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# Diagonal sequence property in Banach spaces with weaker topologies ☆

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**Abstract**

We investigate the diagonal sequence property in Banach spaces with weaker topologies. In particular, we present examples of Banach spaces with weaker locally convex topologies which have the diagonal sequence property but are not Fréchet–Urysohn. The examples answer negatively a question of Averbukh and Smolyanov. We give also a very simple proof of the fact that each Banach space contains a subset $A$ whose weak closure includes 0, but 0 is not contained in the weak closure of any bounded subset of $A$.


*Keywords:* Banach space; Countable tightness; Diagonal sequence property; Fréchet–Urysohn space

## 1. Introduction

**Definition.** A topological space $X$ has the *diagonal sequence property* if for any double sequence $(x_{mn})$ in $X$ with $\lim_n x_{mn} = x \in X$, $m = 1, 2, \ldots$, it is possible to choose a sequence $(n_m) \subset \mathbb{N}$ such that $\lim_m x_{mn_m} = x$. The space $X$ has the *weak diagonal sequence property* if for any $(x_{mn})$ in $X$ with $\lim_n x_{mn} = x \in X$, $m = 1, 2, \ldots$, it is possible to choose a strictly increasing sequence $(m_k)$ and a sequence $(n_k)$ such that $\lim_k x_{m_k n_k} = x$.

These definitions follow Todorčević and Uzcátegui [16]. The diagonal sequence property coincides with *the condition* $(*)$ of [12, p. 797]. The weak diagonal sequence property is equivalent with the condition $(\alpha_4)$ in [1] and with *the condition* $(**)$ of [12, p. 799]. The weak diagonal sequence property was considered in the survey [15]. In [3, p. 99] the following *property* (AS) is considered: for any $(x_{mn})$ in $X$ with $\lim_n x_{mn} = x \in X$, $m = 1, 2, \ldots$, it is possible to choose strictly increasing sequences $(m_k)$ and $(n_k)$ such that $\lim_k x_{m_k n_k} = x$. The property (AS) is interesting from the point of view of differentiability in topological vector spaces.

Obviously, if $X$ is a metric space, then $X$ has the diagonal sequence property and if a topological space has the diagonal sequence property, then it has the property (AS) and hence, has the weak diagonal sequence property as well.

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A topological space $X$ is called a *Fréchet–Urysohn space* if the closure of any subset of $X$ coincides with its sequential closure. We do not use the term 'Fréchet space', as in [16], because in topological vector spaces it has another meaning.

In [3, Lemma 3.3] (see also [17, p. 140]) it is proved that a Fréchet–Urysohn topological vector space has the property (AS). In [12, Theorem 4] (which refers neither to [3] nor to [17]) it is proved that a Fréchet–Urysohn topological group has the weak diagonal sequence property. In [12, p. 798] it is noted also that a Fréchet–Urysohn topological space may not have the weak diagonal sequence property. In [7, Lemma 1.3] it is proved that a Fréchet–Urysohn Hausdorff topological group has the property (AS). It is an open problem, whether the diagonal sequence property is equivalent to the weak diagonal sequence property for topological groups [12] or for topological vector spaces.

**Historical remark.** The first example of a non-metrizable Fréchet–Urysohn Hausdorff topological vector space is mentioned in [4]; namely, it is stated there without proof that the $\Sigma$-product (in the modern terminology) of uncountably many metrizable topological vector spaces is such a space. Later and independently N. Noble proved in [11, Theorem 2.1] that the $\Sigma$-product of first countable topological spaces really is a Fréchet–Urysohn topological space.

In the paper [4], M. Balanzat states that for every topological vector space the following proposition is true: For each null sequence $(x_n)$ there exists a subsequence $(x_{n_k})$ and an unbounded scalar sequence $(\lambda_k)$ such that $\lambda_k x_{n_k} \to 0$. The proposition is trivially wrong. However, from the context one can suppose that the proposition contains a misprint and one has to read 'for every Fréchet–Urysohn space', which is, of course, true, by the Averbukh–Smolyanov lemma. So, one can suppose that a version of the weak diagonal sequence property and a version of Averbukh–Smolyanov lemma was considered in [4]. Unfortunately, this paper contains no proofs.

In [3, Problem 5, p. 111] the following question is posed: Let $E$ be a topological vector space with the property (AS), is then $E$ a Fréchet–Urysohn space? This question is recalled in [7, Remark 1, p. 744], where it is noted also that Nyikos [12, Example 5] answers a similar question in negative in the context of topological groups. We provide, in particular, counterexamples to the Averbukh–Smolyanov question for Banach spaces with weaker locally convex topologies. Most results of this note are valid for more general classes of topological vector spaces, but we restricted ourselves to Banach spaces, attempting to keep simple proofs.

Let us recall some topological definitions. A topological space $X$ is *sequential* if every sequentially closed subset of $X$ is closed; the space $X$ has *countable tightness* if the closure of any its subset coincides with the countable closure. Each Fréchet–Urysohn space is sequential and each sequential space has countable tightness (see e.g. [7]). A sequential topological group with the diagonal sequence property is Fréchet–Urysohn [12].

All linear spaces are supposed to be infinite dimensional.

## 2. Banach space in weak topologies

Let $X$ be a Banach space. A linear subspace $F$ of the dual space $X^*$ is called *total* if $\forall x \in X$, $x \neq 0$ there is $f \in F$ such that $f(x) \neq 0$. Denote by $w(X, F)$ the weakest topology on $X$ in which all functionals from $F$ are continuous and put

$$\|x\|_F := \sup\bigl\{\bigl|f(x)\bigr|\colon\ f \in F\colon\ \|f\| = 1\bigr\}, \quad x \in X.$$

**Proposition 1.** *Let $X$ be a Banach space and $F$ be a norm closed total subspace of $X^*$. Then $X$ is not sequential in the topology $w(X, F)$.*

**Proof.** Take a countably dimensional linear subspace $Y$ of $X$. Cover the unit sphere $S$ of $Y$ by a sequence of norm compact sets $K_n \subset S$ and put $a_n = \inf\{\|y\|_F\colon\ y \in K_n\}$.

By compactness, $a_n > 0$; put $A_n = \frac{n}{a_n} K_n$, $n = 1, 2, \ldots$, and $A = \bigcup_n A_n$.

By the construction, $0 \notin A$. Let us show that the set $A$ is sequentially closed. Take a $w(X, F)$-convergent sequence $(y_k)$ in $A$. There are two possibilities.

(a) There are an integer $n$ and a subsequence $(y'_k) \subset (y_k)$ which belongs to $A_n$. Since $A_n$ is norm compact, it is $w(X, F)$-compact, hence $w(X, F)$-closed. So, $w(X, F)$-limit of $(y'_k)$, hence of $(y_k)$, belongs to $A_n \subset A$.

(b) There are an increasing sequence $(n_k)$ and a subsequence $(y'_k) \subset (y_k)$ such that $y'_k \in A_{n_k}$. Then $\|y'_k\|_F \geqslant n_k$, so the sequence $(\|y'_k\|_F)$ is unbounded. If one will look at $y'_k$ as at functionals on $F$, then by the Uniform Boundedness Principle, one will obtain that $(y'_k)$, hence $(y_k)$, cannot $w(X,F)$-converge.

On the other hand, any $w(X,F)$-neighborhood of 0 contains some finite-codimensional subspace, consequently has nonzero intersection with $Y$, hence intersects $A$. Therefore, 0 belongs to the $w(X,F)$-closure of $A$. □

**Remark.** Proposition 1 is close to Corollary 4.4 in [6]. Our arguments are elementary, but mentioned Corollary 4.4 follows from more strong results. This corollary implies also that Proposition 1 can be false if the subspace $F$ is not norm closed, for example, if $F$ is total and countable dimensional.

Cascales and Raja [6] also considered the question:

[Q] Let $X$ be a Banach space. Does there exist a set $A \subset X$ whose weak closure contains 0 but there is no bounded set $B \subset A$ whose weak closure contains 0?

The question was inspired by the classical J. von Neumann example of a sequence $(e_n)$ in a Hilbert space such that 0 is a cluster point of $(e_n)$, but no subsequence of $(e_n)$ converges to 0 in the weak topology.[1] Cascales and Raja provide a simple (they say, topological) answer to [Q]. We provide a very simple, purely Banach space geometric, answer.

**Proposition 2.** *The answer to the question* [Q] *is 'yes'.*

**Proof.** It is enough to prove the proposition for a countably dimensional normed space $X$. Let us cover the unit sphere $S$ of $X$ by a sequence of norm compact sets $K_n \subset S$, and put $A_n = nK_n$.

Each bounded subset of $A = \bigcup_1^\infty A_n$ is contained in a union of finitely many $A_n$, so its weak closure cannot contain 0.

On the other hand, any weak neighborhood of 0 contains a ray with origin in 0, hence intersects some $A_n$. □

**Temporary definition.** We say that a locally convex topology $\tau$ on a linear space $X$ is *non-weak* if there exists a $\tau$-continuous seminorm $p$ and an infinite dimensional subspace $Y$ of $X$ so that $(Y,p)$ is a normed space.

**Proposition 3.** *Let $X$ be a Hausdorff non-weak locally convex topological vector space. Then there is a set $A \subset X$ such that* 0 *belongs to the weak closure of $A$ and there is no bounded set $B \subset A$ such that* 0 *belongs to the weak closure of $B$.*

The proof repeats the proof of Proposition 2. This proposition is close to Theorem 4.2 of [6].

A Banach space, as any metric space, has the diagonal sequence property. Now we will consider the following questions. Let $X$ be a Banach space and $\tau$ be a weaker Hausdorff locally convex topology on $X$. When does $(X,\tau)$ have the diagonal sequence property?

**Proposition 4.** *Let $X$ be a Banach space and $\tau$ be a strictly weaker Hausdorff locally convex topology on $X$. If $\|\cdot\|$ and $\tau$ have the same convergent sequences then $(X,\tau)$ has the diagonal sequence property, but is not sequential.*

**Proof.** The first part of the proposition is obvious. To prove the second part, we will show that the unit ball of $X$ is not $\tau$-sequential. The unit sphere $S$ of $X$ is norm closed, hence $\tau$-sequentially closed. Obviously, $0 \notin S$, but, because every convex symmetric $\tau$-neighborhood of 0 is norm unbounded, it intersects $S$. So, 0 belongs to the $\tau$-closure of $S$. □

A subspace $F$ of a Banach space $X$ is said to be *norming* if the norm $\|x\|_F$ is equivalent to the original norm $\|x\|$ of the space $X$.

[1] Recently Aron, Garcia and Maestre [2] have proved that given a separable Banach space $X$, there is a weakly dense sequence $(x_n)$ in $X$ such that $\|x_n\| \to \infty$.

**Remark.** The conditions of Proposition 4 guarantee that if $\tau = w(X, F)$, then the subspace $F$ of $X^*$ is norming.

Let us recall that a Banach space $X$ has the *Schur property* if the weak convergence in $X$ coincides with the norm convergence. For example, the space $\ell_1(\Gamma)$, where $\Gamma$ is an arbitrary set, has the Schur property (see e.g. [9, p. 92]). There are separable Banach spaces with the Schur property which are not isomorphic to any subspace of $\ell_1$ [5].

**Corollary 1.** *Let $X$ be a Banach space with the Schur property. Then the (locally convex Hausdorff topological vector) space $(X, w(X, X^*))$ has the diagonal sequence property, but it is not sequential.*

**Remark.** Since each Fréchet–Urysohn space is sequential, this corollary provides the first elementary example answering negatively the Averbukh–Smolyanov question.

**Proposition 5.** *Let $X$ be a Banach space and $\tau$ be a weaker Hausdorff locally convex topology on $X$ such that $F = (X, \tau)^*$ is a norm closed norming subspace of $X^*$. If $X$ contains a $\tau$-convergent but not $\|\cdot\|$-convergent sequence then $X$ does not have the weak diagonal sequence property in the topology $\tau$.*

**Proof.** By the conditions, there is $c > 0$ and a sequence $x_n \in X$ which $\tau$-converges to 0, but $\|x_n\| > c$ for each $n$. Then, for every $m$, the sequence $x_{mn} = mx_n$, $n = 1, 2, \ldots,$ $\tau$-converges to 0. Each sequence $x_{m_k n_k}$, $k = 1, 2, \ldots,$ is $\|\cdot\|$-unbounded, hence, $\|\cdot\|_F$-unbounded. If one will look at $x_{mn}$ as functionals on $F$, then by the Uniform Boundedness Principle, one will obtain that $(x_{m_k n_k})$ cannot $w(X, F)$-converge, hence cannot $\tau$-converge. $\square$

**Remark.** We do not know whether the condition to $F$ be norming is essential here. The next proposition shows that norm closedness of $F$ is essential.

**Proposition 6.** *For every separable Banach space $X$ there exists a norming (but not necessarily norm closed) subspace $F \subset X^*$ such that $X$ has the diagonal sequence property, but is not sequential in the topology $w(X, F)$.*

**Proof.** There exists a bounded linear one-to-one operator $T : X \to \ell_1$ with dense range such that $F = T^*(\ell_1^*) \subset X^*$ is norming. This follows e.g. from the Markushevich theorem [9, p. 8].

Let $(x_{mn})$ and $x$ be from the definition of diagonal sequence property. Then, for each $m$, the sequence $(Tx_{mn})_{n=1}^{\infty}$ weakly converges to $Tx$. Since $\ell_1$ has the Schur property, it norm converges to $x$. Hence, some sequence $(Tx_{m,n_m})_{m=1}^{\infty}$ weakly converges to $Tx$. It implies that the sequence $(x_{m,n_m})_{m=1}^{\infty}$, $w(X, F)$-converges to $x$. Therefore, $X$ has the diagonal sequence property in the topology $w(X, F)$.

Let $S$ be the intersection of the unit sphere of $\ell_1$ with $TX$. It is weakly sequentially closed in $TX$, hence $T^{-1}S$ is $w(X, F)$-sequentially closed. Moreover, 0 belongs to $w(X, F)$-closure of $T^{-1}S$. Therefore, $X$ is not sequential in the topology $w(X, F)$. $\square$

Proposition 6 gives the second answer to the Averbukh–Smolyanov question.

**Corollary 2.** *If a Banach space $X$ is not hereditarily $\ell_1$ then $X$ endowed with the weak topology does not have the weak diagonal sequence property.*

**Proof.** Let $Y$ be an infinite dimensional subspace of $X$ which does not contain any subspace isomorphic to $\ell_1$. By the Rosenthal $\ell_1$ theorem, there exists a sequence $y_n \in Y$ which weakly, but not norm, converges to 0. So, one can apply Proposition 5. $\square$

**Corollary 3.** *A dual Banach space $X^*$ endowed with the weak* topology does not have the weak diagonal sequence property.*

**Proof.** By the Josefson–Nissenzweig theorem (see e.g. [9, p. 101]), there exists a weakly* null sequence $(f_n) \subset X^*$ for which $\|f_n\| \nrightarrow 0$. So, one can apply Proposition 5 taking $X = X^*$ and $\tau = w(X^*, X)$. $\square$

**Corollary 4.** *A dual Banach space $X^*$ endowed with the Mackey topology $\mu(X^*, X)$ has the diagonal sequence property if and only if $X$ does not contain $\ell_1$ isomorphically.*

**Proof.** By a result of Ørno and Valdivia [9, p. 94], every $\mu(X^*, X)$-convergent sequence in $X^*$ is norm convergent if and only if $X$ does not contain $\ell_1$ isomorphically. So, one may use Proposition 5. □

**Corollary 5.** *A dual Banach space $X^*$ endowed with the Mackey topology $\mu(X^*, X)$, which does not contain $\ell_1$ isomorphically, is sequential if and only if $X$ is reflexive.*

The proof is a combination of Corollary 4 and Proposition 4. It provides one more counterexample to the Averbukh–Smolyanov question.

## 3. Unit ball of Banach space in weak topologies

The results of Section 2 provide a reason to consider in weaker topologies not the whole Banach space but its unit ball. Moreover, if we attempt to keep an orientation to differentiability, it is quite natural to consider mappings which are defined not on the whole space but on some bounded subset. Let us begin with

**Remarks.** A topological space $X$ is said to be *separably metrizable* if each of its separable subspaces is metrizable (Yost [18] called this 'the Veech property').

Let $X$ be a Banach space, $B$ its unit ball and $F$ be a total subspace of the dual $X^*$.

1. If $B$ is $w(X, F)$-separably metrizable, then $B$ has the diagonal sequence property in the topology $w(X, F)$. The ball $B$ is separably metrizable in the weak topology if and only if $X$ is Asplund, i.e. every separable subspace of $X$ has separable dual (see e.g. [8, Chapter 3]). A dual ball $B^*$ is separably metrizable in the weak* topology if and only if for every countable subset $F$ of $X^*$ the quotient $X/F^\top$ is separable (see e.g. [8, Chapter 3]; $F^\top$ denotes the annihilator in $X$).

2. There are Banach spaces whose unit ball does not have the weak diagonal sequence property in the weak topology. It can be e.g. $(\sum_m H_m)_{\ell_1}$, where $H_m$ are separable Hilbert spaces. As $(x_{mn})_{n=1}^\infty$, $m = 1, 2, \dots,$ one can take an orthonormal basis of $H_m$ and put $x = 0$.

3. There are Banach spaces whose unit ball has the diagonal sequence property in the weak topology but is not sequential. It can be e.g. the unit ball of $\ell_1$ (see the proof of Proposition 4).

In the next two propositions we repeat ideas of [12]. Denote by $A_{(1)}$ the sequential closure of a subset $A$ of a topological space $(X, \tau)$ and $A_{(2)} := (A_{(1)})_{(1)}$.

**Proposition 7.** *Let $\tau$ be a weaker Hausdorff topology on a Banach space $X$. If the unit ball $B$ is sequential and has the weak diagonal sequence property in the topology $\tau$, then it is Fréchet–Urysohn in this topology.*

**Proof.** Suppose $B$ is not $\tau$-Fréchet–Urysohn. Then there is a subset $A$ of $B$ such that $A_{(2)}$ contains a point $x$ not in $A_{(1)}$. Since multiplication by a positive scalar and translation change nothing, one may assume $x = 0$ and $A_{(2)} - A_{(2)} \subset B$ without loss of generality.

Let $(x_m)$ be a sequence of points of $A_{(1)}$, $\tau$-converging to 0. For each $m$ let $x_{mn}, n = 1, 2, \dots,$ be a sequence of points of $A$, $\tau$-converging to $x_m$. Then $x_{mn} - x_m \to 0$ as $n \to \infty$ for each $m$. By the weak diagonal sequence property, it is possible to pick $(m_k)$ and $(n_k)$ so that $x_{m_k n_k} - x_{m_k} \to 0$ as $k \to \infty$. Then $x_{m_k n_k}$ also $\tau$-converges to 0, contradicting the assumption that 0 is not in $A_{(1)}$. □

**Proposition 8.** *Let $\tau$ be a weaker Hausdorff topology on a Banach space $X$. If the unit ball $B$ is $\tau$-Fréchet–Urysohn, then it has the weak diagonal sequence property in this topology.*

**Proof.** Let $(x_{mn})$ and $x$ be from the definition of weak diagonal sequence property. Similarly to previous proposition we may assume $x = 0$ and $\{x_{mn}\} + \{x_{mn}\} \subset B$. For each positive integer $k$, the sequence $(x_{1,k} + x_{kn})_{n=1}^\infty$ converges to the element $x_{1,k}$, and so the union $A$ of all these sequences has 0 in its closure. By hypothesis there is a sequence in $A$ converging to 0. It cannot contain finitely many $k$ only, so has a subsequence of a form $(x_{1,k_i} + x_{k_i n_i})_{i=1}^{\infty}$. Then $(x_{k_i n_i})$ also converges to 0. $\square$

Now we present an example of a dual space $X^*$ whose ball $B^*$ is weakly* separably metrizable (hence $(B^*, w^*)$ has the diagonal sequence property), but does not have weak* countable tightness (hence, $(B^*, w^*)$ is not Fréchet–Urysohn).

Let $X = C[0, \omega_1]$ be the space of continuous functions on the segment of ordinals $[0, \omega_1]$. Its dual $X^*$ is isometric to the space $\ell_1[0, \omega_1]$ [14, p. 338] with the natural duality

$$\langle x, f\rangle = \sum_{0\leqslant\alpha\leqslant\omega_1} x(\alpha) f(\alpha), \quad x \in X,\ f \in X^*, \quad \text{and the norm} \quad \|f\| = \sum_{0\leqslant\alpha\leqslant\omega_1} \bigl|f(\alpha)\bigr|.$$

Denote by $f_\alpha$, $\alpha \in [0, \omega_1]$, the Dirac measure with support in the point $\alpha$. In the dual space $\ell_1[0, \omega_1]$ consider the hyperplane $H$ norm spanned on all $f_\alpha$, $\alpha \in [0, \omega_1)$. Write $\operatorname{supp} x = \{\alpha\colon\ x(\alpha) \neq 0\}$.

**Proposition 9.** *The dual unit ball $(B^*, w^*)$ does not have countable tightness but is separably metrizable.*

**Proof.** Since the hyperplane $H$ is weakly* countably closed, norming and $f_{\omega_1} \notin H$ (see e.g. [13]), $(B^*, w^*)$ does not have countable tightness.

Let $(e_n)$ be a countable subset in $B^*$. Each $e_n$ has a form $e_n = h_n + a_n f_{\omega_1}$, $\operatorname{supp} h_n \subset [0, \alpha_n]$, $\alpha_n < \omega_1$, $a_n \in \mathbb{R}$. Put $\alpha = \sup_n \alpha_n$. Then $(e_n)$ belongs to the subspace $F + \operatorname{lin} f_{\omega_1}$ where the subspace $F = (f : \operatorname{supp} f \in [0, \alpha])$ is dual to the separable quotient $X/(x : \operatorname{supp} x \in (\alpha, \omega_1])$. The adding of one-dimensional subspace does not alter the weak* closedness, therefore $F + \operatorname{lin} f_{\omega_1}$ is isometric to a dual of some separable quotients of $X$, hence $(B^*, w^*)$ is separably metrizable. $\square$

## 4. Separably Banach space

The whole space $(X^*, w^*)$ from Proposition 9 (by Corollary 3) does not have the diagonal sequence property. We will introduce in $X^*$ a stronger topology to receive the diagonal sequence property (moreover, separable metrizability) but in such a way that it does not have countable tightness. The main idea is based on the following notion.

**Definition.** We say that a topological vector space $E$ is *separably Banach* if each closed separable subspace $F$ of $E$, with induced from $E$ topology, is isomorphic to a Banach space.

This notion is a Banach space counterpart of the separably metrizable topological space.

It is easy to check (see the proof of Corollary 6 below) that any separably Banach topological vector space has the diagonal sequence property. Moreover, any separably Banach topological vector space is countably Fréchet–Urysohn. Let us recall that a topological space $X$ is *countably Fréchet–Urysohn* if the closure of each countable subset of $X$ equals to its sequential closure [10]. We will find a separably Banach locally convex topological vector space which has not countable tightness, hence is not Fréchet–Urysohn. Note, that the corresponding Nyikos' non-Fréchet–Urysohn (moreover, non-sequential) topological group [12], which has the weak diagonal sequence property, is *countable*, hence has the countable tightness. The space $X$ from Corollary 1 is not separably Banach, because no infinite dimensional Banach space is metrizable in the weak topology. This space, as any Banach space in the weak topology, has countable tightness (the Kaplansky property) [8, Chapter 4].

Now we introduce on $\ell_1[0, \omega_1]$ a locally convex topology $\tau$, intermediate between the norm and the weak* topology $w(\ell_1[0, \omega_1], C[0, \omega_1])$ as follows. The topology $\tau$ is generated by the system of seminorms

$$p_\alpha(f) = \sum_{0\leqslant\beta\leqslant\alpha} \bigl|f(\beta)\bigr|, \quad \alpha \in [0, \omega_1) \quad \text{and} \quad p_{\omega_1}(f) = \bigl|\langle u, f\rangle\bigr|, \quad f \in \ell_1[0, \omega_1],$$

where $u \in C[0, \omega_1]$: $u(\alpha) = 1$ for each $\alpha$. Since all seminorms $p_\alpha$ are $\|\cdot\|$-continuous, $\tau$ is weaker then the norm. Every function of $C[0, \omega_1]$ has a form $x = y + au$, where $\operatorname{supp} y \subset [0, \alpha]$ for some $\alpha < \omega_1$ and $a \in \mathbb{R}$. The element $y$ (as a functional on $\ell_1[0, \omega_1]$) is $p_\alpha$-continuous and $u$ is $p_{\omega_1}$-continuous. Therefore, the weak* topology $w(\ell_1[0, \omega_1], C[0, \omega_1])$ is weaker than $\tau$. Put $E = (\ell_1[0, \omega_1], \tau)$.

**Proposition 10.** *The space $E$ does not have countable tightness, but is separably Banach.*

**Proof.** 1. As we noted in the proof of Proposition 9, the hyperplane $H \subset \ell_1[0, \omega_1]$ is weakly* countably closed, hence is $\tau$-countably closed; moreover, $f_{\omega_1} \notin H$.

We must check that $f_{\omega_1}$ belongs to the $\tau$-closure of $H$. If $f_{\omega_1}$ does not belong to this closure then there is an ordinal $\alpha$ and $a > 0$ such that for the functional $q_\alpha(f) = p_\alpha(f - f_{\omega_1})$

$$q_\alpha(h) > a \quad \text{for all } h \in H.$$

This is impossible, because for $\alpha < \omega_1$, $q_\alpha(h) = 0$ if $\operatorname{supp} h \subset (\alpha, \omega_1)$ and $q_{\omega_1}(h) = 0$ if $h \in H$ and $\langle u, h \rangle = 1$. So, $E$ does not have countable tightness.

2. Let $(e_n)$ be a countable subset in $E$. Each $e_n$ has the form $e_n = h_n + a_n f_{\omega_1}$, $\operatorname{supp} h_n \subset [0, \alpha_n]$, $\alpha_n < \omega_1$, $a_n \in \mathbb{R}$. Put $\alpha = \sup_n \alpha_n$. Then $(e_n)$ belongs to the subspace $F + \operatorname{lin} f_{\omega_1}$ where $F = (f\colon \operatorname{supp} f \in [0, \alpha])$. On the subspace $F$ all seminorms $p_\beta$, $\beta < \alpha$ and $p_{\omega_1}$ are weaker than $p_\alpha$, and the seminorms $p_\beta$, $\beta > \alpha$ all coincide with $p_\alpha$. So, on $F$ the topology $\tau$ coincides with the norm $p_\alpha$ and hence $F$ is isomorphic to a Banach space. The addition of one-dimensional subspace changes nothing, hence $(F + \operatorname{lin} f_{\omega_1}, \tau)$ is isomorphic to a Banach space too. Therefore, $E$ is separably Banach. □

**Corollary 6.** *The space $E$ has the diagonal sequence property.*

**Proof.** Let $e_{mn}, e \in E$, be elements from the definition of the diagonal sequence property (we change the letters $x$'s to $e$'s). By Proposition 10, they belong to some Banach subspace of $E$. But each Banach space has the diagonal sequence property, which proves the corollary. □

**Question.** Let $E$ be a separably Banach topological vector space such that each of its separable subspaces is reflexive. Is $E$ Fréchet–Urysohn? Does $E$ have countable tightness?

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