Finally, 

\[ \sum_{n=0}^{\infty} \Delta R_n = O(1) \sum_{n=0}^{\infty} \left| \frac{P_n}{P_{n+1}} \right| \frac{1}{k+1} - \frac{1}{k+2} = O(1). \]

Consequently, the condition for the \([WN, P_{n+1}]\)-method is satisfied. It remains to apply Theorem A.

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**M-BASES IN SEPARABLE AND REFLEXIVE BANACH SPACES**

A. N. Plichko

Let \( E \) be a normed linear space, \( E' \) its dual. A set \( \{e_\alpha, f_\alpha\}_\alpha \) is called a total biorthogonal system if 

\[ f_\alpha(e_\beta) = \delta_\alpha \beta \quad \text{and} \quad \{f_\alpha\} \text{ is total in } E'. \]

If, in addition, \( \{e_\alpha\} \) is total in \( E \), \( \{e_\alpha, f_\alpha\} \) is called an M-basis (Markushchevich basis). We say that an M-basis is bounded if 

\[ \sup_\alpha \|f_\alpha\| \|e_\alpha\| < \infty, \]

\( \varepsilon \)-normalized if 

\[ \sup_\alpha \|f_\alpha\| \|e_\alpha\| \leq 1 + \varepsilon, \]

and normed if \( \varepsilon = 0 \).

The following two problems are posed in [1, p. 205]: 1) does there exist a normed M-basis, in every separable Banach space, and 2) does there exist a bounded M-basis in such a space.

We prove a proposition which is weaker than the first problem, but stronger than the second, along with other closely related assertions. In this paper we use the following notation: \([M]\) is the closure of the linear span of the set \( M \), \( M^\perp = \{f \in E': f(M) = 0\} \) if \( M \subseteq E \) and \( M^\perp = \{x \in E : x(M) = 0\} \) if \( M \subseteq E' \); \( d(M, N) \) is the distance between the sets \( M \) and \( N \).

**Separable Spaces.** The following statement was announced in [2].

**THEOREM 1.** For every separable normed linear space \( E \) and \( \varepsilon > 0 \), there exists an \( \varepsilon \)-normalized M-basis.

**Proof.** Consider \( \{n\} \) a strictly increasing sequence of natural numbers and write \( P_i = \{n; \sum_{s=1}^{i} n_s < n \leq \sum_{s=1}^{i+1} n_s\} \). On the basis of Lemma 1 of [3], we choose a biorthogonal system \( \{x_n, g_n\} \) in which 

\[ \|x_n\| = \|g_n\| = 1 \text{ and} \]

a) \( d(x, \{x_n : n \in P_s, s \geq i\}) > \|x\|/2 \) for \( x \in \{x_n : n \in P_s, s < i\} \), and hence \( d(x, \{x_n : n \notin P_i\}) > \|x\|/8 \) for \( x \in \{x_n : n \in P_i\} \);

b) \( \frac{1}{2} \|x\|_2 \leq \|x\| \leq \frac{3}{2} \|x\|_2 \), where \( x = \sum_{n \in P_i} \alpha_n x_n \), \( \|x\|_2 = \sqrt{\sum_{n \in P_i} (\alpha_n)^2} \);

c) \( \{x_n\}^\infty + \{g_n\}^\perp = E, \{x_n\}^\infty \cap \{g_n\}^\perp = 0 \).
According to Theorem 3 in [4], \(\{x_n, g_n\}\) can be extended to an M-basis using sets \(\{y_j\} \subset \{g_n\}^\perp\) and \(\{h_j\} \subset \{(x_n)\}^\perp\) with \(\|y_j\| = 1\).

Let \(d(y_j, h_j) = a_j\). We choose a double sequence \(\{n_j^k\}, \{n_j^k\} \subset \{n_i\}\) in such a way that for all \(k\) and \(j\)

\[
\frac{6}{\sqrt{n_j^k}} \leq \varepsilon; \quad \frac{(n_j^k - 1)\varepsilon}{8} - \frac{2\varepsilon}{\varepsilon} > \begin{cases} 
\frac{n_j^{k-1}}{a_j} & \text{for } k \neq 1, \\
\sqrt{\frac{n_j^1}{a_j}} & \text{for } k = 1.
\end{cases}
\]

(1)

We denote by \(P_j^k\) that \(P_j\) for which \(n_j^k = n_i\). We put

\[
e_n = \begin{cases} 
x_n & \text{for } n \notin \bigcup_{j=0}^k P_j^k, \\
x_n + \sqrt{n_j^1} y_j & \text{for } n \notin P_j^k, \\
x_n + \left( \sum_{s \in P_j^k} x_s \right) / n_j^{k-1} & \text{for } n \in P_j^k, \quad k \neq 1,
\end{cases}
\]

(2)

and \(D_n = \{\{e_m\}_m \setminus \{e_n\}\}.\) Choose \(\{f_n\}\) such that \(f_m(e_n) = \delta_{mn}\) (it will be shown below that \(e_n \notin D_n\), and hence such a choice is possible).

### Totality of \(\{e_n\}\). Applying (2), condition b), and (1) in succession, we obtain

\[
\left\| \sum_{n=1}^{\infty} \left( \sum_{s \in P_j^k} e_n \right) n_j^1 + \frac{1}{\sqrt{n_j^1}} y_j \right\| \leq \frac{1}{n_j^1} \left( \sum_{n \in P_j^k} x_n \right) \left\| \leq \frac{3}{2} \sqrt{n_j^1} \rightarrow 0.
\]

Thus \(\{y_j\} \subset [e_n]_m\); by (2) and condition c), \(\{x_n\} \subset [e_n]_m^\infty\) and \(\{e_n\}_m^\infty = E\).

### Totality of \(\{f_n\}\). By the construction of \(\{f_n\}\) and the density of \(\{e_n\}\) in the space \(E\), the totality of \(\{f_n\}\) is equivalent to the equality \(\bigcap_m [e_n]_m = 0\).

But \(\bigcap_m [e_n]_m = ((e_n) \bigcup \{h_j\})^\perp = 0\) and hence \(\{f_n\}_m^\perp = 0\).

### Inequality \(\sup_n \|f_n\| \leq 1 + \varepsilon\). We first show that

\[
d(e_n, D_n) \geq 1 - \varepsilon/2.
\]

(3)

Choose any element \(z\) in the linear span of the set \(\{e_m\}_m \setminus \{e_n\}\):

\[
z = \sum_{e \in P_j^k} z_e, \quad z_e \in [e_s] \setminus [e_n].
\]

(4)

If \(n \notin \bigcup_{j=0}^k P_j^k\), then (3) follows at once from the construction; assume that \(n \in P_j^k\). By (2), condition b), and (1), \(\|e_n - z\| \geq \|x_n - z\| - \varepsilon / 4\). Assume that

\[
\|x_n - z\| < 1 - \varepsilon/4.
\]

(5)

Then by (2) there exists a term \(z_j^t = \sum_{e \in P_j^t} \alpha_e e_s\) in the sum (4) for which

\[
b_j^t = \left( \sum_{e \in P_j^t} \alpha_e \right) / n_j^t > \varepsilon/4.
\]

(6)

1. We show that there exists a term \(z_j^t = \sum_{e \in P_j^t} \alpha_e e_s\) in sum (4) for which

\[
\left| \sum_{s \in P_j^t} \alpha_e \right| \geq \begin{cases} 
n_j^{t-1} & \text{for } k \neq 1, \\
\sqrt{n_j^t/a_j} & \text{for } k = 1.
\end{cases}
\]

(7)
Indeed, applying (5), condition a), and condition b) successively, we get:

\[
1 - \varepsilon/4 > \| x_n - z \| = \left\| x_n - \sum_{i \in \mathcal{P}_i} (b_i + \alpha_i) x_i \right\|
\]

\[
\left( z - b_i^{k+1} \sum_{i \in \mathcal{P}_i} x_i - \sum_{i \in \mathcal{P}_i} \alpha_i x_i \right) \geq \frac{1}{8} \left\| x_n - \sum_{i \in \mathcal{P}_i} (b_i + \alpha_i) x_i \right\| \geq \frac{1}{16} \left\| x_n - \sum_{i \in \mathcal{P}_i} (b_i + \alpha_i) x_i \right\| \geq \frac{1}{16} \sqrt{\sum_{i \in \mathcal{P}_i} (b_i + \alpha_i)^2}.
\]

Hence \((1 - \varepsilon/4)^2 > 2^{-8} \sum_{i \in \mathcal{P}_i} (b_i + \alpha_i)^2\). Making some algebraic transformations, we have

\[
-2b_i^{k+1} \sum_{i \in \mathcal{P}_i} \alpha_i \geq (n_i - 1)(b_i)^2 - 2\varepsilon (1 - \varepsilon/4)^2.
\]

Dividing the last inequality by \(2|b_i^{k+1}|\) and using (6), we obtain

\[
\left| \sum_{i \in \mathcal{P}_i} \alpha_i \right| \geq \frac{n_i^{k-1}}{\sqrt{n_i}/a_i} \quad \text{for } k \neq 1,
\]

\[
\left| \sum_{i \in \mathcal{P}_i} \alpha_i \right| \geq \frac{n_i^{k-2}}{\sqrt{n_i}/a_i} \quad \text{for } k = 2.
\]

It follows from the choice of \(z\) that \(a_n = 0\) and hence (7) is proved. For \(k = 1\) we go to Part 2 below, for \(k = 1\), to Part 3 below.

2. Thus,

\[
| b_i^{k-1} | \geq 1,
\]

where \(b_i^{k-1} = \left( \sum_{i \in \mathcal{P}_i} \alpha_i \right) / n_i^{k-1} \); put \(u_i^{k-1} = b_i^{k-1} \sum_{i \in \mathcal{P}_i} x_i \). We use once more arguments similar to those of 1. We show that there exists a term \(z_i^{k-1} = \sum_{i \in \mathcal{P}_i} \alpha_i y_i\) in the sum (4) for which

\[
\left| \sum_{i \in \mathcal{P}_i} \alpha_i \right| \geq \frac{n_i^{k-2}}{\sqrt{n_i}/a_i} \quad \text{for } k \neq 2,
\]

\[
\left| \sum_{i \in \mathcal{P}_i} \alpha_i \right| \geq \frac{n_i^{k-1}}{\sqrt{n_i}/a_i} \quad \text{for } k = 2.
\]

Indeed, applying (5) and conditions a) and b) in succession, we obtain

\[
1 - \varepsilon/4 > \| x_n - z \| = \left\| x_n - u_i^{k-1} - \sum_{i \in \mathcal{P}_i} \alpha_i x_i - \left( z_i^{k-1} - \sum_{i \in \mathcal{P}_i} \alpha_i x_i \right) \right\|
\]

\[
\geq \frac{1}{8} \left\| u_i^{k-1} + \sum_{i \in \mathcal{P}_i} \alpha_i x_i \right\| \geq \frac{1}{16} \left\| u_i^{k-1} + \sum_{i \in \mathcal{P}_i} \alpha_i x_i \right\| = \frac{1}{16} \sqrt{\sum_{i \in \mathcal{P}_i} (b_i^{k-1} + \alpha_i)^2}.
\]

Hence \((1 - \varepsilon/4)^2 > 2^{-8} \sum_{i \in \mathcal{P}_i} (b_i^{k-1} + \alpha_i)^2\). Making some algebraic transformations, we have

\[
-2b_i^{k-1} \sum_{i \in \mathcal{P}_i} \alpha_i \geq n_i^{k-1} (b_i^{k-1})^2 - 2\varepsilon (1 - \varepsilon/4)^2.
\]

Dividing this inequality by \(2|b_i^{k-1}|\) and bearing in mind (9) and (1), we get (10). For \(k = 2\), we go back to the beginning of Part 2, replacing \(k\) by \(k - 1\); for \(k = 2\), we go to Part 3 below.

3. After finitely many steps, we come to the conclusion that sum (4) contains a term \(z_i^{k} = \sum_{i \in \mathcal{P}_i} \alpha_i x_i + y\),

\[
y = \left( \sum_{i \in \mathcal{P}_i} \alpha_i \right) y_i / n_i^{1/2} \quad \text{and} \quad \left| \sum_{i \in \mathcal{P}_i} \alpha_i \right| \geq n_i^{1/2} / a_i.\]

Since \(x_n - (z - y) \in h^{1/2}\), we have \(\| x_n - (z - y) - y \| \geq \alpha_j \| y \| \geq 1\), and this contradicts (5), so that (3) holds. As is easily verified, \(\| f_n \| \leq 1 / d(e_n, D_n)\), and therefore

\[
\| f_n \| / e_n \| \leq (1 + \varepsilon/4) / d(e_n, D_n) \leq (1 + \varepsilon/2) / (1 - \varepsilon/2) \leq 1 + \varepsilon.
\]
The last inequality holds for \( \varepsilon < 1/2 \), which causes no loss of generality. The theorem is proved.

**COROLLARY 1.** In any separable normed linear space \( E \) we can for every \( \varepsilon > 0 \) introduce a norm \( \|x\| \leq \|x\| \leq (1 + 2\varepsilon)\|x\| \) such that the space \((E, \|\cdot\|)\) possesses a normed M-basis.

Indeed, let \( \{e_n, f_n\} \) be the system constructed in Theorem 1; then as the norm \( \|\cdot\| \) we can take the gauge function of the set \( \{x \in E : \|x\| \leq 1, f_n(x) \leq f_n(e_n), n = 1, \infty\} \).

**COROLLARY 2.** Let \( l_1 \subset c_0 \) be the natural imbedding of the space of absolutely summable sequences in the space of sequences converging to zero. Every separable Banach space \( E \) is \( \varepsilon \)-isometric to a space \( E_1 \) intermediate between \( l_1 \) and \( c_0 \) (i.e., \( l_1 \subset E_1 \subset c_0 \), both imbeddings being dense and \( \|x\|_{E_1} \leq \|x\|_{l_1} \) for \( x \in l_1 \)).

WCG-Spaces. A Banach space \( E \) is said to be a WCG-space (weakly compactly generated) if it is generated by a set \( U, [U] = E \) which is compact in the weak topology \( \sigma(E, E') \). In particular, separable and reflexive spaces are WCG-spaces.

**THEOREM 2.** In any WCG-space there exists a bounded M-basis.

Proof. We denote by \( \text{dens} \ E \) the smallest cardinality of the everywhere dense subsets of \( E \), and let \( \alpha_0 \) be the first ordinal number with the same power as \( \text{dens} \ E \).

Let \( \{n_i\} \) be a strictly increasing sequence of natural numbers, and let the \( P_i \) be the same as in Theorem 1. It follows from [3] that in any WCG-space we can choose a bounded biorthogonal system \( \{x_{n_i}^{P_i}, f_{n_i}^{P_i}\} \leq \alpha_0 < \alpha_0 \) such that for all \( \alpha \) and \( P_i \):

a) \( d(x, [x_{n_i}^{P_i} : n_i \leq P_i \wedge \beta_1 = \alpha]) > \|x\|/8 \) for \( x \in [x_{n_i}^{P_i} : n_i \leq P_i \wedge \beta_2 = \alpha] \);

b) \( c \|x\|_2 \leq \|x\| \leq C \|x\|_2 \), where \( x = \sum_{\alpha \in P} a^\alpha_{n_i} x^\alpha_{n_i} \), \( \|x\|_2 = \sum_{\alpha \in P} (a^\alpha_{n_i})^2, 0 < c \leq C < \infty \);

c) \( M \cup N = 0, [M + N] = E, \) where \( M = \{x_{n_i}^{P_i} : n_i \in N\}, N = \{f_{n_i}^{P_i} : n_i \} \).

Consider the quotient space \( E/M \). The image \( \hat{N} \) of the subspace \( N \) under the canonical mapping \( K: E \to E/M \) is an everywhere dense subspace of \( E/M \). Carrying out the same arguments as in [5], it can be shown that there exists an M-basis \( \{\hat{y}_{\beta}, g_{\beta}\} \leq \alpha_0 \), \( \hat{y}_{\beta} \in \hat{N} \) in \( E/M \). Choose representatives \( y_{\beta} \in N \) in \( \hat{y}_{\beta} \). To each \( y_{\beta} \) we associate in one-to-one fashion some sequence \( \{x_{n_i}^{P_i}\} \). If we further carry out exactly the same construction as in Theorem 1, we can construct a bounded M-basis in the space \( E \). The theorem is proved.

**COROLLARY 3.** In any reflexive Banach space there exists a bounded M-basis.

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