Indag. Mathem., N.S., 18 (2), 251–268 June 18, 2007

Copies of $c_0(\Gamma)$ and $\ell_\infty(\Gamma)/c_0(\Gamma)$ in quotients of Banach spaces with applications to Orlicz and Marcinkiewicz spaces

by Anatolij Plichko [a] and Marek Wójtowicz [b]

[a] *Instytut Matematyki, Politechnika Krakowska, ul. Warszawska 24, 31-155 Kraków, Poland*
[b] *Instytut Matematyki, Uniwersytet Kazimierza Wielkiego, Pl. Weyssenhoffa 11, 85-072 Bydgoszcz, Poland*

Communicated by Prof. J.J. Duistermaat at the meeting of September 25, 2006

ABSTRACT

Let X be a Banach space, let Y be its subspace, and let Γ be an infinite set. We study the consequences of the assumption that an operator T embeds $\ell_\infty(\Gamma)$ into X isomorphically with $T(c_0(\Gamma)) \subset Y$. Under additional assumptions on T we prove the existence of isomorphic copies of $c_0(\Gamma^{\aleph_0})$ in X/Y, and complemented copies $\ell_\infty(\Gamma)/c_0(\Gamma)$ in X/Y. In concrete cases we obtain a new information about the structure of X/Y. In particular, $L_\infty[0,1]/C[0,1]$ contains a complemented copy of ℓ_∞/c_0, and some natural (lattice) quotients of real Orlicz and Marcinkiewicz spaces contain lattice-isometric and positively 1-complemented copies of (real) ℓ_∞/c_0.

1. INTRODUCTION

Let X be a Banach space, let Y be a closed subspace of X, and let Γ be an infinite set. The present paper deals with the structure of the space X/Y and is motivated by two recent results: by Rusu [23] and the second named author [29]. In [23, pp. 86–87] it is proved implicitly that *if a subspace Y of ℓ_∞ contains an isomorphic copy of c_0 but not ℓ_∞, then $c_0(\mathbf{R})$ embeds isomorphically into ℓ_∞/Y*. On the other hand, in [29] the existence of lattice copies of $\ell_\infty(\Gamma)/c_0(\Gamma)$ in some quotients of Banach lattices was examined. Regarding the structure of X/Y we refer the reader to the monograph [5] by Castillo and Gonzalez describing the state in this setting up till 1997; the survey paper [21] by Plichko and Yost (of 2000) complements it partially.

In this paper we study the consequences of the assumption that an operator T embeds $\ell_\infty(\Gamma)$ into X isomorphically with

MSC: Primary: 46B25; secondary: 46B26, 46B42

E-mails: aplichko@usk.pk.edu.pl (A. Plichko), mwojt@ukw.edu.pl (M. Wójtowicz).

(1) $T(c_0(\Gamma)) \subset Y.$

In Theorem 1 (extending the above-mentioned result by Rusu, and complementing the classical Drewnowski–Roberts theorem that *the non-containment of* ℓ_∞ *is a three-space property* [5, Theorem 3.2.f]) we show, in particular, that X/Y contains a copy of $c_0(\Gamma^{\aleph_0})$ provided that Y contains no copy of ℓ_∞. In Theorem 2 we strengthen relation (1) between Y and $c_0(\Gamma)$ obtaining the existence of a continuous injection R from $\ell_\infty(\Gamma)/c_0(\Gamma)$ into X/Y, and a projection from X/Y onto the range of R (under an additional assumption on T). The latter result is narrowed in Theorem 3 to the case studied in [29], giving that some quotients of Banach lattices contain (positively) complemented copies of ℓ_∞/c_0.

These general results well apply to concrete cases and yield new information about the structure of quotients of classical spaces. For example, the space $L_\infty[0,1]/C[0,1]$ contains a complemented copy of ℓ_∞/c_0 (Corollary 5), and an application of Theorem 3 to an Orlicz space $X = L_\phi(\mu)$ and Y its order continuous part $E_\phi(\mu)$ gives that $L_\phi(\mu)/E_\phi(\mu)$ contains a lattice-isometric and positively 1-complemented copy of ℓ_∞/c_0 whenever $L_\phi(\mu) \neq E_\phi(\mu)$ (Corollary 9).

The interested reader may apply further the result by Partington, which does not appear in our statements, that *every isomorphic copy of* ℓ_∞/c_0 *contains an isometric copy of* ℓ_∞ (see [18]; its lattice version is addressed in [29]); such an additional conclusion one obtains, e.g., in Corollary 5, Theorem 2(ii), and Corollary 8. Moreover, if Γ is uncountable then $\ell_\infty(\Gamma)/c_0(\Gamma)$, endowed with the natural quotient norm, contains a lattice-isometric copy of $\ell_\infty(\Gamma)$, see [28]; this complements part (i) of Theorem 3.

The main results of this paper are given in Sections 2, 3 and 4, and their proofs are included in the last section.

The terminology we use is standard and is that of [16,17]. All spaces and subspaces are assumed to be linear and norm-closed, and all (linear) operators are continuous. A subspace U of X is said to be 1-complemented if there is a projection P from X onto U with $\|P\| = 1$. The term "copy" means "isomorphic copy". The letters Q and q, respectively, will denote the familiar quotient mappings $X \to X/Y$ and $\ell_\infty(\Gamma) \to \ell_\infty(\Gamma)/c_0(\Gamma)$, respectively.

By $\mathcal{L}_\infty[0,1]$ we denote the linear space of all Lebesgue-measurable functions on the interval $[0,1]$ that are bounded almost everywhere. Then $\mathcal{N}$ denotes the subspace of $\mathcal{L}_\infty[0,1]$ of the functions that vanish almost everywhere on $[0,1]$, and $\mathcal{L}^b_\infty[0,1]$ is the subspace of all *bounded* elements f of $\mathcal{L}_\infty[0,1]$ (i.e., $\|f\|_\infty := \sup_{t\in[0,1]} |f(t)| < \infty$). By S we denote the natural quotient mapping $\mathcal{L}_\infty[0,1] \to \mathcal{L}_\infty[0,1]/\mathcal{N}$, and the latter space is denoted by $L_\infty[0,1]$. Obviously, $(\mathcal{L}_\infty[0,1], \|\ \|_\infty)$ is a closed subspace of $\ell_\infty[0,1]$ containing $C[0,1]$ as a closed subspace. Moreover, $L_\infty[0,1]$, endowed with the "ess sup"-norm, is a Banach space and S restricted to $C[0,1]$ is an isometry preserving disjointness. It is worth to notice that $S(\mathcal{L}^b_\infty[0,1]) = L_\infty[0,1]$ (see (9) in Section 5).

By **N** and **R** we denote the sets of positive integers and real numbers, respectively.

We start with a generalization of the above-mentioned result by Rusu. We recall that Y denotes a subspace of a Banach space X.

Theorem 1. *Let $T:\ell_\infty(\Gamma)\to X$ be an isomorphic embedding that fulfills condition* (1). *Then $c_0(\Gamma^{\aleph_0})$ embeds isomorphically into X/Y if one of the following conditions holds*:

(a) *Y does not contain a copy of ℓ_∞,*
(b) *Y does not contain a copy of $\ell_\infty(\Gamma)$ and* $\operatorname{card}(\Gamma)=\operatorname{card}(\Gamma)^{\aleph_0}$.

(Of course, if $\operatorname{card}(\Gamma)=\operatorname{card}(\Gamma)^{\aleph_0}$ then condition (b) is weaker than (a).) The theorem covers partially also the case when the cardinality $\alpha:=\operatorname{card}(\Gamma)$ fulfills inequality $\alpha^{\aleph_0}>\alpha$. Because $\alpha=2^{\aleph_0}$ implies that $\alpha^{\aleph_0}=\alpha$, we then have two cases:

(j) $\aleph_0\leqslant\alpha<2^{\aleph_0}$, and
(jj) $\alpha>2^{\aleph_0}$.

In case (j) we apply condition (a). In case (jj) the set of infinite cardinals $\{\beta<\alpha\colon \beta^{\aleph_0}=\beta\}$ is not empty, and hence we can apply condition (b) for every subset Γ_β of Γ, with $\operatorname{card}(\Gamma_\beta)=\beta$, instead of Γ. That is, we consider the restriction of T to the set of elements of $\ell_\infty(\Gamma)$ with support contained in Γ_β, which form a subspace of $\ell_\infty(\Gamma)$ which is isometric to $\ell_\infty(\Gamma_\beta)$. This is so, for example, for $\alpha:=\sum_{n=1}^{\infty}\alpha_n$, where $\alpha_1=\aleph_0$, and $\alpha_{n+1}=2^{\alpha_n}$, $n=1,2,\ldots$; here we have $\alpha^{\aleph_0}>\alpha$ (see [15, Corollary V.8.3]).

The first corollary of Theorem 1 is a consequence of the result of Bessaga and Pełczyński on copies of c_0 and ℓ_∞ in duals of Banach spaces.

Corollary 1. *Let V be a subspace of X^* such that V contains a copy of c_0. If V does not contain a copy of ℓ_∞, then $c_0(\mathbf{R})$ embeds isomorphically into X^*/V.*

The next result is due to Rusu [23, pp. 86–87]. It immediately follows from Corollary 1 applied to the space $X=\ell_1$. Another argument is given in Section 5.

Corollary 2. *Let Y be a subspace of ℓ_∞ containing a copy of c_0. If Y does not contain a copy of ℓ_∞, then ℓ_∞/Y contains a copy of $c_0(\mathbf{R})$.*

In particular, for every separable subspace Y of ℓ_∞ containing a copy of c_0, the quotient space ℓ_∞/Y contains a copy of $c_0(\mathbf{R})$.

In the next theorem, which is a partial generalization of [29, Proposition 1], we strengthen condition (1), obtaining much stronger results than in Theorem 1.

Theorem 2. (i) *Let* $T : \ell_\infty(\Gamma) \to X$ *be an isomorphic embedding such that*

$$(2) \qquad T\big(c_0(\Gamma)\big) = Y \cap \operatorname{Im} T.$$

Then the induced operator $R : \ell_\infty(\Gamma)/c_0(\Gamma) \to X/Y$ *defined by* $R \circ q = Q \circ T$ *is injective and* $\|R\| \leqslant \|T\|$.

(ii) *If moreover there is a projection* P *from* X *onto* $\operatorname{Im} T$ *with*

$$(3) \qquad P(Y) \subset Y,$$

then the operator $\mathbf{P} : X/Y \to X/Y$ *of the form* $\mathbf{P}(Q(x)) = Q(P(x))$ *is a projection onto the range of* R *with* $\|\mathbf{P}\| \leqslant \|P\|$, *and we additionally have* $\|R^{-1}\| \leqslant \|T^{-1}\| \cdot \|P\|$.

Under these assumptions for T *and* P, *the space* X/Y *contains a complemented copy of* $\ell_\infty(\Gamma)/c_0(\Gamma)$.

Notice that, in contrast to Theorem 1, the subspace Y in Theorem 2 (and the corresponding with Y subspaces in the next two corollaries) may now contain a copy of ℓ_∞ because condition (2) refers to a *fixed* operator T.

Remark 1. From the conditions (2) and (3) it follows that $P(Y) = T(c_0(\Gamma))$, i.e., the restriction $P_{|Y}$ is a projection from Y onto $T(c_0(\Gamma))$. On the other hand, it can be easily checked that if P a projection in X with $P(Y) = T(c_0(\Gamma)) \subset Y$ and $P(X) = T(\ell_\infty(\Gamma))$ then condition (2) is fulfilled, and this allows us to construct a continuous injection R as in part (i) of Theorem 2.

The corollary below follows from part (i) of Theorem 2.

Corollary 3. *Let* $T : \ell_\infty(\Gamma) \to X$ *be an isomorphic embedding that fulfills condition* (2). *Then the space* X/Y *does not possess an equivalent strictly convex norm, and it contains a copy of* $c_0(\Gamma^{\aleph_0})$.

Moreover, if $\operatorname{card}(\Gamma) \geqslant 2^{\aleph_0}$ *then* X/Y *contains a copy of* $\ell_\infty(\Gamma')$ *for some* $\Gamma' \subset \Gamma$ *with* $\operatorname{card}(\Gamma') \geqslant 2^{\aleph_0}$, *and an isometric copy of* ℓ_∞.

The next corollary is an immediate consequence of the preceding corollary and the following observation: if an operator S embeds isomorphically c_0 into a Banach space X, then its second conjugate S^{**} embeds isomorphically ℓ_∞ into X^{**} with $S^{**}(\ell_\infty) \cap \iota(X) = \iota(S(c_0))$, where ι denotes the canonical embedding of X into X^{**}. The corollary is also a completion of [29, Corollary 5] dealing with the quotient $E^{**}/\iota(E)$ for E a Banach lattice.

Corollary 4. *If* X *contains an isomorphic copy of* c_0, *then the quotient space* $X^{**}/\iota(X)$ *does not possess an equivalent strictly convex norm, and it contains a copy of* $c_0(\mathbf{R})$ *and an isometric copy of* ℓ_∞.

It is obvious that the above corollary is essential when X does not contain a complemented copy of c_0 (thus, X must be nonseparable). The examples of such spaces are furnished by $\ell_\infty(\Gamma)$, ℓ_∞/c_0, and $C(K)$-spaces with few operators [13, 20], among others.

The last corollary of this section illustrates part (ii) of Theorem 2 for the spaces $X = L_\infty[0,1]$ and $X = \mathcal{L}_\infty^b[0,1]$, and $Y = C[0,1]$.

Corollary 5. *Let X denote $\mathcal{L}_\infty^b[0,1]$ or $L_\infty[0,1]$, and let Y denote its closed subspace (isometric in the second case to) $C[0,1]$. Then X/Y contains a complemented copy of ℓ_∞/c_0.*

More exactly, there is an isomorphism $R:\ell_\infty/c_0 \to X/Y$ with $\|R\|\cdot\|R^{-1}\| \leqslant 2$ and a projection P from X/Y onto the range of R with $\|P\| = 2$.

3. THE CASE OF BANACH LATTICES

Let us examine now how Theorem 2 works within the class of *real* Banach lattices. For the basic notions and results regarding Banach lattices we refer the reader to the monographs [2] and [17]. For the convenience of the reader we recall some definitions.

In this section, the term "lattice copy" means "both lattice and topological copy", and "lattice-isometric copy" means "both lattice and isometric copy". A linear lattice E is called Dedekind [σ-]complete if every [countable, resp.] subset V of E bounded from above has a supremum $\sup V$ in E; and it is called *super Dedekind complete* if in addition the "sup" of V is attained on a countable subset V_0 of V. If $E = (E, \|\ \|)$ is a Banach lattice, then its topological dual E^* is a Dedekind complete Banach lattice, and the real Banach function spaces (e.g., Orlicz and Marcinkiewicz spaces) are the examples of super Dedekind complete Banach lattices. We recall that for every $x \in E$ we have $\|x\| = \||x|\|$, where $|x|$ denotes the modulus of x, and hence some calculations in E may be done on the positive part E^+ of E. By E_a we denote the *order continuous part* of E, i.e., the largest ideal in E such that the norm restricted to E_a is order continuous: $E_a = \{x \in E\colon\ |x| \geqslant x_s \downarrow 0 \text{ implies } \|x_s\| \to 0\}$. The ideal E_a is both Dedekind complete and norm-closed in E, and it does not contain lattice copies of ℓ_∞ (see [17, Proposition 2.4.10, Corollary 2.4.3]). The Banach lattice E is said to have the *Fatou property* if for every increasing net $(x_i)_{i\in I}$ in E^+ with $x = \sup_{i\in I} x_i$ it follows that $\|x\| = \sup_{i\in I}\|x_i\|$; the examples are furnished by dual Banach lattices [17, Proposition 2.4.19] and some function spaces [26, p. 144]. If E, F are two Banach lattices then an injective operator $T : E \to F$ is called a *lattice isomorphism* provided that $Tx \geqslant 0$ iff $x \geqslant 0$ (equivalently, $|T(x)| = T(|x|)$ for all $x \in E$), and T is called a *lattice-topological isomorphism* provided that it is, additionally, a homeomorphism. An ideal M of E is said to be *order dense* if for every $x \in E^+$ there is $y \in M^+ \setminus \{0\}$ with $y \leqslant x$. For some function spaces E the order continuous part E_a is always proper and order dense in E (see the next section), and hence E/E_a is of infinite dimension.

If M is a norm-closed ideal of a Banach lattice E, then the quotient space E/M, endowed with the quotient norm, becomes a Banach lattice. If in the hypotheses of

Theorem 2 we put $X = E$, $Y = M$, and T a lattice-topological isomorphism then from [29, Proposition 1] we obtain a much stronger conclusion:

(∗) *condition* (2) *alone implies that the operator R in part* (i) *of Theorem* 2 *is a lattice isomorphism with* $\operatorname{Im} R$ *a norm-closed sublattice of E/M* (hence, R is additionally a topological isomorphism) *and* $\|R^{-1}\| \leqslant \|T^{-1}\|$;

in particular,

(∗∗) *if T is a lattice isometry then R is a lattice isometry too.*

Further, from the form of the projection $\mathbf{P}$ in Theorem 2(ii) we obtain that

(∗∗∗) *if P is positive then $\mathbf{P}$ is positive as well.*

From the statements (∗), (∗∗), (∗∗∗) and Remark 1 we immediately obtain a Banach-lattice version of Theorem 2.

Theorem 3. *Let E be a Banach lattice, and let M be a norm-closed ideal of E.*

If $T : \ell_\infty(\Gamma) \to E$ is a lattice-topological isomorphism, and P is a positive projection from E onto its norm-closed sublattice $\operatorname{Im} T$, *with $P_{|M}$ a projection onto $T(c_0(\Gamma)) \subset M$, then*

(i) *the operator R, defined as in part* (i) *of Theorem* 2, *maps $\ell_\infty(\Gamma)/c_0(\Gamma)$ onto a norm-closed sublattice V of E/M with $\|R\| \leqslant \|T\|$ and $\|R^{-1}\| \leqslant \|T^{-1}\|$; moreover, V is the range of a positive projection $\mathbf{P}$ in E/M, defined as in part* (ii) *of Theorem* 2, *with $\|\mathbf{P}\| \leqslant \|P\|$.*

(ii) *In particular, if* $\operatorname{Im} T$ *is positively* 1*-complemented in E then V is positively* 1*-complemented in E/M, and if T is an isometry then V is a lattice-isometric copy of $\ell_\infty(\Gamma)/c_0(\Gamma)$.*

This theorem will be applied to the case when M equals the order continuous part E_a of E. It was shown in [29, Theorems 1 and 2] that the quotient Banach lattice E/E_a contains lattice copies of ℓ_∞/c_0 whenever E is Dedekind σ-complete with E_a order dense in E and $E \neq E_a$. We shall show below these copies are positively complemented in E/E_a.

Let us recall that the assumption $E \neq E_a$, appearing in a few next results, implies (for E Dedekind σ-complete) that E contains a lattice copy of ℓ_∞ (see [2, Theorem 14.9]).

The first corollary of Theorem 3 says generally about the quality of lattice copies of ℓ_∞/c_0 inside E/E_a, strengthening [29, Corollary 3]; its proof is immediate.

Corollary 6. *Let E be a Dedekind σ-complete Banach lattice with $E \neq E_a$ and E_a order dense in E. Then E/E_a contains a positively complemented lattice copy of ℓ_∞/c_0.*

The next theorem is a nontrivial consequence of Theorem 3 and deals with the existence and complementability of lattice-isometric and lattice-almost isometric copies of ℓ_∞/c_0 in E/E_a whenever E possesses the Fatou property. It strengthens the results obtained in [29, Theorem 2]. For clarity and further applications of Theorem 4, we consider only the case $\Gamma = \mathbf{N}$ (similar conclusions, for Γ uncountable, can be obtained by combining the proofs of our Corollary 7 and Theorem 2 in [29]). To shorten the text, we say that a Banach lattice F contains *lattice-almost isometric copies* of another Banach lattice G (see [29, p. 151]) provided that, for every $\varepsilon > 0$ there is a lattice-topological isomorphism T_ε from G onto a sublattice V_ε of F with $\|T_\varepsilon\| \cdot \|T_\varepsilon^{-1}\| < 1+\varepsilon$; and if, additionally, there is a positive projection P_ε from F onto V_ε with $\|P_\varepsilon\| < 1+\varepsilon$, then the copies are said to be *positively-almost* 1*-complemented* in F.

Theorem 4. *Let E be a Dedekind σ-complete Banach lattice with $E \neq E_a$ and E_a order dense in E. Assume also that E has the Fatou property. Then*

(i) *E/E_a contains lattice-almost isometric copies of ℓ_∞/c_0 that are positively-almost* 1*-complemented in E/E_a;*
(ii) *if, additionally, E contains a lattice-isometric copy of ℓ_∞ then E/E_a contains a lattice-isometric and positively* 1*-complemented copy of ℓ_∞/c_0.*

The last result of this section follows from part (ii) of Theorem 4. Since its proof depends on the existence of lattice-isometric copies of ℓ_∞ in E whenever E_a is an M-ideal of E [9, Theorem 3], we refer the reader to Section 5 for a comment on that property.

We recall that a closed subspace Y of a Banach space X is an M-ideal if there is a projection $P : X^* \to X^*$ with range $Y^\perp$ (the annihilator of Y in X^*) such that $\|x^*\| = \|Px^*\| + \|(I-P)x^*\|$ for all x^* in X^*.

Theorem 5. *Let E be a super Dedekind complete Banach lattice with $E \neq E_a$ and E_a order dense in E. If E has the Fatou property and E_a is an M-ideal in E, then E/E_a contains a lattice-isometric and positively* 1*-complemented copy of ℓ_∞/c_0.*

4. APPLICATIONS TO ORLICZ AND MARCINKIEWICZ SPACES

In this section we shall apply the last theorem of the previous section to two concrete Banach function lattices with the Fatou property; similar results can be obtained for other function spaces (see [9, p. 526]).

The first application deals with Orlicz spaces.

Let (Ω, Σ, μ) be a σ-finite measure space, and let $L_0(\mu)$ denote the linear lattice of all (classes of real) μ-measurable functions on Ω. A function $\varphi : [0,\infty) \to [0,\infty)$ is called an Orlicz function if it is convex, continuous, with $\varphi(0) = 0$ and $\varphi \not\equiv 0$. The function φ determines a functional $\varrho_\varphi : L_0(\mu) \to [0,\infty]$ defined by the rule $\varrho_\varphi(f) = \int_\Omega \varphi(|f(\omega)|)\,d\mu(\omega)$. The subspace

$$L_\varphi(\mu) = \{f \in L_0(\mu) : \varrho_\varphi(rf) < \infty \text{ for some } r > 0\}$$

of $L_0(\mu)$ is called an Orlicz space. It is a super Dedekind complete Banach lattice with respect to the *Luxemburg norm* $\|f\|_\varphi := \inf\{t > 0\colon \varrho_\varphi(f/t) \leqslant 1\}$, and its order continuous part $(L_\varphi(\mu))_a$ equals

$$E_\varphi(\mu) = \{f \in L_0(\mu)\colon \varrho_\varphi(rf) < \infty \text{ for all } r > 0\}$$

(see [26, p. 145]). It is known that $L_\varphi(\mu)$ has the Fatou property, that if $L_\varphi(\mu) \neq E_\varphi(\mu)$ (i.e., if φ does not fulfill the so called Δ_2-condition; see e.g. [7, p. 14]), then $E_\varphi(\mu)$ is order dense in $L_\varphi(\mu)$ (cf. [7, Theorem 1.25]), and $L_\varphi(\mu)$ contains a lattice-isometric copy of ℓ_∞ (see the proof of Theorem 1.89 in [7]; cf. [9, p. 526]). It is also known that $E_\varphi(\mu)$ is an M-ideal in $L_\varphi(\mu)$ (see [7, Theorems 1.47 and 1.48]).

Now from Theorem 4(ii) (or, from Theorem 5) we immediately obtain a strengthening of [29, Corollary 6].

Corollary 7. *Let φ be a finite Orlicz function. If $L_\varphi(\mu) \neq E_\varphi(\mu)$, then the quotient Banach lattice $L_\varphi(\mu)/E_\varphi(\mu)$ contains a lattice-isometric and positively 1-complemented copy of ℓ_∞/c_0.*

This result is, in a sense, not surprising because $L_\varphi(\mu)/E_\varphi(\mu)$ is lattice-isometric to a sublattice of a $C(K)$-space for some K compact Hausdorff [26, Theorems 10 and 11].

Let us now consider $L_\varphi(\mu)$ endowed with another (equivalent) norm $\|\ \|^O$, called the *Orlicz norm*: $\|f\|^O := \sup\{\int_\Omega f \cdot g\, d\mu\colon \varrho_{\varphi^*}(g) \geqslant 1\}$, where φ^* is the complementary function of φ (see [7, Theorem 1.38(4)]). The symbol $L_\varphi^O(\mu)$ will denote the Banach space $(L_\varphi(\mu), \|\ \|^O)$; the previous symbol $L_\varphi(\mu)$ will still denote the Orlicz space endowed with the Luxemburg norm. Then we have (see [7, Theorem 1.45], with the same proof for the general case): $(E_{\varphi^*}(\mu))^* = L_\varphi^O(\mu)$. It follows that $L_\varphi^O(\mu)$, as a dual Banach lattice, is a (super Dedekind complete) Banach lattice with the Fatou property. However, if φ is strictly monotone then the Orlicz norm $\|\ \|^O$ is strictly monotone (see [10], i.e., $\|f_1\|^O < \|f_2\|^O$ whenever $0 \leqslant f_1 \leqslant f_2$ and $f_1 \neq f_2$); therefore $L_\varphi^O(\mu)$ cannot contain lattice-isometric copies of ℓ_∞. In this case, from part (i) of Theorem 4 we immediately obtain

Corollary 8. *Let φ be a finite and strictly monotone Orlicz function. If $L_\varphi(\mu) \neq E_\varphi(\mu)$, then the quotient Banach lattice $L_\varphi^O(\mu)/E_\varphi(\mu)$ contains lattice-almost isometric and positively-almost 1-complemented copies of ℓ_∞/c_0.*

The second application of Theorem 5 deals with Marcinkiewicz spaces. Now we restrict our considerations to the function space $L_0 := L_0(I, \mathcal{B}, \lambda)$, where $I = (0, 1)$, λ is the Lebesgue measure on the σ-algebra $\mathcal{B}$ of the Lebesgue measurable subsets of I. If $f \in L_0$ then f^* denotes the *decreasing rearrangement* of f defined by the formula $f^*(t) := \inf\{s > 0\colon m_f(s) \leqslant t\}$, $t > 0$, where m_f is the distribution function of f: $m_f(s) = \lambda\{r \in I\colon |f(r)| > s\}$. Further, let Ψ be a strictly increasing concave function $\Psi\colon [0, 1] \to [0, \infty)$, with Ψ continuous at $0 = \Psi(0)$ (a more general case is considered in [12]). Then the *Marcinkiewicz space* $M(\Psi)$ is the set of all $f \in L_0$ such that the number

$$\|f\|_\Psi = \sup_{t>0} \frac{1}{\Psi(t)} \int_0^t f^* \, d\lambda$$

is finite, and $\| \ \|_\Psi$ is a norm on $M(\Psi)$. It is well known that $(M(\Psi), \| \ \|_\Psi)$ is a (super Dedekind complete) Banach lattice with the Fatou property [3,14]. By $M_0(\Psi)$ we denote a subspace of $M(\Psi)$ consisting of all f satisfying

$$\lim_{t\to 0+} \frac{1}{\Psi(t)} \int_0^t f^* \, d\lambda = 0.$$

The properties of $M_0(\Psi)$ which are useful for our purposes are collected in the lemma below (proper references are given in Section 5).

Lemma 1. (a) *We have* $M_0(\Psi) \neq \{0\}$ *if and only if* $\inf_{t>0} \frac{t}{\Psi(t)} = 0$.
(b) *Let* $M_0(\Psi) \neq \{0\}$. *Then*

(i) $M_0(\Psi)$ *is an order continuous part of* $M(\Psi)$,
(ii) $M_0(\Psi)$ *is order dense in* $M(\Psi)$ *with* $M_0(\Psi) \neq M(\Psi)$, *and*
(iii) $M_0(\Psi)$ *is an* M*-ideal in* $M(\Psi)$.

By way of example, every function $\Psi_p(t) := t^p$, with $0 < p < 1$, fulfills the equivalent condition in part (a) of the lemma, while $\overline{\Psi}(t) = \min\{1/2, t\}$ does not; hence the quotient Banach lattices $M(\Psi_p)/M_0(\Psi_p)$ are nontrivial and of infinite dimension, and $M(\overline{\Psi})/M_0(\overline{\Psi})$ is isometric to $M(\overline{\Psi})$.

From Lemma 1 and Theorem 5 we immediately obtain a somewhat unexpected (in the context of the remark following Corollary 7) information about the structure of $M(\Psi)/M_0(\Psi)$.

Corollary 9. *Let* $M_0(\Psi) \neq \{0\}$. *Then the quotient Banach lattice* $M(\Psi)/M_0(\Psi)$ *contains a lattice-isometric and positively* 1*-complemented copy of* ℓ_∞/c_0.

5. THE PROOFS

We recall that the letters Q and q, respectively, denote the natural quotient mappings $X \to X/Y$ and $\ell_\infty(\Gamma) \to \ell_\infty(\Gamma)/c_0(\Gamma)$, respectively.

By e_γ, e_n, and e_f, respectively, we denote the familiar γth, nth, and fth unit vectors of the spaces $\ell_\infty(\Gamma)$, ℓ_∞, and $\ell_\infty(F)$, respectively, where F is an infinite set (another, in general, than **N** or Γ).

For A an infinite subset of Γ the symbol $\ell_\infty^A(\Gamma)$ will denote the isometric copy of $\ell_\infty(A)$ of the elements of $\ell_\infty(\Gamma)$ with support included in A; the symbol $c_0^A(\Gamma)$ has a similar meaning.

Proof of Theorem 1. We follow partially an idea of the proof of Theorem 1 in [23] (given only for Γ countable; an application in our proof of a Rosenthal theorem makes it more general and simple).

Put $\alpha = \operatorname{card}(\Gamma)$, and let F be a set of the cardinality $\alpha^{\aleph_0}$. By Tarski's theorem (see its proof in [25, p. 121]), there is a class $\{\mathcal{G}_f\colon f \in F\}$ of infinite *countable* subsets of Γ with $\mathcal{G}_{f_1} \cap \mathcal{G}_{f_2}$ finite for $f_1 \neq f_2$.

If $\alpha^{\aleph_0} = \alpha$ (when we consider condition (b)), then there exists a class $\{H_f\colon f \in F\}$ of pairwise disjoint subsets of Γ with $\operatorname{card}(H_f) = \alpha$ for all $f \in F$, and then we define $\Gamma_f := \mathcal{G}_f \cup H_f$. Note that this implies that $\operatorname{card}(\Gamma_f) = \alpha$. We can arrange that

$$(4) \qquad \Gamma_{f_1} \cap \Gamma_{f_2} \quad \text{is finite when } f_1 \neq f_2.$$

Indeed, fixing $f_0 \in F$, we can choose H_{f_0} and apply Tarski's theorem to H_{f_0} obtaining $H_{f_0} = \bigcup_{f\in F} \mathcal{G}_f$. Then the elements of the class $\{\Gamma_f\colon f \in F \setminus \{f_0\}\}$ fulfill condition (4). In the other case that $\alpha^{\aleph_0} > \alpha$ (which can be assumed in condition (a)), when such a class $\{H_f\colon f \in F\}$ does not exist, we define $\Gamma_f := \mathcal{G}_f$, $f \in F$. In this case we also have (4), but now $\operatorname{card}(\Gamma_f) = \aleph_0$.

Now assume that Y contains no copy of $\ell_\infty(\Gamma)$ if $\alpha^{\aleph_0} = \alpha$, and that Y contains no copy of ℓ_∞ otherwise. Using the class $\{\Gamma_f\colon f \in F\}$ introduced above we will show the existence of an isomorphism R from $c_0(F)$ into X/Y. Because $\operatorname{card}(F) = \operatorname{card}(\Gamma^{\aleph_0})$, this would prove our theorem.

Because $\ell_\infty^{\Gamma_f}(\Gamma)$ is isometric to $\ell_\infty(\Gamma)$ if $\alpha^{\aleph_0} = \alpha$, and isometric to ℓ_∞ otherwise, the assumption implies that $T(\ell_\infty^{\Gamma_f}(\Gamma))$ is not contained in Y. Therefore we can find, for every $f \in F$, an element x_f in the unit ball of $\ell_\infty(\Gamma)$ such that $x_f \in \ell_\infty^{\Gamma_f}(\Gamma)$ and $Q(Tx_f) \neq 0$. For the sets $F_n := \{f \in F\colon \|Q(Tx_f)\| \geqslant 1/n\}$ we have $F_n \subset F_{n+1}$, $n = 1, 2, \ldots$, and since $\alpha^{\aleph_0}$ cannot be represented as the sum of an infinite strictly increasing sequence of cardinal numbers [15, Corollary V.8.3], we have $\operatorname{card}(F_{n_0}) = \alpha^{\aleph_0}$ for some n_0. Since the spaces $c_0(F)$ and $c_0(F_{n_0})$ are isometric, without loss of generality we may assume that

$$(5) \qquad \text{the number} \quad a := \inf\bigl\{\bigl\|Q(Tx_f)\bigr\|\colon f \in F\bigr\} \quad \text{is positive.}$$

Let e_f be the fth unit vector of $c_0(F)$. Now we consider the operator R from $c_0(F)$ into X/Y of the form

$$R\Bigl(\sum_{f\in F} t_f e_f\Bigr) = \sum_{f\in F} t_f Q(Tx_f).$$

We shall prove first it is well defined and continuous. To this end, for B a finite subset of F, we define an auxiliary finite (by (4)) set Δ_B by the formula

$$\Delta_B := \bigcup_{f_1\neq f_2,\, f_1, f_2\in B} \Gamma_{f_1} \cap \Gamma_{f_2},$$

and for x_f fixed, $f \in B$, we put $x_f(\Delta_B) := \sum_{\gamma\in\Delta_B} x_f(\gamma)e_\gamma$ (where $x_f(\emptyset) = 0$), and $v_f(B) := x_f - x_f(\Delta_B)$. Then $v_f(B) \in \ell_\infty^{\Gamma_f}(\Gamma) \setminus c_0^{\Gamma_f}(\Gamma)$, and since $x_f(\Delta_B) \in c_0^{\Gamma_f}(\Gamma) \subset c_0(\Gamma)$, we obtain $T(x_f(\Delta_B)) \in Y$ (by (1)), whence

$$(6) \qquad Q\big(Tv_f(B)\big) = Q(Tx_f), \quad f \in B.$$

From the construction of Δ_B it follows that the elements $v_f(B)$ have pairwise disjoint supports (because $\operatorname{supp}(v_f(B)) \subset \Gamma_f \setminus \Delta_B$, for all $f \in B$; see (4)) with $\|v_f(B)\| \leqslant \|x_f\| \leqslant 1$. It implies that

$$(7) \qquad \Big\|\sum_{f\in B} t_f v_f(B)\Big\| \leqslant \max_{f\in B} |t_f|,$$

for all scalars t_f, $f \in B$. From (6) and (7) we obtain

$$\Big\|\sum_{f\in B} t_f Q(Tx_f)\Big\| = \Big\|\sum_{f\in B} t_f Q\big(Tv_f(B)\big)\Big\| \leqslant \|T\| \max_{f\in B} |t_f|,$$

which proves that, for every element $(t_f)_{f\in F}$ of $c_0(F)$, the series $\sum_{f\in F} t_f Q(Tx_f)$ converges in X/Y, and hence the operator R is well defined. It is continuous because $\|R\| \leqslant \|T\|$.

We thus have shown that R maps $c_0(F)$, where $\operatorname{card}(F) = \operatorname{card}(\Gamma)^{\aleph_0}$, into the Banach space X/Y with $\|R(e_f)\| = \|Q(Tx_f)\| \geqslant a > 0$ for all $f \in F$ (by (5)). The result of Rosenthal [22, Theorem 3.4] asserts that in this case there is a subset G of F with $\operatorname{card}(G) = \operatorname{card}(F)$ such that R restricted to $c_0^G(F)$ is an isomorphism. Since $c_0^G(F)$ and $c_0(F)$ are isometric, the latter conclusion on R shows finally that the space X/Y contains a copy of $c_0(F)$. The proof is complete.

Proof of Corollary 1. It is enough to apply the following variant of the well-known theorem of Bessaga and Pełczyński [16, Proposition 2.e.8]: if T maps isomorphically c_0 into X^* then there is an isomorphism S from ℓ_∞ into X^* such that $S(c_0) \subset \operatorname{Im} T$ (see [30, Theorem]).

Proof of Corollary 2. Let Y be a subspace of ℓ_∞ which contains a subspace V isomorphic to c_0. By the theorem of Lindenstrauss and Rosenthal ([16, Theorem 2.f.12(i)]), there is an automorphism S of ℓ_∞ such that $SV = c_0$. If Y (hence SY) does not contain subspaces isomorphic to ℓ_∞ then, by Theorem 1, $\ell_\infty/SY = S(\ell_\infty)/SY$ contains a subspace isomorphic to $c_0(\mathbf{R})$. Finally, ℓ_∞/Y contains a copy of $c_0(\mathbf{R})$.

Proof of Theorem 2. The letters ξ and η will denote arbitrary (fixed) elements of $\ell_\infty(\Gamma)$ and $c_0(\Gamma)$, respectively.

Part (i). Condition (2) implies that the formula on R well defines a mapping from $\ell_\infty(\Gamma)$ into X/Y with the required properties (cf. [29, p. 153]).

Part (ii). It is easy to check that the formula on $\mathbf{P}$ defines a projection in X/Y. Moreover, $\xi + \eta = T^{-1}(T\xi + T\eta) = T^{-1}(P(T\xi + T\eta)) = T^{-1}(PT\xi + y) =$

$T^{-1}(PT\xi + Py)$ for some $y \in P(Y)$ (by (3) and Remark 1). Hence $\|q(\xi)\| \leqslant \|T^{-1}\| \cdot \|P\| \cdot \|T\xi + y\|$. On the other hand, since $T(c_0(\Gamma)) = P(Y)$, the latter inequality holds for all $y \in Y$, whence $\|q(\xi)\| \leqslant \|T^{-1}\| \cdot \|P\| \cdot \|Q(T\xi)\| = \|T^{-1}\| \cdot \|P\| \cdot \|R(q(\xi))\|$. It immediately implies that $\|R^{-1}\| \leqslant \|T^{-1}\| \cdot \|P\|$, as claimed.

Proof of Corollary 3. Let R be the operator defined in Theorem 2. If X/Y had an equivalent strictly convex norm $\| \, \|_0$, say, then the space $W := \ell_\infty(\Gamma)/c_0(\Gamma)$ would possess an equivalent strictly convex norm $||| \, |||$ of the form $|||q(\xi)||| = \|q(\xi)\|_W + \|R(q(\xi))\|_0$, where $\| \, \|_W$ is the natural quotient norm on W, but this is impossible (see [4,18]).

Moreover, since W contains a copy V of $\ell_\infty(\Gamma)$ (see [28, Corollary 1.3]), let us assume for simplicity that $V = \ell_\infty(\Gamma)$. Then for the sets $\Gamma_n := \{\gamma \in \Gamma\colon \|R(e_\gamma)\| \geqslant 1/n\}$ we have $\bigcup_{n=1}^{\infty} \Gamma_n = \Gamma$, and hence, by our assumption (that $\mathrm{card}(\Gamma) \geqslant 2^{\aleph_0}$), there is n_0 such that $\mathrm{card}(\Gamma_{n_0}) \geqslant 2^{\aleph_0}$. By Rosenthal's result [22, Proposition 1.2 and Remark 1 on p. 17], the latter condition implies there is $\Gamma' \subset \Gamma_{n_0}$ with $\mathrm{card}(\Gamma') = \mathrm{card}(\Gamma_{n_0})$ such that the operator R restricted to an isometric copy of $\ell_\infty(\Gamma')$ in V is an isomorphism. Thus, X/Y contains a copy of $\ell_\infty(\Gamma')$ with $\mathrm{card}(\Gamma') \geqslant 2^{\aleph_0}$ indeed. The last assertion of Corollary 3 follows from the fact that every isomorphic copy of $\ell_\infty(\mathbf{R})$ contains an isometric copy of ℓ_∞ (see [19, Corollary on p. 207]).

Proof of Corollary 5. Let $\theta_n = n/(n+1)$, $n = 1, 2, \ldots$, and let (x_n) be a sequence of positive and pairwise disjoint elements of $C[0,1]$ with $1 = x_n(\theta_n) = \|x_n\|$ for all n's.

We first consider the case $X = \mathcal{L}^b_\infty[0,1]$ and $Y = C[0,1]$. The operator $T : \ell_\infty \to \mathcal{L}^b_\infty[0,1]$ of the form

$$(8) \qquad T(t_n) = (p)\sum_{n=1}^{\infty} t_n x_n,$$

where (p) denotes the pointwise sum, is well defined and T is an isometry. Moreover, we have $T(c_0) \subset C[0,1] = Y$, because the series in (8) converges *uniformly* for $(t_n) \in c_0$, and $T(c_0) = [x_n]$ (the norm-closure of $\mathrm{lin}\{x_n\colon n \in \mathbf{N}\}$). Let us now consider the operator P from $\mathcal{L}^b_\infty[0,1]$ onto $\mathrm{Im}\, T$ defined by the formula

$$Px = (p)\sum_{n=1}^{\infty} \big(x(\theta_n) - x(1)\big) x_n.$$

It is easy to check that P is a projection with $\|P\| = 2$. Moreover, if $u \in C[0,1]$ then the series $\sum_{n=1}^{\infty}(u(\theta_n) - u(1))x_n$ is norm-convergent in $Y = C[0,1]$; hence $P(Y) = T(c_0) \subset Y$. By Remark 1, the operators T and P fulfill the assumptions (i) and (ii) of Theorem 2, and hence the required result follows.

For $X = L_\infty[0,1]$, we shall apply both the previous constructions of T and P and a *function lifting* $\phi : L_\infty[0,1] \to \mathcal{L}^b_\infty[0,1]$. We recall that ϕ is a linear mapping preserving multiplication (hence disjointness) with

$$(9) \qquad \|\phi\| = 1 \quad \text{and} \quad \phi S f = f \quad \text{for all } f \in \mathcal{L}^b_\infty[0,1],$$

where $S:\mathcal{L}_\infty[0,1]\to L_\infty[0,1]$ is the natural quotient map (see Section 1), and that such ϕ does exist (see [11, pp. 34–35, 46]; cf. [24, pp. 1140–1141]). Let us put $Y = S(C[0,1])$. Since S restricted to $C[0,1]$ is an isometry preserving disjointness, the operator $\widetilde{T} := ST$ is an isometry from ℓ_∞ into $L_\infty[0,1]$, with $\widetilde{T}(c_0) \subset S(C[0,1]) = Y$. Moreover, by (9), the operator $\widetilde{P} := SP\phi$ is a projection from $X = L_\infty[0,1]$ onto $\operatorname{Im}\widetilde{T}$ (an isometric copy of ℓ_∞) with $\|\widetilde{P}\| = 2$, and $\widetilde{P}(Y) = \widetilde{T}(c_0) \subset Y$. By Remark 1 and Theorem 2, the result holds true also for the case $X = L_\infty[0,1]$ and $Y = S(C[0,1])$.

The remaining proofs deal with positive operators on a Banach lattice E. We recall that in this case it is enough to define an additive and positively homogeneous operator T_0, say, on the cone E^+; then T_0 extends to E to a linear operator T by the formula $T(x) = T_0(x^+) - T_0(x^-)$, $x \in E$ (see [2, Theorem 1.7]).

Proof of Theorem 4. Part (i) depends on the following property which can be derived from the proof of Partington's result [18, Theorem 3]: *If a Banach lattice E contains a lattice copy of ℓ_∞ then E contains lattice-almost isometric copies of* ℓ_∞ (cf. [6, Theorem 3]; we recall that here we only consider *real* Banach lattices). Thus, fixing $\varepsilon > 0$, there is a lattice isomorphism $S_\varepsilon : \ell_\infty \to E$ with

$$(10)\qquad 1/(1+\varepsilon)\sup_{n\geqslant 1}|t_n| \leqslant \big\|S_\varepsilon\big((t_n)\big)\big\| \leqslant \sup_{n\geqslant 1}|t_n| = \big\|(t_n)\big\|_{\ell_\infty},$$

for all $(t_n) \in \ell_\infty$. Let us put $x_n = S_\varepsilon(e_n)$, and $\mathbf{1} = \sup_{n\geqslant 1} e_n$. Since E has the Fatou property, for every $n \in \mathbf{N}$ we can find $u_n \in E_a$ with

$$(11)\qquad 0 \leqslant u_n \leqslant x_n \quad \text{and} \quad \|u_n\| \geqslant 1/(1+\varepsilon)^2.$$

By (11), for every $(t_n) \in \ell_\infty^+$ we obtain

$$(12)\qquad \sup_{m\geqslant 1}\sum_{n=1}^{m} t_n u_n \leqslant \sup_{m\geqslant 1} S_\varepsilon\left(\sum_{n=1}^{m} t_n e_n\right) \leqslant S_\varepsilon\big((t_n)\big) \leqslant \big\|(t_n)\big\|_{\ell_\infty} S_\varepsilon(\mathbf{1})$$

(the suprema exist because E is Dedekind σ-complete). From (10), (11) and (12) we get

$$(13)\qquad 1/(1+\varepsilon)^2\big\|(t_n)\big\|_{\ell_\infty} \leqslant \left\|\sup_{m\geqslant 1}\sum_{n=1}^{m} t_n u_n\right\| \leqslant \big\|(t_n)\big\|_{\ell_\infty},$$

for all $(t_n) \in \ell_\infty$ (because $x_n \wedge x_m = 0$, for all $n \neq m$, and hence, by (11), the elements of the sequence (u_n) are pairwise disjoint; it follows that $|\sum_{n=1}^m t_n u_n| = \sum_{n=1}^m |t_n| u_n$ for all real numbers t_n, $n = 1, 2, \ldots$). From the latter remark, and from (12) and (13) it follows that the formula

$$(14)\qquad T_\varepsilon\big((t_n)\big) := \sup_{m\geqslant 1}\sum_{n=1}^{m} t_n^+ u_n - \sup_{m\geqslant 1}\sum_{n=1}^{m} t_n^- u_n,$$

defines a lattice-topological isomorphism T_ε from ℓ_∞ to E with

$$(15) \qquad \|T_\varepsilon\| \cdot \|T_\varepsilon^{-1}\| \leqslant (1+\varepsilon)^2.$$

From (13) we also obtain that for every $(t_n) \in c_0$ the series $\sum_{n=1}^\infty t_n u_n$ is norm-convergent in E, and hence in the (norm-closed) ideal E_a. Thus,

$$(16) \qquad T_\varepsilon(c_0) \subset E_a.$$

Let (Q_n) be the sequence of positive projections in E of the form $Q_n(x) = \sup\{x \wedge ku_n \colon k \in \mathbf{N}\}$, $x \geqslant 0$ (since E is Dedekind σ-complete, Q_n exists for every n; see [2, Theorem 3.13]), and let (f_n) be a sequence of positive elements of E^* with $f_n(u_m) = \delta_{nm}$ and $\|f_n\| \leqslant (1+\varepsilon)^2$ (the existence of such f_n's follows from (11)). By (12), the operator P_ε defined by the formula

$$(17) \qquad P_\varepsilon(x) := \sup_{m \geqslant 1} \sum_{n=1}^{m} f_n(Q_n x) u_n, \quad x \geqslant 0,$$

is a positive projection in E with

$$(18) \qquad P_\varepsilon(E) = T_\varepsilon(\ell_\infty) \quad \text{and} \quad \|P_\varepsilon\| \leqslant (1+\varepsilon)^2$$

(cf. [27, p. 37]). Moreover, if $0 \leqslant x \in E_a$, then for all n we have $Q_n(x) \leqslant x$ which follows that $Q_n(x) \in E_a$ (because E_a is an ideal of E) and hence, by [2, Theorem 12.13], $\lim_{n\to\infty} \|Q_n x\| = 0$. We thus obtain $P_\varepsilon(E_a) \subset T(c_0)$, but obviously $P_\varepsilon(x) = x$ for all $x \in T_\varepsilon(c_0)$, whence

$$(19) \qquad P_\varepsilon(E_a) = T_\varepsilon(c_0)$$

(cf. [2, Theorem 1.8]). From (16), (18), (19) and part (i) of Theorem 3 we obtain the required result for part (i) of Theorem 4.

Part (ii). Let $S \colon \ell_\infty \to E$ be a lattice isometry, and let $\varepsilon \in (0,1)$ be fixed. We put $x_n = S(e_n)$, $n = 1, 2, \ldots$, and choose positive $u_n \leqslant x_n$ with $1 - \varepsilon/n \leqslant \|u_n\|$. As in the proof of item (i), we find a sequence $(f_n) \subset (E^*)^+$ with $f_n(u_m) = \delta_{nm}$ and $\|f_n\| \leqslant 1/(1-\varepsilon/n)$ for all n's. Let T_ε be the operator defined, for our sequence (u_n), by the above formula (14). Let R be the operator mapping ℓ_∞ into E/E_a defined in item (ii) of Theorem 3, i.e., $R(q(\xi)) = Q(T_\varepsilon(\xi))$, $\xi \in \ell_\infty$. In the proof of Theorem 2 in [29] it has been shown that R is a lattice isometry. It proves the first part of our item (ii).

Further, let us consider the projection P_ε defined for our sequences (u_n) and (f_n) by the formula (17), and let $\mathbf{P}_\varepsilon$ be the positive projection from E/E_a onto the range of R of the form $\mathbf{P}_\varepsilon(Qx) = Q(P_\varepsilon(x))$ (see item (ii) of Theorem 3). We claim that $\mathbf{P}_\varepsilon$ fulfills the second part of item (ii), i.e., $\|\mathbf{P}_\varepsilon\| = 1$; equivalently, for all $x \in E^+$, $w \in E_a$ and $k \in \mathbf{N}$ the following inequality holds

$$(20) \qquad \|\mathbf{P}_\varepsilon(Qx)\| \leqslant \|x + w\|/(1 - \varepsilon/k).$$

The proof of (20) will be based on the following property

(#) *Let E be a Dedekind σ-complete Banach lattice, let (u_n) be a sequence of positive and pairwise disjoint elements of E, and let (t_n), (s_n) be two sequences of real numbers with $t_n \geqslant 0$ for all n's such that $\sup_{m\geqslant 1}\sum_{n=1}^{m} t_n u_n$ exists in E and the series $\sum_{n=1}^{\infty} s_n u_n$ is norm-convergent in E. Then*

$$\left|\sup_{m\geqslant 1}\sum_{n=1}^{m} t_n u_n + \sum_{n=1}^{\infty} s_n u_n\right| = \sup_{m\geqslant 1}\sum_{n=1}^{m} |t_n + s_n| u_n. \tag{21}$$

To prove (21), we shall use the notion of *order convergence* in E. We recall (see [2, p. 30]) that a sequence (a_n) in E is order convergent to an element $a \in E$ (in symbols, $a_n \stackrel{(o)}{\to} a$) whenever there exists a sequence $(v_n) \subset E^+$ with $v_n \downarrow 0$ and $|a_n - a| \leqslant v_n$ for all n's. It is obvious that if $a_n \stackrel{(o)}{\to} a$ and $b_n \stackrel{(o)}{\to} b$ then $a_n + b_n \stackrel{(o)}{\to} a+b$, and hence (by inequality $\||x| - |y|\| \leqslant |x - y|$ for all $x, y \in E$)

$$|a_n + b_n| \stackrel{(o)}{\to} |a + b|. \tag{22}$$

We put $A_m = \sum_{n=1}^{m} t_n u_n$, $A = \sup_{m\geqslant 1} A_m$, $B_m = \sum_{n=1}^{m} s_n u_n$, and $B = \sum_{n=1}^{\infty} s_n u_n$. Then we have $A_m \uparrow A$, whence $A_m \stackrel{(o)}{\to} A$, and $B_m \stackrel{(o)}{\to} B$ because

$$|B_m - B| = \sum_{n=m+1}^{\infty} |s_n| u_n \downarrow 0.$$

By (22) and the remark following (13), we thus obtain

$$\sum_{n=1}^{m} |t_n + s_n| u_n = |A_m + B_m| \stackrel{(o)}{\to} |A + B|.$$

On the other hand, the sequence $(|A_m + B_m|)$ is increasing, and hence $|A + B| = \sup_{m\geqslant 1} |A_m + B_m| = \sup_{m\geqslant 1}\sum_{n=1}^{m} |t_n + s_n| u_n$. The proof of (#) is complete.

Now we shall prove inequality (20). We fix $x \in E^+$ and $w \in E_a$, and we consider the elements $A = P_\varepsilon(x)$ and $B = P_\varepsilon(w)$. We notice first that, by (17), we have here $A = \sup_{m\geqslant 1}\sum_{n=1}^{m} t_n u_n$, where $t_n = f_n(Q_n x) \geqslant 0$ for all n's, and $B = \sum_{n=1}^{\infty} s_n u_n$, where $s_n = f_n(Q_n w)$ for all n's, and that the series defining B is norm-convergent in E_a because $\lim_{n\to\infty} t_n = 0$ (see (16) and (19), and the remark preceding (19)). We define next, for $k = 1, 2, \ldots$, the four elements: $A^{(k)} := \sup_{m\geqslant k}\sum_{n=k}^{m} t_n u_n$, $A_m^{(k)} := \sum_{n=k}^{m} t_n u_n$, $B^{(k)} = \sum_{n=k}^{\infty} s_n u_n$, and $B_m^{(k)} = \sum_{n=k}^{m} s_n u_n$. Since $u_n \in E_a$ for all n's, we have

$$A_m^{(k-1)} \in E_a \quad \text{and} \quad B, B_m^{(k-1)}, B^{(k)} \in E_a \quad \text{for all } m \geqslant k \geqslant 1. \tag{23}$$

Then, by (23), for every k fixed we have:

$$\begin{aligned}\|\mathbf{P}_\varepsilon(Qx)\| &= \inf_{y\in E_a} \|y + P_\varepsilon(x)\| = \inf_{y\in E_a} \|y + A + B\| \\ &= \inf_{y\in E_a} \|y + A^{(k)} + A_1^{(k-1)} + B^{(k)} + B_1^{(k-1)}\|\end{aligned}$$

$$= \inf_{y \in E_a} \left\| y + A^{(k)} + B^{(k)} \right\| \leqslant \left\| A^{(k)} + B^{(k)} \right\|$$
$$= \left\| \left| A^{(k)} + B^{(k)} \right| \right\|.$$

The forms of the elements $A^{(k)}$ and $B^{(k)}$ fulfill the assumptions in (#), whence

$$\text{(24)} \qquad \left| A^{(k)} + B^{(k)} \right| = \sup_{m \geqslant k} \sum_{n=k}^{m} \left| f_n\big(Q_n(x+w)\big) \right| u_n.$$

Since E has the Fatou property and the sums $\sum_{n=k}^{m} |f_n(Q_n(x+w))| u_n$ increase with m (for k fixed), we get the equality

$$\text{(25)} \qquad \left\| \sup_{m \geqslant k} \sum_{n=k}^{m} \left| f_n\big(Q_n(x+w)\big) \right| u_n \right\| = \sup_{m \geqslant k} \left\| \sum_{n=k}^{m} \left| f_n\big(Q_n(x+w)\big) \right| u_n \right\|.$$

Moreover, for all n's we have $\|Q_n\| = 1$ (as $0 \leqslant Q_n x \leqslant x$ for all $x \geqslant 0$), and $\| \sum_{n=k}^{m} u_n \| \leqslant \|S(\mathbf{1})\| = 1$ (as u_n's have been chosen with $0 \leqslant u_n \leqslant x_n$). Hence, from (24) and (25) we obtain further estimations on $\|\mathbf{P}_\varepsilon(Qx)\|$:

$$\text{(26)} \qquad \left\| \mathbf{P}_\varepsilon(Qx) \right\| \leqslant \sup_{m \geqslant k} \Big(\max_{k \leqslant n \leqslant m} \left| f_n\big(Q_n(x+w)\big) \right| \Big) \leqslant \Big(\sup_{m \geqslant k} \|f_m\| \Big) \|x + w\|.$$

Since the functionals f_n's have been chosen with $\|f_n\| \leqslant 1/(1 - \varepsilon/n)$, from (26) we finally obtain (20). The proof of part (ii) of Theorem 4 is complete.

Proof of Theorem 5. Here we apply the result below, due to Hudzik [9, Theorem 3], from which Theorem 5 follows immediately. However, the reader should note that in Hudzik's paper the term "monotone completeness" corresponds to what we call the "Fatou property" (as defined in the Meyer–Nieberg monograph [17]); see also [1, p. 282] for a comment on the name "monotone completeness" which is often called "the Levi property".

Lemma 2. *Let E be a super Dedekind complete Banach lattice with $E \neq E_a$ and E_a order dense in E. If E_a is an M-ideal in E then E contains a lattice-isometric copy of ℓ_∞.*

We shall present a shorter (than in [9]) proof of the lemma. Since E_a is an M-ideal, it is *proximal*, i.e., for every $x \in E$ there is $y \in E_a$ with $\|Q(x)\| = \|x - y\|$ (see [8, Proposition II.1.1]). In particular, there exists $x \in E$ with $\|x\| = 1$ and $\|Q(x)\| = 1$. By [9, Theorem 2], the latter property immediately implies that E contains a lattice-isometric copy of ℓ_∞. □

Proof of Lemma 1. Part (a) is included in [12, Theorem 2.3(i)].

Part (b). Item (i) and the first part of item (ii) are included in [12, Theorem 2.3(ii)]. For a proof of the second part of our item (ii) observe that the function $f = \Psi'$ is laying in $M(\Psi) \setminus M_0(\Psi)$. Item (iii) is included in [12, Theorem 2.4].

ACKNOWLEDGEMENT

The authors thank the referee for remarks and comments which improved the quality of this paper.

REFERENCES

[1] Abramovich Y.A., Wickstead A.W. – When each continuous operator is regular II, Indag. Math. N.S. **8** (1997) 281–294.

[2] Aliprantis C.D., Burkinshaw O. – Positive Operators, Academic Press, New York, 1985.

[3] Bennett C., Sharpley R. – Interpolation of Operators, Academic Press, Boston, 1988.

[4] Bourgain J. – ℓ_∞/c_0 has no equivalent strictly convex norm, Proc. Amer. Math. Soc. **78** (1980) 225–226.

[5] Castillo J.M.F., González M. – Three-space Problems in Banach Space Theory, Springer-Verlag, Berlin, 1997.

[6] Chen J. – The lattice-almost isometric copies of l^1 and l^∞ in Banach lattices, Acta. Math. Acad. Paedagog. Nyházi. (N.S.) **22** (2006) 73–76.

[7] Chen S. – Geometry of Orlicz spaces, Dissertationes Math. **361** (1996).

[8] Harmand P., Werner D., Werner W. – M-Ideals in Banach Spaces and Banach Algebras, Springer-Verlag, Berlin, 1993.

[9] Hudzik H. – Banach lattices with order isometric copies of ℓ_∞, Indag. Math. N.S. **9** (1998) 521–527.

[10] Hudzik H., Kurc W. – Monotonicity properties of Musielak–Orlicz spaces and dominated best approximation in Banach lattices, J. Approx. Theory **95** (1998) 353–386.

[11] Ionescu Tulcea A., Ionescu Tulcea C. – Topics in the Theory of Lifting, Springer-Verlag, Berlin, 1969.

[12] Kamińska A., Lee H.J. – M-ideal properties in Marcinkiewicz spaces, Ann. Soc. Math. Pol., Ser. I, Comment. Math. 2004, Spec. Iss. 123–144.

[13] Koszmider P. – Banach spaces of continuous functions with few operators, Math. Ann. **330** (2004) 151–183.

[14] Krein S.G., Petunin Yu.I., Semenov E.M. – Interpolation of Linear Operators, Amer. Math. Soc. Transactions of Math. Monogr., vol. 54, Providence, 1982.

[15] Kuratowski K., Mostowski A. – Set Theory, Polish Scientific Publishers, Warszawa, 1976.

[16] Lindenstrauss J., Tzafriri L. – Classical Banach Spaces, I, Springer-Verlag, Berlin, 1977.

[17] Meyer-Nieberg P. – Banach Lattices, Springer-Verlag, Berlin, 1991.

[18] Partington J.R. – Subspaces of certain Banach sequence spaces, Bull. London Math. Soc. **13** (1981) 162–166.

[19] Partington J.R. – Equivalent norms on spaces of bounded functions, Israel J. Math. **35** (1980) 205–209.

[20] Plebanek G. – A construction of a Banach space $C(K)$ with few operators, Topology Appl. **143** (2004) 217–239.

[21] Plichko A., Yost D. – Complemented and uncomplemented subspaces of Banach spaces, Extr. Math. **15** (2000) 335–371.

[22] Rosenthal H.P. – On relatively disjoint families of measures, with some applications to Banach space theory, Studia Math. **37** (1970) 13–36.

[23] Rusu Gh. – On certain subspaces of a class of Banach spaces, Bul. Acad. Ştiinţe Repub. Mold. Mat. **33** (2) (2000) 85–91.

[24] Strauss W., Macheras N.D., Musiał K. – Liftings, in: E. Pap (Ed.), Handbook of Measure Theory, Elsevier, 2002, pp. 1131–1184.

[25] Walker R.C. – The Stone–Čech Compactification, Springer-Verlag, Berlin, 1974.

[26] Wnuk W. – On the order-topological properties of the quotient space L/L_A, Studia Math. **79** (1984) 139–149.

[27] Wnuk W. – Banach Lattices with Order Continuous Norms, Polish Scientific Publishers, Warszawa, 1999.

[28] Wójtowicz M. – The lattice-isometric copies of $\ell_\infty(\Gamma)$ in quotients of Banach lattices, Int. J. Math. Math. Sci. **2003** (47) (2003) 3003–3006.
[29] Wójtowicz M. – The lattice copies of $\ell_\infty(\Gamma)/c_0(\Gamma)$ in a quotient of a Banach lattice, Indag. Math. N.S. **16** (2005) 147–155.
[30] Wójtowicz M. – Isomorphic and isometric copies of $\ell_\infty(\Gamma)$ in duals of Banach spaces and Banach lattices, Comment. Math. Univ. Carolinae **47** (2006) 467–471.

(Received January 2006)