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M. I. Ostrovskii, A. M. Plichko

ON THE UKRAINIAN TRANSLATION OF "THÉORIE DES OPÉRATIONS LINÉAIRES" AND MAZUR'S UPDATES OF THE "REMARKS" SECTION

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The paper contains comments to the Ukrainian translation of the classical S. Banach's book "Théorie des opérations linéaires" published in 1948 and to S. Mazur's updates for the "Remarks" section of this translation.

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Статья содержит комментарии к украинскому переводу классической книги С. Банаха 'Théorie des opérations linéaires", опубликованной в 1948 году, и обновлениям С. Мазура его раздела "Замечаний" к этому переводу.

In 1939, after the beginning of the World War II, Banach's school in Lwów (this is the Polish name of the city; it is Ukrainian name is Lviv and, during the Soviet times, the Russian name Lvov was widely used) was in the Soviet Union in the period September 1939 – June 1941. At that time S. Banach started an active interaction with Soviet mathematicians. In particular, during these years a Ukrainian translation of his monograph *Théorie des opérations linéaires* [1] was planned and prepared.

One of the unusual features of this translation [2] is that it is entitled *Курс функціонального аналізу* (*лінійні операції*) (*A course of functional analysis* (*linear operations*)) and its title-page contains the statement "Approved by the Ukrainian Ministry of Higher Education as a College Textbook". We think that the reason for this change was the fact that at those times it was necessary to get an approval for publication of any book from Soviet authorities and it was very difficult to get an approval for a Ukrainian translation of a research monograph.

Another important feature of the translation is that S. Mazur updated the "Remarks" section for this translation.

The Ukrainian translation played an important role in spreading Banach's ideas in the Soviet Union. As an example of this influence we would like to mention that a famous expert of the Banach space theory, Mikhail I. Kadets, was attracted to this area just because he bought a copy of [2], which was among those few mathematical books which were available in Ukrainian bookstores at that time.

We decided to mark the 60th anniversary (in 2008) of the publication of the Ukrainian translation [2] with a publication of an English translation of Mazur's updates. We believe

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that these updates are of interest to everyone interested in the history of functional analysis. An additional reason for this publication is the fact that the "Remarks" section of the English translation of Banach's book [3] contains only minor additions to the corresponding section of the French edition, so it does not reflect the updates which were published almost 40 years earlier. More information on the history of the Ukrainian translation of [1] can be found in [7].

The purpose of our publication is to describe the difference between the updates in [2] and the "Remarks" section of [1], rather than to present a literal translation of the "Remarks" section from [2]. We also correct misprints and imprecise references. Our translation is expected to be read with the French version at hand, but for reader's convenience we add our English translations of some parts of the French edition, short explanations and comments. Some additional information on topics touched upon in this note can be found in the paper [4]. Note, however, that not all of the remarks in [4] are correct.

We have no official documents which confirm that the updates belong to S. Mazur. Our belief that this is the case for at least most of them is based on the following: (1) Many of the updates are devoted to systematic presentation of S. Mazur's results obtained after the publication of [2]; (2) At the end of the introduction to [1] S. Banach wrote that he thanks S. Mazur for "his help in editing the remarks"; (3) G. Köthe [4, p. 489] writes: He [S. Mazur] wrote the "Remarques" at the end of [1]. There is only one piece of evidence showing that S. Banach also participated in writing, or at least, editing the updates, see our footnote to the translation of updates to Chapter X.

The library of the Lviv University has a copy of [1] with handwritten notes containing updates in the "Remarks" section, probably inserted by Mazur. However, the whole original handwritten text of the updates is apparently lost.

We believe that the updates were prepared in 1940. This opinion is supported by the following facts. The agreement between the publisher and the translator (M. Zarycki) was signed in 1940, and the translation has been completed in 1940. The bibliography of the updates does not contain any sources published after 1940. The updates contain information on new unpublished results of S. Mazur, but do not mention his related results obtained after 1940, for example, his results with W. Orlicz on ordered (B_0) spaces obtained in 1942–43 (and published in 1953).

We also would like to mention an important difference between the usage of topological notions in [1] and in the updates to the "Remarks" section of [2]. In [1] topology is not used, only convergence of sequences. In Mazur's updates, topology plays an important role.

The publication of the Ukrainian translation [2] was delayed, it appeared only in 1948. It seems that on the final stages of its preparation, the translation was not carefully checked by any of the experts. A supporting evidence for this opinion is, for example, a footnote in [2, p. 181] (corresponds to [1, p. 213]). It refers to a non-existent paper in volume **IV** of *Studia Mathematica* (volume **IV** was published in 1933). To the best of our knowledge Banach's proof of the statement for which the reference is given contained a gap, and the first correct proof is due to O. C. McGehee [5]. See [6] for additional information on this matter.

In S. Mazur's updates our comments are printed in small letters in square brackets. Our English translations of excerpts from the French edition, included for reader's convenience, are printed in small letters in quotation marks.

Acknowledgement. We would like to express our gratitude to the referee whose criticism has improved the paper.

Translation and comments by M.I. Ostrovskii and A.M. Plichko

Glossary of terminology used in [8] and in this translation
Θ denotes zero in a topological group, vector space, etc.
$|x|$ denotes a quasinorm or a seminorm, and $\|x\|$ denotes a norm of x

Types of spaces

space of type	(G)	is a	complete metric group
	(F)	$--$	complete linear metric space with a separately continuous multiplication, an invariant metric d and a quasinorm $\lvert x\rvert = d(x,\Theta)$
	(F^*)	$--$	linear metric space with a separately continuous multiplication and an invariant metric
	(F^*_ϑ)	$--$	topological vector space whose topology is determined by a set of quasinorms of cardinality $\aleph_\vartheta$
	(B)	$--$	Banach space
	(B_0)	$--$	Fréchet space
	(B^*_0)	$--$	metrizable locally convex space
	(B^*_ϑ)	$--$	locally convex space whose topology is determined by a set of seminorms of cardinality $\aleph_\vartheta$

'Linear operator' means 'continuous linear operator'
$\overline{E}$ means the dual space (now usually denoted by E^*)
'Regularly closed subspace Γ of a dual space $\overline{E}$' is what is now called a weakly closed subspace (that is, such that for every $f \notin \Gamma$ there is $x \in E$ for which $f(x) \neq 0$ and $x(\Gamma) \equiv 0$)
'Regular space' means 'reflexive space'
'Weak closure' is what is now called 'weak sequential closure'
'weak closure in the dual space' is what is now called 'weak* sequential closure'

CHAPTER I

§1. [The text below replaces the second part of the paragraph at the top of [8, p. 230] (starting "M. D. van Dantzig a montré . . ."). It explains the statement that in a metric group, where $d(x_n - x_m, \Theta) \to 0$ as $n, m \to \infty$ implies the convergence of (x_n), one can introduce an equivalent metric in which the group becomes a space of type (G). For details see comments in S. Kakutani, *Selected papers*, I, Birkhäuser, Boston e.a., 1986.]

This follows from the fact that in each metric group E there is a metric d, equivalent to the original one, such that $d(x, y) = d(x + z, y + z)$ for each $z \in E$ (i.e., invariant metric); see

S. Kakutani, *Über die Metrisation der topologischen Gruppen*, Proc. Imp. Acad. Jap. Tôkyô, **12**, 1936, 82–84 and S. Kakutani, *Über ein Metrisationsproblem*, Proc. Phys.-Math. Soc. Japan, III. Ser., **20**, 1938, 85–90.

If in a metric group E the conditions $x_n, y_n \in E$, $\lim x_n = \Theta$ imply $\lim(y_n + x_n - y_n) = \Theta$, then there is a metric d in E, equivalent to the original one, such that $d(x, y) = d(x + z, y + z) = d(z + x, z + y)$ for every $z \in E$ (i.e., right side invariant metric); see

D. van Dantzig, *Zur topologischen Algebra, I. Komplettierungtheorie*, Math. Ann., **107**, 1932, 587–626.

CHAPTER II

§2. ...S. Kakutani (*Two fixed-point theorems concerning bicompact convex sets*, Proc. Imp. Acad. Jap. Tôkyô, **14**, 1938, 242–245) deduces [the Hahn-Banach] Theorem 1 as a corollary of a general fixed point theorem.

R.P. Agnew and A.P. Morse (*Extensions of linear functionals, with applications to limits, integrals, measures and densities*, Ann. of Math., **39**, 1938, 20–30) give the following generalization of Theorem 1:

Suppose that we are given

1° a functional $p(x)$ on a vector space E, such that $p(x+y) \leq p(x)+p(y)$ and $p(tx) = tp(x)$ for $x, y \in E$, $t \geq 0$;

2° a homogeneous additive functional $f(x)$ on a subspace $G \subset E$ such that $f(x) \leq p(x)$ for $x \in G$;

3° a solvable group Φ of one-to-one linear transformations of the space E into itself which transform the subspace G into itself, such that $p(\varphi(x)) = p(x)$ for $\varphi \in \Phi$, $x \in E$, and $f(\varphi(x)) = f(x)$ for $\varphi \in \Phi$, $x \in G$.

Then there exists a homogeneous additive functional $F(x)$ defined on E such that $F(x) \leq p(x)$ for $x \in E$, $F(x) = f(x)$ for $x \in G$, and $F(\varphi(x)) = F(x)$ for $\varphi \in \Phi$, $x \in E$.

From here, as an immediate corollary, we obtain:

Theorem. *Let*

1° *$p(x)$ be a functional defined on a vector space E, such that $p(x+y) \leq p(x)+p(y)$ and $p(tx) = tp(x)$ for $x, y \in E$, $t \geq 0$;*

2° *Φ be a solvable group of one-to-one linear transformations of the space E into itself, for which Θ is a fixed point, and $p(\varphi(x)) = p(x)$ for $\varphi \in \Phi$, $x \in E$.*

Then there exists a homogeneous additive functional $F(x)$ defined on E, such that $F(x) \leq p(x)$ and $F(\varphi(x)) = F(x)$ for $\varphi \in \Phi$, $x \in E$.

§3. [In the text below the Agnew-Morse theorem is used to establish the existence of the generalized integral (item 1), measure (item 2), limit of a function (item 3), and limit of a sequence (item 4).]

The theorem of Agnew and Morse (see §2) allows, by a remark of these authors, to extend the conclusions of §3; thus:

- in item 1, condition 3) can be replaced by condition $\overline{3}$) $\int x(rs+s_0)ds = \int x(s)ds$ for $r \neq 0$;
- in item 2, condition 3) can be replaced by $\overline{3}$) $\mu(A) = \mu(B)$, if a set A is obtained from a set B by the transformation $\varphi(s) = rs + s_0$, where $r \neq 0$ and s_0 is arbitrary;
- in item 3, one can add to conditions 1)–4) condition 5) $\lim\limits_{s\to\infty} x(rs) = \lim\limits_{s\to\infty} x(s)$ for $r > 0$;
- in item 4, one can add the condition $\lim\limits_{n\to\infty} \xi_{r\cdot n} = \lim\limits_{n\to\infty} \xi_n$ for positive integers r.

[An English translation of an excerpt from [8, p. 231] ([2, p. 196]): "The theorem of item 4 immediately implies that one can assign to each subset S of positive integers $\mathbb{N}$ a measure $m(S)$ in such a way that 1) $m(S) > 0$; 2) $m(S_1 \cup S_2) = m(S_1) + m(S_2)$ if $S_1 \cap S_2 = \varnothing$; 3) $m(S+r) = m(S)$, $r \in \mathbb{N}$; and 4) $m(\mathbb{N}) = 1$"].

Using the remark of Agnew and Morse mentioned above, one can add to the conditions 1)–4) one more:

5) $m(S_1) = \frac{1}{r}m(S_2)$, if the set S_1 is obtained from the set S_2 by the transformation $\varphi(s) = rs$, where r is a positive integer.

[Now the Agnew-Morse theorem is a well-known and widely used result, see, e.g., P. Bandyopadhyay, A.K. Rou, *Uniqueness of invariant Hahn-Banach extensions*, Extracta Math., **22**, 2007, 93–114].

CHAPTER III

§1. ... S. Mazur observed that in each space E of type (F) the multiplication of scalars and elements is [jointly] continuous, i.e.

$$\text{if } \lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} h_n = h,\ \textit{where } x_n, x \in E,\ h_n, h \in \mathbb{R},\ \textit{then } \lim_{n\to\infty} h_n x_n = hx. \quad (1)$$

This observation is a corollary of the following theorem of S. Mazur and W. Orlicz (*Über Folgen linearer Operationen*, Studia Math., **4**, 1933, 152–157):

[**Uniform Boundedness Principle.**] *If a sequence of linear operations $\{U_n(x)\}$ defined on a space E of type (F), with values in a space E_1 of type (F), is bounded at each point, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|U_n(x)| \le \varepsilon \ \ (n = 1, 2, \dots) \ \ for \ \ |x| \le \delta,\ x \in E.$$

Here a sequence $\{z_n\}$ of elements in the space of type (F) is called *bounded*, if $\lim_{n\to\infty} t_n z_n = \Theta$ for every scalar sequence $\{t_n\}$ convergent to 0.[1]

Therefore a space E of type (F) is a complete linear metric space, in which operations (the addition of elements and the multiplication of scalars by elements) are [jointly] continuous, and

$$\forall x, y \in E d(x, y) = d(x - y, \Theta). \quad (2)$$

[From [8, p. 232]: "It is unknown whether each complete metric linear space E with continuous operations has an equivalent metric in which E becomes space of type (F)".]

S. Mazur (*O zbiorach i funkcjonałach wypukłych w przestrzeniach linjowych* (in Polish). Habilitation thesis, Lwów, 1936, 1–20) proved that every space of such type has an equivalent metric satisfying condition (2).[2]

In the above thesis, S. Mazur considers general topological linear spaces, i.e. linear and topological spaces, in which operations are continuous, and classifies them in the following way.

Let E be a linear space, and Φ be a class of cardinality $\aleph_\vartheta$ of functionals $d(x, y)$ defined for $x, y \in E$, such that

1) $d(x, y) = 0$ for all $d \in \Phi$ if and only if $x = y$;
2) $d(x, y) = d(y, x)$;
3) $d(x, z) \le d(x, y) + d(y, z)$;
4) $d(x, y) = d(x - y, \Theta)$;
5) if $\lim_{n\to\infty} d(x_n, x) = 0,\ \lim_{n\to\infty} h_n = h$ then $\lim_{n\to\infty} d(h_n x_n, hx) = 0$.

[1]For a space of type (F) this definition coincides with the classical definition of a bounded set. For detailed history of the definition of bounded sets see D. Przeworska-Rolewicz, S. Rolewicz, *Historical remarks on bounded sets.* European mathematics in the last centuries, Wrocław, 2005, 87–97. (*All footnotes in this note are due to translators.*)

[2]Without the guarantee of completeness. The Banach's problem was solved by V.L. Klee (*Invariant metrics in groups* (*solution of a problem of Banach*). Proc. Amer. Math. Soc., **3**, 1952, 484–487. For details see S. Rolewicz, *Metric linear spaces*, PWN, Warszawa, 1985.

Let us call the set of all points $x \in E$ satisfying $d_n(x, a) < \varepsilon$ $(n = 1, 2, \dots m)$, where $d_n \in \Phi$ and $\varepsilon > 0$ a *neighborhood* of $a \in E$. We obtaine a topological linear space, which is called a *space of type* (F^*_ϑ). The converse is also true:

Theorem. *Every topological linear space is isomorphic to some space of type* (F^*_ϑ).

[Mazur's thesis contains no proofs.]

We say that two topological linear spaces are *isomorphic*, if there exists an isomorphic transformation, i.e. simultaneously additive and homeomorphic one, of one space onto the other. If a class Φ satisfying conditions 1)–5), contains one functional only, then it is a metric on E, and the obtained metric linear space is called a *space of type* (F^*). This space is of type (F), if and only if it is complete. The following result holds (see J.V. Wehausen, *Transformations in linear topological spaces*, Duke Math. J., **4**, 1938, 157–169):

Theorem. *A topological linear space E is isomorphic to a space of type (F^*) if and only if in E the 1^{st} axiom of countability is satisfied.*

CHAPTER IV

§1. ... A topological linear space E is called *locally convex*, if each open set $A \subset E$ contains an open convex subset. S. Mazur, in his Habilitation Thesis, gives the following classification of locally convex topological linear spaces.

Let E be a linear space, and Φ be a class of functionals $\varphi(x)$, defined on E, such that Φ has cardinality $\aleph_\vartheta$ and the following properties hold:

1) $\varphi(x) = 0$ for all $\varphi \in \Phi$ if and only if $x = \Theta$;
2) $\varphi(x + y) \leq \varphi(x) + \varphi(y)$;
3) $\varphi(tx) = |t|\varphi(x)$.

Let $a \in E$. The set of all $x \in E$ such that $\varphi_n(x - a) < \varepsilon$ $(n = 1, 2, \dots m)$, where $\varphi_n \in \Phi$ and $\varepsilon > 0$ is called a *neighborhood* of a. We get a locally convex linear topological space. A space constructed in this way is called a *space of type* (B^*_ϑ). The following result holds.

Theorem. *Every locally convex linear topological space is isomorphic to a space of type* (B^*_ϑ).

This theorem was independently discovered by J. von Neumann (*On complete topological spaces*, Trans. Amer. Math. Soc., **37**, 1935, 1–20).

A. Kolmogoroff (*Zur Normierbarkeit eines allgemeinen topologischen linearen Raumes*, Studia Math., **5**, 1934, 29–33) proved

Theorem. *A linear topological space E is isomorphic to a normed linear space, if and only if E contains a bounded open convex set.*

A subset A of a linear topological space is called *bounded* if for every neighborhood V of Θ there exists $\varepsilon > 0$ such that $\varepsilon A \subset V$.

In a space E of type (B^*_0) [i.e. when $\vartheta = 0$] there exists a metric generating a topology equivalent to the original one, and the condition $d(x, y) = d(x - y, \Theta)$ is satisfied. A metric satisfying this condition can be obtained by letting $d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\varphi_n(x-y)}{1+\varphi_n(x-y)}$, where $\{\varphi_n\}_{n=1}^{\infty}$ are all functionals of the class Φ. If this metrics is complete, then E is called a *space of type* (B_0). Spaces of type (B^*_0) and, in particular, spaces of type (B_0), were an object of extensive study by S. Mazur and W. Orlicz.[3] They proved that many theorems about normed

[3] see their papers *Sur les espaces métriques linéaires,* **I**, **II**, Studia Math., **10**, 1948, 184–208; **13**, 1953, 137–179.

linear spaces and, in particular about spaces of type (B), are also valid for spaces of type (B_0^*) and (B_0), respectively.

[Mazur and Orlicz presented some of their results on spaces of type (B_0^*) to the Lwów Mathematical Society in 1933. In the "Remarks" to [2] Mazur mentioned their results on spaces of type (B_0) many times. Because of the war Mazur and Orlicz were able to publish these results only in 1948–53. Most of results of Mazur and Orlicz on (B_0) spaces mentioned in [2] were proved in the paper cited in our footnote; but not all. The paper of Mazur and Orlicz is independent on the well known paper by J. Diedonné and L. Schwartz (*La dualité dans les espaces* (F) *et* (LF), Ann. Inst. Fourier, **1**, 1950, 61–101).]

These authors [that is, Mazur and Orlicz] proved:

Theorem. *A space E of type (F) is isomorphic to a space of type (B_0) if and only if the following condition is satisfied:*

if $x_n \in E$, $x_n \to \Theta$ and $\sum_{n=1}^{\infty} |t_n| < \infty$, then the series $\sum_{n=1}^{\infty} t_n x_n$ converges.

A space E of type (F) is isomorphic to a space of type (B) if and only if there exists $r > 0$ such that the following condition is satisfied:

if $x_n \in E$, $|x_n| < r$ and $\sum_{n=1}^{\infty} |t_n| < \infty$, then the series $\sum_{n=1}^{\infty} t_n x_n$ converges.

The space[4] (s) is a space of type (B_0); the space[5] (S) is not isomorphic to a space of type (B_0).

Examples of spaces of type (B_0) which are not isomorphic to spaces of type (B) are:

1) The space (C^*) [$= C(\mathbb{R})$] of all continuous functions in $(-\infty, +\infty)$; $\varphi_n(x) = \max\{|x(t)|: -n \le t \le n\}$.
2) The space $(C^{(\infty-0)})$ of infinitely differentiable functions on $[0,1]$; $\varphi_n(x) = \max\{|x^{(n-1)}(t)|: 0 \le t \le 1\}$.
3) The space $L^{(p-0)}$, where $p > 1$, of functions on $[0,1]$ such that $\int_0^1 |x(t)|^q dt < \infty$ for every $q < p$; $\varphi_n(x) = \left[\int_0^1 |x(t)|^{q_n} dt\right]^{1/q_n}$, where $q_n < p$, $\lim_{n\to\infty} q_n = p$.
4) The space $l^{(p+0)}$, where $p \ge 1$, of sequences $\{\xi_k\}$ such that $\sum_{k=1}^{\infty} |\xi_k|^q < \infty$ for every $q > p$; $\varphi_n(x) = [\sum_{k=1}^{\infty} |\xi_k|^{q_n}]^{1/q_n}$, where $q_n > p$, $\lim_{n\to\infty} q_n = p$.

§2 and §3. [Form [8, p. 234]: "For a space E of type (F) one can establish the equivalence of the following two properties:

(α) Given a linear functional $f(x)$ defined on a linear subset $G \subset F$, there exists a linear functional $F(x)$, defined on the whole E, such that $F(x) = f(x)$ for every $x \in G$.

(β) Under the same conditions, if, moreover, G is closed then there exists, for every $x_0 \in E\backslash G$, a linear functional $F(x)$ on E such that $F(x_0) \neq 0$, but $F(x) = 0$ for all $x \in G$.

Not all spaces of type (F) have these properties. For example, any linear functionals on (S) must vanish identically." [8, pp. 215–216] contains a text, which is slightly similar to a part of the next remarks to §2 and §3.]

S. Mazur and W. Orlicz observed that all spaces of type (B_0^*) have properties (α) and (β). Theorem 3[6] was generalized in

S. Mazur, *Über convexe Mengen in linearen normierten Räumen*, Studia Math., **4**, 1933, 70–84.

[4] of all scalar sequences

[5] of all measurable functions on $[0,1]$

[6] on extension of a functional from an one dimensional subspace of a normed space with preservation of the norm

This paper contains a proof of the following result: for each convex body K in a normed linear space and for each point x of its boundary there is a support hyperplane to the convex body K containing x. For separable spaces this result was obtained by G. Ascoli (*Sugli spazi lineari metrici e le loro varieta lineari*, Ann. Mat. Pura Appl., IV. Ser. **10**, 1932, 33–81).

[Ascoli and Mazur were not the first in applying the geometrical ideas of Minkowski to infinite dimensional normed spaces. E. Helly did this already in his papers published in 1912 and 1921. The first paper is cited in [8, p. 56] ([2, p. 47]).]

Let E be a linear topological space, $f(x)$ be a non-zero linear functional in E, and α be a scalar. The set $H = \{x \in E : f(x) - \alpha = 0\}$, is called a *hyperplane.* A set $A \subset E$ *lies on one side* of the hyperplane H, if the function $f(x) - \alpha$ does not change its sign on A. A set $W \subset E$ is called a *convex body* if it is convex, closed, and contains interior points. A hyperplane H is a *support hyperplane* of a convex body W, if H contains points of W and W lies on one side of H. So the mentioned above result can be stated as:

Theorem. *Suppose that w belongs to the boundary of a convex body W. Then w is contained in at least one support hyperplane H of W.*

This theorem is valid for an arbitrary normed linear space E. For a *separable*[7] E, the above result can be complemented by the following

Theorem. *The set of boundary points of a convex body W which have no tangent to W hyperplane, that is, points which are contained in at least two support hyperplanes of W, is a subset of the first category in the boundary of W.*

S. Mazur and W. Orlicz observed that these theorems are valid for all spaces of type (B_0^*).

Let us mention here the following result of M. Eidelheit (*Zur Theorie der konvexen Mengen in liearen normierten Räumen*, Studia Math., **6**, 1936, 104–111; see also S. Kakutani, *Ein Beweis des Satzen von M. Eidelheit über konvexe Mengen*, Proc. Imp. Acad. Jap. Tôkyô, **13**, 1937, 93–94):

Theorem. *If convex bodies W_1 and W_2 have no common interior points then there exists a hyperplane H which separates W_1 and W_2.*

Let f be a non-zero functional. We say that a hyperplane $H = \{x : f(x) = \alpha\}$ *separates* two convex bodies if $f(x) - \alpha \leq 0$ for all x in one of the convex bodies and $f(x) - \alpha \geq 0$ for all x in the other convex body.

[In [8, p. 234] the final remark to §2 and 3 is the statement: some operators between subspaces of Banach spaces do not admit norm preserving extensions. This statement is not accompanied by any references. In [2, p. 201] the remarks to §2 and 3 are substantially longer.]

Some operators between subspaces of Banach spaces do not admit norm preserving extensions (see S. Banach and S. Mazur, *Zur Theorie der linearen Dimension*, Studia Math., **4**, 1933, 100–112). In recent years F. Riesz, L.V. Kantorovich, H. Freudenthal, M. G. Krein and others develop the theory of, so-called, linear semi-ordered spaces. In particular, L. V. Kantorovich proved that many theorems about linear functionals on linear normed spaces (in particular, the extension theorem) are valid for linear operations on linear normed spaces with values in a linear semi-ordered regular space.

Following L. V. Kantorovich we say that a set E is a *linear semi-ordered space*, if it is a linear space with an order relation $x > y$ such that the following conditions are satisfied (w, x, y, z are elements of E, t is a scalar):

[7] complete

1) if $x > y$, then $x \neq y$;
2) if $x > y$ and $z > w$, then $x + z > y + w$;
3) for any two elements $x, y \in E$ there exists an element z such that $z \geq x$ and $z \geq y$;
4) if $x > y$ and $t > 0$, then $tx > ty$;
5) for every set $A \subset E$ there exists the upper bound $\sup A$.

The notions of an upper bound $\sup A$ of a set $A \subset E$ and of a lower bound $\inf A$ are defined in the same way as for real numbers. We say that $x \in E$ or $x = \pm\infty$ is a *limit* of a sequence $\{x_n\} \subset E$ and write $\lim_{n\to\infty} x_n = x$, if $\inf(\sup(x_n, x_{n+1}, \dots)) = \sup(\inf(x_n, x_{n+1}, \dots)) = x$. A linear semi-ordered space E is called *regular* if it satisfies the condition: if $A_n \subset E$ and $\lim_{n\to\infty}(\sup A_n) = x$, then there exist finite sets $B_n \subset A_n$ such that $\lim_{n\to\infty}(\sup B_n) = x$.

The following are examples of linear semi-ordered regular spaces:

1. The space of all measurable functions on $[0,1]$, with $x > y$ meaning that $x(t) \geq y(t)$ almost everywhere and $x(t) > y(t)$ on a set of positive measure; $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} x_n(t) = x(t)$ almost everywhere.
2. The space of scalar sequences; with $x > y$ meaning that $\xi_k \geq \eta_k$ for all k and $\xi_k > \eta_k$ for some k; $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} \xi_{nk} = \xi_k$ for all k.
3. The space of functions on $[0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, where $p \geq 1$; the order relation is defined as in Example 1; $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} x_n(t) = x(t)$ almost everywhere and $|x_n(t)| \leq y(t)$ for all $n = 1, 2, \dots$ and some $y \in L^{(p)}$.
4. The space of scalar sequences such that $\sum_{k=1}^\infty |\xi_k|^p < \infty$, where $p \geq 1$: the order relation is defined as in Example 2; $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} \xi_{nk} = \xi_k$ and $|\xi_{nk}| \leq \eta_k$, $n = 1, 2, \dots$ for some $y = \{\eta_k\} \in l^{(p)}$ and all k.

We restrict ourselves by these remarks and refer the reader to the paper L. V. Kantorovitch (*Lineare halbgeordnete Räume*, Matem. Sbornik, New Ser. **2**, 1937, 121–168), and to the bibliography therein.

§4. ...T. H. Hildebrandt and I. J. Schoenberg (*On linear functional operations and the moment problem for a finite interval in one or several dimensions*, Ann. of Math., **34**, 1933, 317–328) gave a simple proof of the theorem on the form of a linear functional in the space (C) [$= C[0,1]$] using Bernstein polynomials.

Out of numerous known statements concerning the general form of linear functionals in spaces essential for applications we would like to mention here some statements due to G. Fichtenholz and L. Kantorovitch (*Sur les opérations linéaires dans l'espace des fonctions bornées*, Studia Math., **5**, 1934, 69–98):

The general form of a linear functional in the space (M) [$= L_\infty[0,1]$] is given by the formula $f(x) = \int_I x(t)\Phi(dt)$, where $\Phi(Z)$ is a bounded additive function defined on the class of all measurable subsets of the segment $I = [0,1]$ such that $\Phi(Z) = 0$ for sets Z of measure zero, and the integral is taken in the Radon sense. Therefore the set of all linear functionals in the space (M) has the cardinality $2^{2^{\aleph_0}}$.

In the space of all bounded on I functions [$= \ell_\infty(I)$], with the norm $\|x\| = \sup\{|x(t)| : 0 \leq t \leq 1\}$, the general form of a linear functional is the same, but the function $\Phi(Z)$ is defined on the class of all subsets of I. The set of all linear functionals in this space has the cardinality $2^{2^{\aleph_0}}$....

§8. ...Numerous statements concerning existence of solutions for systems of linear equations with infinitely many unknowns are given by M. Eidelheit, (*Über lineare Gleichungen in* *separablen Räumen*, Studia Math., **6**, 1936, 117–138 and **8**, 1939, 154–169; *Zur Theorie der Systeme linearer Gleichungen*, Studia Math., **6**, 1936, 139–148 and **7**, 1938, 150–154).

In this connection it is natural to recall the interesting studies in this direction due to G. Köthe and O. Toeplitz, *Lineare Räume mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen*, J. Reine Angew. Math., **171**, 1934, 193–226; G. Köthe, *Lösbarkeitsbedingungen für Gleichungen mit unendlich vielen Unbekannten*, J. Reine Angew. Math., **178**, 1938, 193–213.

CHAPTER V

§1 and §2. [The references to the following papers are added.] S. Mazur and L. Sternbach, *Über Konvergenzmengen von Folgen linearer Operationen*, Studia Math., **4**, 1933, 54–65; S. Banach und S. Mazur, *Eine Bemerkung über die Konvergenzmengen von Folgen linearer Operationen*, Studia Math., **4**, 1933, 90–94; S. Mazur und L. Sternbach, *Über die Borelischen Typen von linearen Mengen*, Studia Math., **4**, 1933. 48–53...

... Each infinite dimensional space of type (B) contains a linear subspace of a prescribed Borel class[8]. It is unknown whether there are spaces of type (F) which contain linear sets (A) which are not Borel, but it is known that in every infinite dimensional space of type (B) there are linear projective sets, (some of them are complements of the sets (A)) which are not Borel; see S. Banach and C. Kuratowski, *Sur la structure des ensembles linéaires*, Studia Math., **4**, 1933, 95–99.[9]

"Each infinite dimensional space of type (F) contains linear sets which do not satisfy the Baire condition.

...If E and E_1 are spaces of type (F), every additive B-measurable operation $U(x)$ defined on a closed linear subspace $G \subset E$ and whose codomain lies in E_1 is continuous by Theorem 4 ([8, p. 23]). However, if the set G is not closed, the operation U may be discontinuous." The next sentence is omitted in the Ukrainian edition: "We know examples for with G is B-measurable, the operation U is discontinuous of the first Baire class; however, we do not know any examples in which it is of a higher Baire class. Similarly, it is not known if the operator U can satisfy the Baire condition without being B-measurable." For example, one can prove that for each infinite dimensional space E of type (B) there exists an additive functional of a prescribed Baire class defined on some linear set $G \subset E$. Generalizations of theorems of §2 and §3 to spaces of type (F) were obtained in the paper of S. Mazur and W. Orlicz cited above (Studia Math., **4**, 1933, 152–157). ...

§3. ... "A space of type (F) is finite dimensional if some/each ball in it is compact", see M. Eidelheit und S. Mazur, *Eine Bemerkung über die Räume vom Typus* (F), Studia Math., **7**, 1938, 159–161. ...

§7. ...[The text below replaces paragraphs 2–4 of remarks to §7 in [8], starting "On ne sait pas ...".] A series of results, related to the theorems of this section, was obtained by S. Mazur and W. Orlicz (*Sur les méthodes linéaires de sommation*, C. R. Acad. Sci., Paris, **196**, 1933, 32–34). They proved, among other things, the following generalization of Theorem 11[10]:

Theorem. *If every bounded sequence, summable by a permanent method A, is summable by a permanent method B, then every bounded sequence, summable by the permanent method*

[8]See V.L. Klee, *On the Borel and projective types of linear subspaces*, Math. Scand., **6**, 1958, 189–199. Also, for each n, each infinite dimensional Banach space contains a subspace whose projective class is exactly n.

[9]Probably, (A) here means 'analytic', although Banach and Kuratowski do not use this notation. Each infinite dimensional space of type (F) contains a linear subspace of projective class P_{2n-1} which is not in P_k, $1 \le k \le 2n-1$ (C. Bessaga and A. Pełczyński, *Selected topics in infinite-dimensional topology*, Monografje Matematyczne, Warszawa, 1975, p. 267–271).

[10]Its statement: if a permanent method B is not weaker than a permanent reversible method A, then each bounded sequence summable by A is summable by B to the same limit.

A, is summable by the permanent method B to the same limit.

Given a method A (corresponding to $\{a_{ik}\}_{i,k=1}^{\infty}$), we denote by A^* the set of all sequences summable by A. We say that a sequence $x_0 = \{\xi_k^0\}$, summable by the method A, has *property* (P) if for every $\varepsilon > 0$ and each positive integer p there exists a convergent sequence $x = \{\xi_k\}$ such that:

$$\left|\sum_{k=1}^{\infty} a_{i,k}(\xi_k^0 - \xi_k)\right| < \varepsilon \ \ (i = 1, 2, \dots), \quad |\xi_k^0 - \xi_k| < \varepsilon \ \ (k = 1, 2, \dots, p),$$

$$\left|\sum_{k=1}^{n} a_{ik}(\xi_k^0 - \xi_k)\right| < \varepsilon \ \ (i = 1, 2, \dots, p;\ n = 1, 2, \dots).$$

Theorem. *Let A be a permanent method and let $x_0 \in A^*$.*

If x_0 has property (P), then $B(x_0) = A(x_0)$ for every permanent method B, which is not weaker than A.

If x_0 has not property (P), then, for every scalar α, there is a permanent method B such that $B^* = A^*$ and $B(x_0) = \alpha$.

If a permanent method A is row-finite, i.e. $a_{ik} = 0$ for $k > k_i$, then a sequence x_0, summable by method A, has property (P) if and only if the conditions

$$\sum_{i=1}^{\infty} \vartheta_i a_{ik} = 0 \ \ (k = p, p+1, \dots) \text{ imply } \sum_{i=1}^{\infty} \vartheta_i (a_{ip}\xi_p^0 + a_{ip+1}\xi_{p+1}^0 + \dots) = 0$$

for every absolutely convergent series $\sum_{i=1}^{\infty} \vartheta_i$ and every positive integer p.

CHAPTER VIII

§3. Given $x_0 \in E$ let $\varphi(f) = |f(x_0)|$ for $f \in \overline{E}$. The class Φ of functionals obtained in this way determines some topology in the space $\overline{E}$, with this topology $\overline{E}$ becomes a locally convex linear topological space. The following result holds:

Theorem. *A linear set $\Gamma \subset \overline{E}$ is regularly closed if and only if it is closed in the above-mentioned topology.*

This result is due to S. Mazur[11]; see also S. Kakutani, *Weak topology and regularity of Banach spaces*, Proc. Imp. Acad. Jap., Tôkyô, 15, 1939, 169–173.

According to S. Mazur and W. Orlicz this theorem and Lemma 3[12] are valid also for spaces of type (B_0).

[The notion of a transfinite closure is not widely known these days[13]. If ϑ is a limit ordinal and c_ξ, $1 \le \xi < \vartheta$, is a transfinite sequence of scalars, then its *transfinite upper limit* $\overline{\lim}_{\xi\to\vartheta} c_\xi$ is defined as $\inf\{t : c_\xi \le t$ starting from some $\xi_t < \vartheta\}$. The definition of the transfinite lower limit $\underline{\lim}_{\xi\to\vartheta} c_\xi$ is similar. S. Banach [8, Lemma 1, p. 118] used the Hahn-Banach theorem to prove

Lemma. *Let f_ξ, $1 \le \xi < \vartheta$ be a transfinite sequence of linear functionals and $\|f_\xi\| \le M$. Then there is a linear functional f, $\|f\| \le M$ such that*

$$\underline{\lim}_{\xi\to\vartheta} f_\xi(x) \le f(x) \le \overline{\lim}_{\xi\to\vartheta} f_\xi(x)$$

[11] no reference is provided.

[12] Its statement: A linear subspace of a dual Banach space is regularly closed if and only if it is transfinitely closed.

[13] However, recently A. Pietsch (*History of Banach spaces and linear operators*, Birkhäuser, Boston, 2007), Section 3.6, recalled this notion and some results proved using it.

for every $x \in E$.

Each functional f satisfying the inequalities of this lemma is called the *transfinite limit* of (f_ξ). A linear subspace $\Gamma \subset \overline{E}$ is called *transfinitely closed* if it contains all its transfinite limits.]

§4 and §5. ... According to S. Mazur and W. Orlicz, Theorem 5[14] is valid for the spaces of type (B_0).

V. Šmulian (*Sur les ensembles régulièrement fermés et faiblement compacts dans l'espace du type* (B), Doklady Akad. Nauk. USSR, (New Ser.), **18**, 1938, 405–407) proved

Theorem. *A linear separable set* $\Gamma \subset \overline{E}$ *is regularly closed if and only if the unit ball of* Γ *is weakly compact* (*in the sense of the weak convergence of functionals*)[1].

[1]The author's footnote: See [2, p. 112] for the definition of the weak compactness in the sense of the weak convergence of functionals. ...

CHAPTER IX

§1 ... S. Mazur and W. Orlicz observed that a space of type (F) is isomorphic to a space of type (B_0) if and only if each weakly convergent sequence in it is bounded. ...

... Theorem 2[15] was strengthened by S. Mazur[16] who proved the following:

Theorem. *If a sequence* $\{x_n\}$ *weakly converges to* x_0 *then for each* $\varepsilon > 0$ *there are scalars* c_n $(n = 1, 2, \ldots, m)$ *such that*

$$c_n \geq 0, \ \sum\nolimits_{n=1}^{m} c_n = 1 \ \ and \ \ \left\|\sum\nolimits_{n=1}^{m} c_n x_n - x_0\right\| < \varepsilon \ .$$

Moreover, each weakly convergent to x_0 sequence $\{x_n\}$ in the spaces $L^{(p)}$ and $l^{(p)}$, where $p > 1$, contains a subsequence $\{x_{n_k}\}$, such that $\lim_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} x_{n_k} = x_0$ [in the norm topology]; see S. Banach and S. Saks, *Sur la convergence forte dans les champs* L^p, Studia Math., **2**, 1930, 51–57. The same statement[17] is valid for every uniformly convex space of type (B); see S. Kakutani, *Weak convergence in uniformly convex spaces*, Tôhoku Math. J., **45**, 1938, 188–193.

A space E of type (B) is called *uniformly convex*, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that the conditions $x, y \in E$, $\|x\| = \|y\| = 1$, $\|x - y\| \geq \varepsilon$, imply $\frac{1}{2}\|x + y\| \leq 1 - \delta$; see J.A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc., **40**, 1936, 396–414.

For the space (C) the above mentioned theorem of S. Mazur was proved by D.C. Gillespie and W.A. Hurwitz, *On sequences of continuous functions having continuous limits*, Trans. Amer. Math. Soc., **32**, 1930, 527–543, and independently by Z. Zalcwasser, *Sur une propriété du champ des fonctions continues*, Studia Math., **2**, 1930, 63–67.[18] It should be mentioned that this [Mazur's] theorem is equivalent to the fact that each convex closed set is weakly closed. A set W is called *weakly closed*, if the weak limit of each sequence from W is contained in W.

[14]Its statement: The notions of regularly closed and weak* sequentially closed subspaces in the dual spaces of separable Banach spaces are equivalent.

[15]Its statement: If a sequence $\{x_n\}$ in a Banach space converges to x weakly then some of linear combinations of $\{x_n\}$ converge to x in the norm topology.

[16]*Über konvexe Mengen in linearen normierten Räumen*, Studia Math., **4**, 1933, 70–84.

[17]Now this property of a space is called the *Banach-Saks property*.

[18]In fact, Mazur notes (on p. 71 in his paper mentioned in footnote 16) that his theorem follows from the mentioned results of Gillespie–Hurwitz and Zalcwasser and the universality of $C[0, 1]$.

M. Krein (*Sur quelques questions de la géométrie des ensembles convexes situés dans un espace linéaire normé et complet*, Doklady Akad. Nauk USSR, (New Ser.), **14**, 1937, 5–7) proved

Theorem. *If a set $Z \subset E$ is separable, weakly compact and weakly closed, then the smallest convex closed set W containing the set Z has the same properties.*[19]

§3. V. Šmulian (*On some geometrical properties of the sphere in a space of type (B)*, Doklady Akad. Nauk USSR, (New Ser.), **24**, 1939, 648–652) proved

Theorem. *Let E be a uniformly convex space of type (B). If a sequence $\{x_n\}$ weakly converges to x_0 and $\lim_{n\to\infty} \|x_n\| = \|x_0\|$, then $\lim_{n\to\infty} x_n = x_0$.*[20]

J. A. Clarkson (Trans. Amer. Math. Soc., **40**, 1936, 396–414) proved that the spaces $L^{(p)}$ and $l^{(p)}$ are uniformly convex when $p > 1$. ...

§4. ... One can consider the following more general notion of weak completeness. Let A be a semi-ordered set such that for every two of its elements $a_1, a_2 \in A$ there exists an element $a_3 \in A$ which exceeds them in the order sense. Let $x(a)$ be a bounded function defined on the set A with values in the given space E, and finally, assume that for every functional $f \in \overline{E}$ there exists a limit $\lim_{a\in A} f(x(a))$ in the Moore sense. The space E is called *weakly complete* if for every set A and every function $x(a)$ satisfying the above conditions, there exists $x_0 \in E$ such that $\lim_{a\in A} f(x(a)) = f(x_0)$ for all $f \in \overline{E}$. Using this notion of the weak completeness, H.H. Goldstine (*Weakly complete Banach spaces*, Duke Math. J., **4**, 1938, 125–131), proved

Theorem. *A space is weakly complete if and only if it is regular.*

...

... [The following reference is added to the remarks on unconditional convergence] A.E. Taylor, *A note on unconditional convergence*, Studia Math., **8**, 1938, 148–153.

CHAPTER X

§1... S. Mazur and W. Orlicz generalized the theory of linear equations to spaces of type (B_0). The method used by these authors is different from the one developed by me[21]. The starting point is a theorem of S. Mazur on the existence of support hyperplanes (see p. 103 of this translation), which is also valid for spaces (B_0). Using this result they proved

Theorem. *Let E be a space of type (B_0).*

A convex Θ-symmetric set $W \subset E$ is norming if and only if it is everywhere dense in some neighborhood of Θ.

A bounded subset $R \subset E$ is norming if and only if each element $x \in E$ has the form $x = \sum_{n=1}^{\infty} \vartheta_n x_n$ for some $\{\vartheta\}_{n=1}^{\infty}$ satisfying $\sum_{n=1}^{\infty} |\vartheta_n| < +\infty$ and some $x_n \in R$.

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[19] In 1928, Mazur proved that the closed convex hull of a norm compact set is norm compact. The fact that the norm closed convex hull of a weakly compact set is weakly compact is now known as the *Krein-Šmulian theorem.*

[20] Now this property is known as the *property H of Kadets.*

[21] This is the only place among all updates of the "Remarks" section which indicates that some of them were written, or at least edited, by S. Banach.

We say that a set $R \subset E$ is *norming* if for every sequence of linear functionals $f_n(x)$ such that $|f_n(x)| \leq N$ $(n = 1, 2, \dots)$ for some constant N and all $x \in R$, there exists $\varepsilon > 0$ and a constant M such that $|f_n(x)| \leq M$ $(n = 1, 2, \dots)$ for $x \in E$, $|x| \leq \varepsilon$. See in this connection the proof of Theorem 1 of this chapter given by S. Mazur and W. Orlicz, it was published in the monograph S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, Monografje Matematyczne, Warszawa-Lwów, 1935, pp. 34–36.

[The mentioned theorem states that if a conjugate operator U^* is invertible then the equation $y = Ux$ has solution for every y. Let us emphasize that the notion of the norming set is introduced not in [8], but in the mentioned book of Kaczmarz–Steinhaus.]

§2. The theory developed by F. Riesz[22] can be generalized to the case when U is not completely continuous, but has a completely continuous iteration U^n see S. Nikolskij, *Lineare Gleichungen im metrischen Raume*, Doklady Acad. Nauk USSR, (New Ser.), **2**, 1936, 315–319.[23]

It is worth mentioning that, for example, a linear operation of the form $U(x) = \int_0^1 K(t,s)x(s)ds$ in the space $L^{(1)}$, where $K(t,s)$ is a bounded function on $0 \leq t, s \leq 1$, can be non-completely-continuous, but having a completely continuous square U^2; see J. Sirvint, *Sur les transformations intégrales de l'espace* L, Doklady Akad. Nauk USSR, (New Ser.), **18**, 1938, 255–257.

[After that [8, p. 241] contains a presentation of the result from S. Mazur, *Über die Nullstellen linearer Operationen*, Studia Math., **2**, 1930, 11-20, which states that for reflexive space E and $\|U\| = 1$ the equations $x = Ux$ and $f = U^*f$ have the same number of linearly independent solutions.]

Using this theorem S. Mazur (*Über die schwache Konvergenz und einen Satz von Birkhoff*, Ann. de la Soc. Polonaise de Math., **12**, 1932, p. 116) deduces

Corollary. *If $U(x)$ is a linear operation in a weakly complete space E of type (B), the space E is such that its bounded sets are weakly compact* [i.e. E is reflexive], *and $\|U\| \leq 1$, then the limit $\lim_{m\to\infty} \frac{1}{m}\sum_{n=1}^{m} U^n(x)$ exists for every $x \in E$.*[24]

This corollary is a generalization of the well-known ergodic theorem of Birkhoff[25] from statistical mechanics in the formulation of J. von Neumann. For the space $L^{(p)}$, $p > 1$, see F. Riesz, *Some mean ergodic theorems*, J. London Math. Soc., **13**, 1938, p. 274–278,

See also K. Yosida, *Mean ergodic theorem in Banach spaces*, Proc. Imp. Acad. Jap. Tôkyô, **14**, 1938, 292–294.[26]

§3 [This section is devoted to regular values and eigenvalues of linear equations. From the previous and next quotations we see that Mazur was interested in ordered spaces. In 1942–43 Mazur and Orlicz generalized some results of N.I. Akhiezer and M.G. Krein (*Some questions in the theory of moments*, Ukrainian State Science Press, Kharkov, 1938, in Russian; English transl. in Transl. Math. Monogr., v. 2, AMS, Providence, R.I., 1962) about ordered spaces to spaces of type (B_0), and published them in 1953. The fact that S. Mazur does not write about the Akhiezer-Krein book and his work with Orlicz on generalization of some results from there indicates that the updates were written before 1942.]

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[22] of abstract Fredholm equations $y = x - Ux$ with completely continuous U.
[23] Further generalizations of this theory have been developed using the notion of a *Riesz operator*. See, for example, Part II in H.R. Dowson, *Spectral Theory of Linear Operators*, London, Academic Press, 1978.
[24] Mazur's paper does not contain a proof of this result. A "reconstruction" of Mazur's proof is contained in [9, p. 505].
[25] *Proof of the ergodic theorem*, Proc. Nat. Acad. USA **17**, 1931, 656–680.
[26] This paper, and S. Kakutani, *Iteration of linear operators in complex Banach spaces*, Imp. Acad. Tokyo, **14**, 1938, 295–300, contain proofs of the Corollary.

See the papers M. Krein, *Sur les opérations linéaires transformant un certain ensemble conique en lui-même*, Doklady Akad. Nauk USSR, (New Ser.), **23**, 1939, 749–752; M.A. Rutman, *Sur une classe specielle d'opérateurs linéaires totalement continus*, Doklady Akad. Nauk USSR, (New Ser.), **18**, 1938, 625–627, and M.A. Rutman, *Sur les opérateurs totalement continus linéaires laissant invariant un certain cône*, Mat. Sbornik, **8**, 1940, 77–93.

CHAPTER XI

§9. ... A space E of type (B) is called *regular*, if each linear functional $F(f)$ in $\overline{E}$ is of the form $F(f) = f(x_0)$, where $x_0 \in E$. If the space E is separable and its unit ball is weakly compact it itself, i.e. is weakly compact and weakly closed, then, by Theorem 13[27], E is regular. As was noted by V. Gantmakher and V. Šmulian (*Sur les espaces linéaires dont la sphére unitaire est faiblement compacte*, Doklady Akad. Nauk USSR, (New Ser.), **17**, 1937, 91–94), regularity of an arbitrary space E implies that its unit ball is weakly compact in itself. These authors also prove the following

Theorem. *A space E is regular if and only if for every transfinite bounded sequence $\{x_\xi\}_{\xi<\vartheta}$ of elements of E there exists $x \in E$ such that*

$$\underline{\lim}_{\xi\to\theta} f(x_\xi) \le f(x) \le \overline{\lim}_{\xi\to\theta} f(x_\xi)$$

for every $f \in \overline{E}$.

If the unit ball of E is weakly compact in itself, then the unit ball of $\overline{E}$ also has this property.

V. L. Šmulian (*On the principle of inclusion in the space of type* (B), Mat. Sbornik, **5**, 1939, 317–328) proved

Theorem. *A space E is regular if and only if the intersection of every decreasing transfinite sequence of its nonvoid closed convex and bounded subsets $\{K_\xi\}_{\xi<\vartheta}$ is nonvoid.*

It should be mentioned also that D. P. Milman (*On some criteria for the regularity of spaces of type* (B), Doklady Akad. Nauk USSR, (New Ser.), **20**, 1938, 243–246),[28] proved:

Each uniformly convex space of type (B) is regular.

CHAPTER XII

§1. There are non-isomorphic separable spaces of type (B) having equal linear dimensions; see S. Banach und S. Mazur, *Zur Theorie der linearen Dimension*, Studia Math., **4**, 1933, 100–112.

§2. R.E.A.C. Paley (*Some theorems on abstract spaces*, Bull. Amer. Math. Soc., **42**, 1936, 235–240) proved that the space $l^{(q)}$ is not isomorphic to a subspace of $L^{(p)}$ if the numbers p and q satisfy one of the following conditions:

$p > 2 > q$; $q > 2 > p$; $2 < q < p$.

It is unknown if this theorem is valid, when $p < q < 2$[29].

[At the end of the "Remarks" section (both in [8] and [2]) there is a table showing presence or absence of different properties in different Banach spaces. Some positions in the tables were empty indicating that

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[27] which states that a separable Banach space is reflexive if its unit ball is weakly sequentially compact.

[28] see also B. J. Pettis, *A proof that every uniformly convex space is reflexive*, Duke Math. J., **5**, 1939, 249–253.

[29] The negative answer to this question is due to M. I. Kadets, *On linear dimension of the space L_p*, Uspehi Mat. Nauk, **13**, 1958, 95–98 (in Russian). More general result was obtained in J. Bretagnolle, D. Dacuhna-Castelle, J.-L. Krivine, *Lois stables et espaces L^p*, Ann. Inst. H. Poincaré, Sect. B (N.S.), **2** (1965/1966) 231–259.

the author did not know the answer at the time of publication. For the Ukrainian translation S. Mazur filled some of the positions which were empty in [8]. Namely:

L_∞, ℓ_∞, $C[0,1]$ and $C^{(p)}[0,1], p \geq 1$ have uncomplemented subspaces.[30]

The spaces $C[0,1]$ and $C^{(p)}[0,1], p \geq 1$ contain closed (separable) subspaces which cannot be mapped onto the whole space by a linear continuous operator[31].

It is worth mentioning that in the English translation (S. Banach, *Theory of linear operations*, North-Holland, Amsterdam, 1987) the table is updated according to the survey by A. Pełczyński and C. Bessaga (*Some aspects of the present theory of Banach spaces*, in: S. Banach, *Œuvres*, Vol. **II**, Travaux sur l'analyse fonctionnelle. Edited by C. Bessaga, S. Mazur, W. Orlicz, A. Pełczyński, S. Rolewicz and W. Żelazko, PWN—Éditions Scientifiques de Pologne, Warsaw, 1979, pp. 223–304), and is much more complete than the table in [2].]

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Department of Mathematics and Computer Science,
St. John's University, 8000 Utopia Parkway,
Queens, NY 11439, USA,
ostrovsm@stjohns.edu
Departamento de Matematicás,
Universidad de Extremadura,
Avenida de Elvas, 06071 Badajoz, Spain,
aplichko@usk.pk.edu.pl

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[30]Because ℓ_1 is uncomplemented in $C[0,1]$ (S. Banach, S. Mazur, Studia Math., **4**, 1933, 100–112) and the space $C^{(p)}[0,1]$ is isomorphic to $C[0,1]$. This was noticed by H. Komatuzaki (*Sur les projections dans certain espaces du type* (B), Proc. Imp. Acad. Jap., **16**, 1940, 274–279).

[31]For example, it is the case for subspaces isomorphic to ℓ_2.