, and this contradicts (5), so that (3) holds. As is easily verified, $\|f_n\| \leq 1/d(e_n, D_n)$, and therefore

$$\|f_n\|\,\|e_n\| \leqslant (1+\varepsilon/4)/d(e_n, D_n) \leqslant (1+\varepsilon/4)/(1-\varepsilon/2) \leqslant 1+\varepsilon.$$

The last inequality holds for $\varepsilon < 1/2$, which causes no loss of generality. The theorem is proved.

COROLLARY 1. In any separable normed linear space E we can for every $\varepsilon > 0$ introduce a norm $\|x\| \leq |||x||| \leq (1 + 2\varepsilon)\|$ such that the space $(E, |||\cdot|||)$ possesses a normed M-basis.

Indeed, let $\{e_n, f_n\}$ be the system constructed in Theorem 1; then as the norm $|||\cdot|||$ we can take the gauge function of the set $\{x \in E: \|x\| \leq 1,\ f_n(x) \leq f_n(e_n),\ n = \overrightarrow{1, \infty}\}$.

COROLLARY 2. Let $l_1 \subset c_0$ be the natural imbedding of the space of absolutely summable sequences in the space of sequences converging to zero. Every separable Banach space E is ε-isometric to a space E_1 intermediate between l_1 and c_0 (i.e., $l_1 \subset E_1 \subset c_0$, both imbeddings being dense and $\|x\|_{c_0} \leq \|x\|_{E_1} \leq \|x\|_{l_1}$ for $x \in l_1$).

WCG-Spaces. A Banach space E is said to be a WCG-space (weakly compactly generated) if it is generated by a set U, $[U] = E$ which is compact in the weak topology $\sigma(E, E')$. In particular, separable and reflexive spaces are WCG-spaces.

THEOREM 2. In any WCG-space there exists a bounded M-basis.

Proof. We denote by dens E the smallest cardinality of the everywhere dense subsets of E, and let α_0 be the first ordinal number with the same power as dens E.

Let $\{n_i\}$ be a strictly increasing sequence of natural numbers, and let the P_i be the same as in Theorem 1. It follows from [3] that in any WCG-space we can choose a bounded biorthogonal system $\{x_\alpha^n, f_\alpha^n\}_{1 \leq \alpha < \alpha_0}^{n=1,\infty}$ such that for all α and P_i:

a) $d(x, [x_\beta^n: n \notin P_i \vee \beta \neq \alpha]) > \|x\|/8$ for $x \in [x_\beta^n: n \in P_i \wedge \beta = \alpha]$;

b) $c\|x\|_{l_2} \leq \|x\| \leq C\|x\|_{l_2}$, where $x = \sum_{n \in P} a_\alpha^n x_\alpha^n$, $\|x\|_{l_2} = \sqrt{\sum_{n \in P} (a_\alpha^n)^2}$, $0 < c \leqslant C < \infty$;

c) $M \cap N = 0, [M + N] = E$, where $M = [x_\alpha^n]_{\alpha, n}$, $N = [f_\alpha^n]_{\alpha, n}^{\perp}$.

Consider the quotient space E / M. The image $\hat{N}$ of the subspace N under the canonical mapping $K: E \to E/M$ is an everywhere dense subspace of E / M. Carrying out the same arguments as in [5], it can be shown that there exists an M-basis $\{\hat{y}_\beta, g_\beta\}_{\beta < \alpha_0}$, $\hat{y}_\beta \in \hat{N}$ in E / M. Choose representatives $y_\beta \in N$ in $\hat{y}_\beta$. To each y_β we associate in one-to-one fashion some sequence $\{x_\alpha^n\}_{n=1}^{\infty}$. If we further carry out exactly the same construction as in Theorem 1, we can construct a bounded M-basis in the space E. The theorem is proved.

COROLLARY 3. In any reflexive Banach space there exists a bounded M-basis.

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