Ukrainian Mathematical Journal, Vol. 58, No. 5, 2006

ONE MOMENT ESTIMATE FOR THE SUPREMUM OF NORMALIZED SUMS IN THE LAW OF THE ITERATED LOGARITHM

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UDC 519.21

For a sequence of independent random elements in a Banach space, we obtain an upper bound for moments of the supremum of normalized sums in the law of the iterated logarithm by using an estimate for moments in the law of large numbers. An example of their application to the law of the iterated logarithm in Banach lattices is given.

1. Introduction

A Banach space B is said *to be of type* 2 if there exists a constant $C = C(B)$ such that, for any finite collection of elements $(x_i) \subset B,$ one has

$$\mathbf{E}\left\|\sum \varepsilon_i x_i\right\|^2 \le C \sum \|x_i\|^2,$$

where $\varepsilon_i,\ i \ge 1,$ are symmetric independent Bernoulli random variables. Let B be a Banach space of type 2 and let $x_i \in B.$ In [1], the following estimate was obtained:

$$\mathbf{E}\left(\sup_{n\ge 1} \frac{\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|}{\chi(n)}\right)^2 \le C \sup_{n\ge 1} \frac{\sum_{i=1}^n \|x_i\|^2}{n}, \tag{1}$$

where $C = C(B),\ \chi(t) = (2tLL(t))^{1/2},$ and $L(t) = \max(1, \ln(t)),\ t > 0$ (inequality (1) is not presented in explicit form in [1], but it easily follows from Lemma 2 in [1]).

In [2], this estimate was used in the proof of the order law of the iterated logarithm and somewhat strengthened for $B = R^1,$ namely, it was shown that

$$\mathbf{E}\left(\sup_{n\ge 1} \frac{\left|\sum_{i=1}^n \xi_i\right|}{\chi(n)}\right)^q \le C_q\, \mathbf{E}\left(\sup_{n\ge 1} \frac{\sum_{i=1}^n \xi_i^2}{n}\right)^{q/2}, \tag{2}$$

where $q > 0$ and (ξ_i) is a sequence of independent random variables in R^1 with $\mathbf{E}\xi_i = 0.$

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Translated from Ukrains'kyi Matematychnyi Zhurnal, Vol. 58, No. 5, pp. 653–665, May, 2006. Original article submitted April 17, 2004; revision submitted December 13, 2004.

0041–5995/06/5805–0737

It should be noted that the problem of the estimation of moments of the quantity

$$\sup_{n\geq 1} \frac{\left|\sum_{i=1}^{n} \xi_i\right|}{d_n},$$

where $d_n > 0,$ has been known for many years. It was studied in connection with the investigation of laws of large numbers and the law of the iterated logarithm (see [1, 3–6]).

Answering some remarks of the referee, we dwell upon one interesting historical aspect of the law of the iterated logarithm.

Let a probability space be the segment $[0,1]$ with Lebesgue measure, let $\nu_n(x)$ be an n-digit in the binary decomposition of the number $x,$ and let

$$\kappa_n = \kappa_n(x) = 2\nu_n(x) - 1.$$

Then, almost surely (a.s), one has

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} \kappa_i}{\chi(n)} = 1, \qquad \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} \kappa_i}{\chi(n)} = -1.$$

In this form, the law of the iterated logarithm was first established by Khinchin in 1924. This was an answer to a problem in number theory investigated by some scientists (Hausdorff, Hardy, and Littlewood).

Kolmogorov made the next important step and generalized the law of the iterated logarithm to a broad class of sequences of independent random variables.

Thus, 80 years ago, the law of the iterated logarithm was extended from number theory to probability theory. This fact and the role of the law of the iterated logarithm are underestimated. In our opinion, the Khinchin law of the iterated logarithm is a fundamental law of general mathematical significance.

In the first part of the paper, we show that estimates analogous to (1) and (2) hold for a broad class of normalizing constants. In the second part, we give an example of application of the obtained moment estimates to the order law of the iterated logarithm in Banach lattices.

Recall the corresponding definitions.

A subset A of a Banach lattice B is called *bounded in order* if there exist elements y and z such that $y \leq x \leq z$ for all $x \in A.$ A Banach lattice B is called *σ-complete* if, for any sequence $x_n \in B$ bounded in order, there exist the least upper bound and the greatest lower bound. In a σ-complete Banach lattice, a sequence (x_n) bounded in order has the *least upper and greatest lower (order) bounds* defined by the equalities

$$\limsup_{n\to\infty} x_n = \inf_m (\sup_{n\geq m} x_n) \qquad \text{and} \qquad \liminf_{n\to\infty} x_n = \sup_m (\inf_{n\geq m} x_n),$$

respectively (see [7, p. 504]). Every separable σ-complete Banach lattice is isomorphic to a certain Banach ideal space [8, p. 25]. This isomorphism preserves both the order and the upper (lower) bounds and enables one to reduce the proof of many results for abstract Banach lattices to a special case of a Banach ideal space. For a sequence of random elements X_i in a separable Banach lattice, $\sup_{i<n} X_i$ and $\inf_{i<n} X_i$ are Borel random elements, and, according to the definition given above, we can consider $\limsup_{n\to\infty} X_n$ and $\liminf_{n\to\infty} X_n.$ A q-concave Banach lattice is an important example of a σ-complete lattice (this follows, e.g., from [8, p. 6]). Recall its definition. Let $1 \leq q < \infty.$ A Banach lattice B is called *q-concave* if there exists a constant $D_q = D_q(B)$ such that, for any finite collection of elements $(x_i) \subset B,$ the following inequality holds:

$$\Big(\sum \|x_i\|^q\Big)^{1/q} \le D_q \left\| \Big(\sum |x_i|^q\Big)^{1/q} \right\|.$$

For what follows, we need the notion of mean ψ-deviation of random variables and its generalization to random elements in a Banach lattice. Recall the corresponding definition (for details, see [2]). A continuous function $\psi(t)$ defined on R^1, even, convex, and positive for $t \neq 0$ is called an *N-function* [9, p. 149] if $\lim_{t\to 0} t^{-1}\psi(t) = 0$ and $\lim_{t\to\infty} t^{-1}\psi(t) = +\infty$. For every N-function $\psi(t)$, the *complementary N-function* is defined by the equality

$$\psi^*(t) = \sup_{s\in R^1} (st - \psi(s)).$$

Let a random element X take values in the separable Banach lattice B and let $\psi(t)$ be an N-function. The least upper bound

$$\mathfrak{S}_\psi X = \sup(x \in K_\psi(X)),$$

where $K_\psi(X) = \big\{\mathbf{E}(\eta X)\colon \eta$ is a random variable, $\mathbf{E}\psi^*(\eta) \le 1\big\}$ and $\mathbf{E}X$ is the Pettis integral, is called the *mean ψ-deviation* of X.

In particular, we define the *mean deviation of order* p, $1 < p < \infty$, of a random element X by the relation $\mathfrak{S}_p X = \sup(x \in K_p(X))$, where $K_p(X) = \{\mathbf{E}(\eta X)\colon \mathbf{E}|\eta|^q \le 1\}$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, and the *mean square deviation* simply by $\mathfrak{S}X$. For random variables in the space R^1, this notion coincides with the classical notion of mean deviation of order p; for a Banach ideal space on the measurable space (T, Λ, μ), it coincides with the pointwise deviation $(\mathbf{E}|X(t)|^p)^{1/p}$, $t \in T$, for $\mathbf{E}X = 0$.

Let X be a random element defined on the probability space $(\Omega, \Sigma, \mathbf{P})$ and taking values in the separable Banach lattice B with $\mathbf{E}X = 0$, let (X_i) be a sequence of its independent copies, and let (b_i) be a number sequence. We set $S_n = \sum_{i=1}^n b_i X_i$ and $v_n = \sum_{i=1}^n b_i^2$. The relations

$$\limsup_{n\to\infty} \frac{S_n}{\chi(v_n)} = \mathfrak{S}X, \qquad \liminf_{n\to\infty} \frac{S_n}{\chi(v_n)} = -\mathfrak{S}X \quad \text{a.s.} \tag{3}$$

can be regarded as a natural generalization of the law of the iterated logarithm in R^1 to Banach lattices.

Equalities (3) are called the *order law of the iterated logarithm* in a Banach lattice B. The case $b_i = 1$ was considered in [2].

For Banach lattices and random elements in Banach spaces, we use the definitions and notation from [8–10]; we restrict ourselves to separable spaces in order to avoid complications related to the problem of the measurability of the sum and supremum of two random elements.

2. Estimates for Moments of the Supremum of Normalized Sums

Theorem 1. *Suppose that B is a separable Banach space of type 2, (X_i) is a sequence of independent random elements in B with $\mathbf{E}X_i = 0$, (d_n) is a sequence of positive numbers, $d_n \uparrow \infty$ as $n \to \infty$, and $q \ge 1$. Then*

$$\mathbf{E}\left(\sup_{n\ge 1} \frac{\left\|\sum_{i=1}^n X_i\right\|}{\chi(d_n)}\right)^q \le C\mathbf{E}\left(\sup_{n\ge 1} \frac{\sum_{i=1}^n \|X_i\|^2}{d_n}\right)^{q/2}, \tag{4}$$

where the constant $C = C(B)$ may also depend on q and the sequence (d_n) but does not depend on (X_i).

Let us formulate an auxiliary lemma on the completeness of some normed spaces, which is necessary for the proof of Theorem 1. The proof of the lemma is standard, and we give it only for the completeness of presentation.

Let B be an arbitrary Banach space with norm $\|\cdot\|$, let (c_n) be a fixed sequence of positive numbers, and let $q \geq 1$. On the linear space of sequences (x_i), $x_i \in B$, we introduce the norms

$$\|(x_i)\|_0 = \sup_{n\geq 1}\left\|c_n \sum_{i=1}^{n} x_i\right\|,$$

$$\|(x_i)\|_1 = \sup_{n\geq 1}\left(c_n \sum_{i=1}^{n} \|x_i\|^2\right)^{1/2}, \qquad \text{and} \qquad \|(x_i)\|_2 = \left(\mathbf{E}\sup_{n\geq 1}\left\|c_n \sum_{i=1}^{n} \epsilon_i x_i\right\|^q\right)^{1/q} \tag{5}$$

(the validity of all properties of a norm for the quantities $\|\cdot\|_0 - \|\cdot\|_2$ is verified directly). Consider the normed spaces $B_j = \{(x_i)\colon \|(x_i)\|_j < \infty\}$, $j = 0, 1, 2.$

Lemma 1. *The spaces* $(B_j, \|\cdot\|_j)$, $j = 0, 1, 2,$ *are Banach spaces.*

Proof. *Space* B_0. Of course, for experts in Banach spaces, B_0 resembles an expanding model. We prove its completeness by using the standard procedure. Let $\mathbf{x}_m = (x_1^m, x_2^m, \dots)$ be a fundamental sequence of elements of B_0. By the definition of norm $\|\cdot\|_0$, for any i the sequence (x_i^m) is fundamental in B. Thus, there exists the limit $\lim_{m\to\infty} x_i^m = x_i$. Furthermore, for $\mathbf{x} = (x_i)$, we have

$$\|\mathbf{x} - \mathbf{x}_m\|_0 = \sup_{n\geq 1} c_n \left\|\sum_{i=1}^{n}(x_i - x_i^m)\right\| = \sup_{n\geq 1}\lim_{k\to\infty} c_n \left\|\sum_{i=1}^{n}(x_i^k - x_i^m)\right\|$$

$$\leq \sup_{k\geq m}\sup_{n\geq 1} c_n \left\|\sum_{i=1}^{n}(x_i^k - x_i^m)\right\| = \sup_{k\geq m}\|\mathbf{x}_k - \mathbf{x}_m\|_0 \to 0 \quad \text{as} \quad m \to \infty.$$

This also implies that $\|\mathbf{x}\|_0 < \infty$ and, hence, $\mathbf{x} \in B_0$. Thus, B_0 is a Banach space.

Space B_1. Consider a sequence of spaces $B^1, B^2, \dots$; an element $x^n \in B^n$ has the form $x^n = (x_1^n, \dots, x_n^n)$, $x_i^n \in B$, and the norm is defined by the relation $\|x^n\| = \left(c_n \sum_{i=1}^{n} \|x_i^n\|^2\right)^{1/2}$. We take the space $l_\infty(B^n)$ that consists of (bounded) sequences $\mathbf{x} = (x^1, x^2, \dots)$ with norm $\|\mathbf{x}\| = \sup_n \|x^n\|$. Since every space B^n is a Banach space, $l_\infty(B^n)$ is also a Banach space. The space B_1 is its "diagonal," i.e., it consists of the sequences $\mathbf{x} = (x^1, x^2, \dots)$ for which $x_1^1 = x_1^2 = x_1^3 = \dots$, $x_2^2 = x_2^3 = x_2^4 = \dots$, etc. It is clear that only a "diagonal" element can be the limit of a sequence of "diagonal" elements. Thus, the subspace B_1 is closed and, hence, complete.

Space B_2. For the probability space $(\Omega, \Sigma, \mathbf{P})$, we consider the space $L_q(B_0)$ of vector functions $X(\omega) = (X_1(\omega),\ X_2(\omega), \dots)$ with Ω in B_0 and with norm $\|X\|_q = \left(\int_\Omega \|X(\omega)\|_0^q d\mathbf{P}\right)^{1/q}$. This space is a Banach space because, as shown above, B_0 is a Banach space (see, e.g., [11, p. 92]). The space B_2 is a subspace of the space $L_q(B_0)$, which consists of vector functions $X(\omega) = (\varepsilon_1(\omega)x_1, \varepsilon_2(\omega)x_2, \dots)$ all components of which are independent. Its completeness is verified as follows: It is obvious that only an element of the form $X = (\varepsilon_i x_i)$ can be the limit element of the sequence $X_m = (\varepsilon_i x_i^m)$ in the norm $\|\cdot\|_q$. If $\|X_m - X\|_q \to 0$, then $\varepsilon_i^m \to \varepsilon_i$

in probability for each i. If ε_i^m, $i \geq 1$, are independent for each m, then the limit random variables ε_i, $i \geq 1$, are also independent. Thus, the subspace B_2 is closed and, hence, complete.

The following generalization of the Pisier lemma is the basis of Theorem 1:

Lemma 2. *Suppose that B is a Banach space of type 2, (d_n) is a sequence of positive numbers, $d_n \uparrow \infty$ as $n \to \infty$, and $q \geq 1$. Then, for any sequence $(x_i) \subset B$, one has*

$$\mathbf{E}\left(\sup_{n\geq 1} \frac{\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|}{\chi(d_n)}\right)^q \leq C\left(\sup_{n\geq 1} \frac{\sum_{i=1}^n \|x_i\|^2}{d_n}\right)^{q/2},$$

where the constant $C = C(B)$ is the same as in Theorem 1.

Proof. On the linear space of sequences (x_i), $x_i \in B$, we introduce norms (5) with $\dfrac{1}{d_n}$ instead of c_n in the first case and with $\dfrac{1}{\chi(d_n)}$ instead of c_n in the second case. As a result, we obtain spaces B_1 and B_2, which are complete by virtue of Lemma 1.

Let us show that the condition

$$\|(x_i)\|_1 < \infty \tag{6}$$

implies that

$$\|(x_i)\|_2 < \infty. \tag{7}$$

First, we assume that $\sum_{i=1}^\infty \|x_i\|^2 = \infty$ and use the inequality

$$\limsup_{n\to\infty} \frac{\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|}{\chi\left(\sum_{i=1}^n \|x_i\|^2\right)} \leq C(B) \quad \text{a.s.} \tag{8}$$

For $B = R^1$, this inequality was established in [12] for $C(B) = 1$. A generalization to Banach spaces of type 2 was given in [13] (Corollary 1); more exactly, the following inequality was proved in [13]:

$$\limsup_{n\to\infty} \frac{\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|}{\chi\left(\mathbf{E}\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|^2\right)} \leq \sqrt{e}.$$

To derive inequality (8) from this relation, it is necessary to use type 2 of the space B. Using inequalities (8) and (6), we get

$$\sup_{n\geq 1} \frac{\left\|\sum_{i=1}^n \varepsilon_i x_i\right\|}{\chi(d_n)} < \infty \quad \text{a.s.} \tag{9}$$

Therefore [10, p. 159], inequality (7) is equivalent to the condition

$$\sup_{n\geq 1} \left(\frac{\|x_n\|}{\chi(d_n)}\right)^q < \infty.$$

Hence, the implication (6) $\Rightarrow$ (7) is true if the last inequality holds, which is the case by virtue of the estimates

$$\sup_{n\geq 1}\left(\frac{\|x_n\|}{\chi\left(d_n\right)}\right)^q \leq \sup_{n\geq 1}\left(\frac{\|x_n\|^2}{d_n}\right)^{q/2} \leq \|(x_i)\|_1^q < \infty. \tag{10}$$

For $\sum_{i=1}^{\infty}\|x_i\|^2 < \infty$, the series $\sum_{n=1}^{\infty}\varepsilon_i x_i$ converges almost surely in the norm of B. In this case, inequality (9) is true because, under these conditions, we have $\chi\left(d_n\right) \to \infty$. Further, it is necessary to repeat the chain of inequalities (10).

Coordinate functionals, or, more exactly, functionals of the form $(0,\dots,0,f_i,0,\dots)$, $f_i \in B^*$, are continuous in both norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Hence, there exists an isolated convex topology weaker than both these norms. It remains to use Lemma 3.3 in [14, p. 24], according to which the natural imbedding $B_1 \to B_2$ is bounded.

Proof of Theorem 1. For symmetric random elements (X_i), inequality (4) follows from Lemma 2 and the Fubini theorem. The general case can be reduced to the symmetric one by the standard symmetrization procedure, according to which we take $X_i^{(s)} = X_i - X_i'$, where (X_i') is an independent copy of (X_i). Then

$$\mathbf{E}\left(\sup_{n\geq 1}\frac{\left\|\sum_{i=1}^{n}X_i\right\|}{\chi(d_n)}\right)^q \leq \mathbf{E}\left(\sup_{n\geq 1}\frac{\left\|\sum_{i=1}^{n}X_i^{(s)}\right\|}{\chi(d_n)}\right)^q$$

$$\leq C\mathbf{E}\left(\sup_{n\geq 1}\frac{1}{d_n}\sum_{i=1}^{n}\|(X_i - X_i')\|^2\right)^{q/2} \leq C_1\mathbf{E}\left(\sup_{n\geq 1}\frac{1}{d_n}\sum_{i=1}^{n}\|X_i\|^2\right)^{q/2},$$

where $C_1 = C_1(q,(d_n))$; here, we have used the following well-known inequality for a random element Y with $\mathbf{E}Y = 0$:

$$\mathbf{E}\|Y\|^q \leq \mathbf{E}\|Y - Y'\|^q,$$

where Y' is an independent copy of Y [11, p. 222].

The theorem is proved.

Corollary 1. *Suppose that a separable Banach lattice B is p-concave for certain $1 \leq p < \infty$, (X_i) is a sequence of independent random elements in B with $\mathbf{E}X_i = 0$, (d_n) is a sequence of positive numbers, $d_n \uparrow \infty$ as $n \to \infty$, and $q \geq 1$. Then*

$$\mathbf{E}\left\|\sup_{n\geq 1}\frac{\left|\sum_{i=1}^{n}X_i\right|}{\chi(d_n)}\right\|^q \leq C\,\mathbf{E}\left\|\sup_{n\geq 1}\left(\frac{\sum_{i=1}^{n}|X_i|^2}{d_n}\right)^{1/2}\right\|^q, \tag{11}$$

where $C = C(q,B,(d_n))$.

Proof. We consider only the case $X_i = \varepsilon_i x_i$, where ε_i, $i \geq 1$, are symmetric independent Bernoulli random variables and $x_i \in B$. The transition from this case to the general case repeats the transition from Lemma 2 to Theorem 1.

First, let B be a p-concave Banach ideal space and let $p \le q$. Then B is also q-concave [8, p. 49]. Further, we use the inequality

$$(\mathbf{E}\|Y\|^q)^{1/q} \le D_q \|(\mathbf{E}|Y|^q)^{1/q}\|, \tag{12}$$

where Y is a random element with values in a q-concave Banach ideal space [15].

Using (12) and Lemma 2 in R^1, we get

$$\left(\mathbf{E}\left\|\sup_{n\ge1}\frac{\left|\sum_{i=1}^n \varepsilon_i x_i\right|}{\chi(d_n)}\right\|^q\right)^{1/q} \le D_q \left\|\left(\mathbf{E}\sup_{n\ge1}\left|\frac{\sum_{i=1}^n \varepsilon_i x_i}{\chi(d_n)}\right|^q\right)^{1/q}\right\| \le C(q,B)\left\|\sup_{n\ge1}\left(\frac{\sum_{i=1}^n |x_i|^2}{d_n}\right)^{1/2}\right\|,$$

which proves estimate (11) in this case.

The case $p > q$ follows directly from the inequality $(\mathbf{E}\|Y\|^q)^{1/q} \le (\mathbf{E}\|Y\|^p)^{1/p}$ and the previous estimates proved for p. The transition to abstract q-concave Banach lattices is carried out with the use of the indicated isomorphism between Banach lattices and Banach ideal spaces [8, p. 25].

The corollary is proved.

Remark 1. 1. Inequalities (1), (2), and (4) resemble the classical Khinchin inequality. However, in contrast to the latter, the opposite inequality in (1), (2), and (4) does not hold in the general case for any constant C. Indeed, in inequality (1) for the real axis, we set

$$x_i = \chi(2^{2n}) \quad \text{for} \quad i = 2^{2n} \qquad \text{and} \qquad x_i = 0 \quad \text{for} \quad i \ne 2^{2n}.$$

Then, after elementary calculations, we get

$$\mathbf{E}\left(\sup_{n\ge1}\frac{\left|\sum_{i=1}^n \varepsilon_i x_i\right|}{\chi(n)}\right)^2 \le \left(\sup_{n\ge1}\frac{\sum_{i=1}^n \chi(2^{2i})}{\chi(2^{2n})}\right)^2 \le 4, \qquad \sup_{n\ge1}\frac{\sum_1^n x_i^2}{n} = \infty.$$

2. For the real axis, in the case where $q > 2$, $d_n = n$, $X_i = \xi_i$ are independent copies of the random variable ξ, and $\mathbf{E}\xi = 0$, using the results of [6] we get

$$\mathfrak{S}_q|\xi| \le \left[\mathbf{E}\left(\sup_{n\ge1}\frac{\left|\sum_{i=1}^n \xi_i\right|}{\chi(n)}\right)^q\right]^{1/q} \le C\mathfrak{S}_q|\xi|,$$

$$\mathfrak{S}_q|\xi| \le \left[\mathbf{E}\left(\sup_{n\ge1}\frac{\sum_{i=1}^n {\xi_i}^2}{n}\right)^{q/2}\right]^{1/q} \le C\mathfrak{S}_q|\xi|\,.$$

Thus, in this case, estimate (4) is exact in a certain sense.

3. If we set $q = 2$ in the conditions of case 2, then, for $\hat{\psi}(t) = \dfrac{|t|^2 \ln(1+|t|^2)}{LL(1+t^2)}$, we get

$$\mathfrak{S}_{\hat{\psi}}|\xi| \le C_1 \left[\mathbf{E} \left(\sup_{n\ge 1} \frac{\left|\sum_{i=1}^{n} \xi_i\right|}{\chi(n)} \right)^2 \right]^{1/2} \le C_2 \mathfrak{S}_{\hat{\psi}}|\xi|,$$

and for $\psi(t) = |t| \ln(1+|t|)$ we obtain (see [6])

$$\mathfrak{S}_{\psi}(\xi^2) \le C_1 \mathbf{E} \sup_{n\ge 1} \frac{\sum_{i=1}^{n} {\xi_i}^2}{n} \le C_2 \mathfrak{S}_{\psi}(\xi^2),$$

i.e., in this case, (4) is a fairly rough upper bound.

4. In connection with case 1, the following question arises: Is it possible to strengthen inequality (4) by replacing d_n on its right-hand side by $d_n f(n)$, where $f(n) \to \infty$ as $n \to \infty$? The example presented below shows that this is not true in the space R^1. We fix a natural m, take $q = 1$ and $d_n = n$, and set $\xi_i = \varepsilon_i a_i$, where

$$a_i = \begin{cases} 0 & \text{for } i \le m, \\ 1 & \text{for } i > m. \end{cases}$$

Then we obtain the following upper bound for the square of the right-hand side of (4):

$$\sup_{n\ge 1} \frac{\sum_{i=1}^{n} a_i^2}{n f(n)} = \sup_{n \ge m} \frac{n-m}{n f(n)} \le \frac{1}{f(m)}. \tag{13}$$

The corresponding lower bound for the left-hand side has the form

$$\mathbf{E} \sup_{n\ge 1} \frac{\left|\sum_{i=1}^{n} \varepsilon_i a_i\right|}{\chi(n)} = \mathbf{E} \sup_{n>m} \frac{\left|\sum_{i=m+1}^{n} \varepsilon_i a_i\right|}{\chi(n)} \ge \mathbf{E} \sup_{n>2m} \frac{\left|\sum_{i=m+1}^{n} \varepsilon_i a_i\right|}{\chi(n-m)} \frac{\chi(n-m)}{\chi(n)} \ge \frac{1}{2} - \delta \tag{14}$$

for certain $\delta > 0$. The last inequality holds because, for sufficiently large m and $n > 2m$, one has

$$\frac{\chi(n-m)}{\chi(n)} \ge \frac{1}{2} - \delta,$$

and, by virtue of the law of the iterated logarithm, for $k = n - m$, we get

$$\limsup_{k\to\infty} \frac{\left|\sum_{i=1}^{k} \varepsilon_i\right|}{\chi(k)} = 1.$$

Estimates (13) and (14) give a negative answer to the question posed.

5. It is of interest to estimate the constant C in inequality (4). Simple arguments show that it can be taken in the form that depends only on a constant of type 2 of the space B and on q and (d_n).

3. Order Law of the Iterated Logarithm for Weighted Sums of Independent Identically Distributed Random Variables

First, we consider two auxiliary lemmas. In the first lemma, an important class of functions is considered. We say that a function $\psi(x)$ satisfies *condition (ID)* (see [16]) if the following assertions are true:

(i) it is strictly increasing on $[0,\infty)$ and

$$\sup_{t>0}\frac{\psi\left(t+1\right)}{\psi\left(t\right)}<\infty;$$

(ii) for $t\geq 1,$ $\psi^{-1}(t)$ is absolutely continuous, and its derivative $\gamma(t)$ satisfies the following relation for $t\geq 0,$ $s\geq 1,$ and certain finite θ and $C>0$:

$$\gamma\left(st\right)\leq Cs^{\theta}\gamma\left(t\right).$$

For example, the functions t^{α} and $e^{\alpha t},$ $\alpha>0,$ satisfy condition (ID) for $\theta=\dfrac{1-\alpha}{\alpha}$ and $\theta=0,$ respectively (if the function $\psi(t)$ is convex, then we can take $\theta=0;$ see [16]).

Lemma 3 [16]. *Suppose that (ξ_i) is a sequence of independent identically distributed random variables with expectation zero and $P\left(n,\varepsilon\right):=\mathbf{P}\left\{\left|\sum_{i=1}^{n}\xi_i\right|>n\varepsilon\right\}.$ If the sequence (n_k) satisfies the condition*

$$\lim_{k\to\infty}\frac{n_k}{\psi\left(\lambda k\right)}=1\quad \textit{for certain}\quad \lambda>0,$$

and the function $\psi(t)$ satisfies condition (ID), then the following relations are equivalent:

(i) $\forall\varepsilon>0\ \sum_{k}P\left(n_k,\varepsilon\right)<\infty;$

(ii) $\mathbf{E}h\left(\xi_i\right)<\infty,$ *where* $h\left(t\right)=\displaystyle\int_0^t s\gamma\left(s\right)ds.$

Assume that $v_n=\sum_{i=1}^{n}b_i^2,$ the sequence (b_n) satisfies the condition

$$b_n^2\uparrow,\quad \frac{v_n}{b_n^2}\uparrow\quad \text{as}\quad n\to\infty, \tag{15}$$

and the indices $m_n\uparrow\infty$ are chosen so that, for certain $1<q<Q,$ we have

$$q\leq\frac{v_{m_{n+1}}}{v_{m_n}}\leq Q \tag{16}$$

[under condition (15), these indices always exist (see [17, p. 330])].

The lemma below is a certain generalization of Gaposhkin's result [18] to weighted sums of identically distributed independent random variables.

Lemma 4. *For the sequence* (ξ_n) *of independent copies of a random variable* ξ *with* $\mathbf{E}\xi = a,$ *a sequence* (b_n) *that satisfies condition (15),* m_n *defined by (16), and* $\overline{m}_n = m_n - m_{n-1},$ *the following relations are equivalent:*

(i) $\sum_n \mathbf{P}\left\{\left|\frac{1}{\overline{m}_n}\sum_{i=1}^{\overline{m}_n}\xi_i - a\right| > \varepsilon\right\} < \infty \ \ \forall\varepsilon > 0;$

(ii) $\frac{1}{m_n}\sum_{i=m_{n-1}+1}^{m_n}\xi_i \to a$ *a.s.;*

(iii) $\frac{1}{v_n}\sum_{i=1}^{n} b_i^2\xi_i \to a$ *a.s.*

Proof. The equivalence of relations (i) and (ii) is well known [17, p. 330].

Let us prove the equivalence (ii) $\Leftrightarrow$ (iii) $\big($in fact, we need the implication (ii) $\Rightarrow$ (iii)$\big)$. Under our assumptions, it follows from the results of [18] that, as $n \to \infty,$ one has

$$\frac{1}{\overline{m}_n}\sum_{i=m_{n-1}+1}^{m_n}\xi_i - \alpha_n \to 0 \quad \text{a.s.} \tag{17}$$

for a certain nonrandom α_n if and only if

$$\frac{1}{v_n}\sum_{i=1}^{n} b_i^2\xi_i - \beta_n \to 0 \quad \text{a.s.} \tag{18}$$

for a certain nonrandom β_n.

Assume that relation (ii) is true. Then relation (17) holds, and, hence, (18) is also true for a certain β_n. If $\beta_n \nrightarrow a,$ then there exists a subsequence $\beta_{n_k} \to a + \delta,\ \ \delta \neq 0$ (δ may be equal to ∞).

Recall that a sequence of random variables (ζ_n) is called *jointly absolutely integrable* if

$$\forall\varepsilon > 0\ \ \exists\delta > 0\ \ \forall n\colon \quad \mathbf{P}(A) < \delta \ \Rightarrow \ \int_A |\zeta_n(\omega)|d\mathbf{P} < \varepsilon.$$

It is easy to verify that the sequence $\frac{1}{v_{n_k}}\sum_{i=1}^{n_k} b_i^2\xi_i$ is jointly absolutely integrable. Therefore, by virtue of the Vitali theorem [19, p. 144], we get

$$a = \mathbf{E}\left(\frac{1}{v_{n_k}}\sum_{i=1}^{n_k} b_i^2\xi_i\right) \to a + \delta,$$

which is impossible. Therefore, relation (iii) is true.

The proof of the implication (iii) $\Rightarrow$ (ii), in fact, repeats the previous reasoning without changes.

The lemma is proved.

We can now formulate a theorem on the order law of the iterated logarithm.

Let X be a random element in a Banach lattice B. Assume that there exist a random variable $\tau \in L_2(\Omega)$ and an element $u \in B_+$ for which

$$|X| \leq \tau u \quad \text{a.s.} \tag{19}$$

By virtue of [2], this condition is sufficient for the existence of the mean square deviation $\mathfrak{S}X.$

We impose the following restrictions on the sequence $(b_n):$

$$b_n^2 = o\left(\frac{v_n}{LL\left(v_n\right)}\right) \tag{20}$$

and

$$\lim_{n\to\infty} \frac{\bar{m}_n}{\psi\left(n\right)} = 1, \tag{21}$$

where $\psi(x)$ is a certain function that satisfies condition (ID) (see Lemma 3), where the sequence $(\bar{m}_n)$ is defined in Lemma 4.

Theorem 2. *Suppose that a separable Banach lattice B is q-concave for certain $q < \infty,$ a symmetric random element X in B satisfies condition (19), and $X_n,\ n \geq 1,$ are its independent copies. Also assume that a sequence (b_n) satisfies conditions (15), (20), and (21) and*

$$\mathbf{E}h\left(\tau^2\right) < \infty \tag{22}$$

for the function

$$h\left(t\right) = \int\limits_0^t s\left(\psi^{-1}(s)\right)' ds.$$

Then the law of the iterated logarithm (3) is true.

Remark 2. In the general case, condition (22) cannot be weakened. Indeed, for $b_i = 1,\ v_n = n,\ \bar{m}_{n+1} = m_n = 2^n,$ and $\psi(t) = 2^t,$ we have $h(t) = \dfrac{t}{\ln 2}$ in $R^1,$ and inequality (22) means that $\mathbf{E}\tau^2 < \infty.$ Thus, we obtain the Hartman–Wintner theorem [17, p. 381]. It is well known that, in this classical case, the condition $\mathbf{E}\tau^2 < \infty$ is also necessary.

Proof of Theorem 2. We prove only the first equality in the law of the iterated logarithm (3) (the second one is proved by analogy). Taking into account the indicated order isomorphism between Banach lattices and a Banach ideal space [8, p. 25], we restrict ourselves to the case where B is a Banach ideal space on the measurable space $(T, \Lambda, \mu).$ Moreover, we can assume that $\mu(T) = 1.$ Then $X = (X(t), t \in T) \in B$ almost surely and $\mathfrak{S}X = (\sigma(t), t \in T) \in B.$ For $m \geq 1,$ we set

$$U_m = \left(U_m(t) = \sup_{n\geq m} \frac{S_n(t)}{\chi(v_n)},\ t \in T\right).$$

It is easy to show that

$$\|U_1\| < \infty \quad \text{a.s.} \tag{23}$$

and

$$\mu\{t \in T\colon \lim_{m\to\infty} U_m(t) = \sigma(t)\} = 1 \quad \text{a.s.} \tag{24}$$

Relations (23) and (24) guarantee that the general conditions of order convergence in a Banach ideal space [9, p. 368] are satisfied.

First, we prove inequality (23). Since the random element X is symmetric, it can be represented in the form $X = \varepsilon \hat{X}$, where ε and $\hat{X}$ are independent, $\hat{X}$ is a copy of X, and ε is a symmetric Bernoulli random variable. Relation (23) holds if

$$\mathbf{E}_{\hat{X}} \|U_1\|^q < \infty \quad \text{a.s.}, \tag{25}$$

where $\mathbf{E}_{\hat{X}}(\eta)$ denotes the mathematical expectation of the random variable η for a fixed sequence $(\hat{X}_i)$. Let us prove inequality (25). Using Corollary 1 and taking (19) into account, we get

$$\begin{aligned}(\|U_1(t)\|^q)^{1/q} &\le C\mathbf{E}_{\hat{X}} \left\| \sup_{n\ge 1} \left(\frac{1}{v_n} \sum_{i=1}^{n} b_i^2 \hat{X}_i^2\,(t) \right)^{1/2} \right\| \\ &\le C \left\| \sup_{n\ge 1} \left(\frac{1}{v_n} \sum_{i=1}^{n} b_i^2 \hat{X}_i^2\,(t) \right)^{1/2} \right\| \le C \sup_{n\ge 1} \left(\frac{1}{v_n} \sum_{i=1}^{n} b_i^2 \tau_i^2 \right)^{1/2} \|u\| \quad \text{a.s.},\end{aligned} \tag{26}$$

where τ_i are independent copies of the random variable τ in condition (19).

By virtue of conditions (21) and (22) and Lemma 3, we have

$$\sum_n \mathbf{P} \left\{ \left| \frac{1}{\overline{m}_n} \sum_{i=1}^{\overline{m}_n} \tau_i^2 - \mathbf{E}\tau^2 \right| > \varepsilon \right\} < \infty \quad \forall \varepsilon > 0.$$

According to Lemma 4, this inequality is equivalent to the relation

$$\frac{1}{v_n} \sum_{i=1}^{n} b_i^2 \tau_i^2 \to \mathbf{E}\tau^2 \quad \text{a.s.}$$

Hence,

$$\sup_{n\ge 1} \frac{1}{v_n} \sum_{i=1}^{n} b_i^2 \tau_i^2 < \infty \quad \text{a.s.}$$

This inequality, together with (26), yields (25).

Let us show that condition (24) is also satisfied. To this end, we prove that, almost everywhere on T, the law of the iterated logarithm holds in R^1:

$$\limsup_{n\to\infty} \frac{S_n(t)}{\chi\,(v_n)} = \sigma(t) \quad \text{a.s.} \tag{27}$$

Then, using (27) and the Fubini theorem, we immediately obtain condition (24).

We fix $t \in T$. Since b_n^2 increases and the random variables $X_n(t)$ are identically distributed, we have

$$b_n^2 X_n^2\,(t) \nrightarrow 0$$

whence

$$\sum_{i=1}^{\infty} b_i^2 X_i^2(t) = \infty \quad \text{a.s.}$$

By analogy with (8), we get (see also [12])

$$\limsup_{n\to\infty} \frac{|S_n(t)|}{\chi\left(\sum_{i=1}^{n} b_i^2 X_i^2(t)\right)} \le 1 \quad \text{a.s.} \tag{28}$$

By virtue of (19), almost everywhere on T we have $|X_i(t)| \le \tau_i u(t)$ a.s. Therefore, the random variables $X_i(t)$ satisfy condition (22). Then, as in the case of $\tau_i,$ using Lemmas 3 and 4 we get

$$\frac{1}{v_n} \sum_{i=1}^{n} b_i^2 X_i^2(t) \to \sigma^2(t) \ \text{ a.s.} \quad \text{as} \quad n \to \infty.$$

With regard for (28), the last relation yields

$$\limsup_{n\to\infty} \frac{|S_n(t)|}{\chi(v_n)} \le \sigma(t) \quad \text{a.s.} \tag{29}$$

Further, let $I(A)$ denote the random variable equal to 1 for $\omega \in A$ and 0 otherwise. We use the equality

$$\forall \varepsilon > 0: \quad \lim_{n\to\infty} \frac{1}{v_n} \sum_{i=1}^{n} b_i^2 \mathbf{E}\left(X_i^2(t) I\left\{b_i^2 X_i^2(t) > \varepsilon \frac{v_i}{LL(v_i)}\right\}\right) = 0,$$

which follows from condition (20) and the fact that the random variables $X_i(t)$ are identically distributed. It is known [20] that this equality yields the following lower bound in the law of the iterated logarithm:

$$\limsup_{n\to\infty} \frac{|S_n(t)|}{\chi(v_n)} \ge \sigma(t) \quad \text{a.s.}$$

The last relation and (29) mean that equality (27) is satisfied, and, hence, the law of the iterated logarithm (3) is true.

The theorem is proved.

Using Theorem 2 and known results concerning the law of large numbers and the law of the iterated logarithm in R^1, we obtain several corollaries, which give simple sufficient conditions for the validity of the law of the iterated logarithm (3). It is worth noting that, in (i) and (ii), the condition of the symmetry of a random element X is replaced by the condition $\mathbf{E}X = 0$.

Corollary 2. *Suppose that a separable Banach lattice B is q-concave for certain $q < \infty$, a random element X satisfies inequality (19) in B, and (X_n) is a sequence of independent copies of X. Assume that one of the following conditions is satisfied:*

(i) $\mathbf{E}X = 0$ *and the sequence* (b_n) *satisfies the conditions*

$$b_n^2 \uparrow, \quad b_n^2 = O\left(\frac{v_n}{n}\right);$$

(ii) $\mathbf{E}X = 0,$ *the sequence* (b_n) *satisfies condition (15), and, for certain* $p > 1,$ *one has*

$$b_n^2 = O\left(v_n \left(\ln v_n\right)^{\frac{1}{1-p}}\right), \quad \mathbf{E}\tau^{2p} < \infty;$$

(iii) the random element X *is symmetric, the sequence* (b_n) *satisfies conditions (15) and (20), and, for certain* $\lambda > 0,$ *one has*

$$\mathbf{E}\exp(\lambda\tau^2) < \infty.$$

Then the order law of the iterated logarithm (3) is true.

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