Anatolij Plichko

PROOF OF MILMAN'S THEOREM ON EXTENSION OF M-BASIC SEQUENCE

Abstract. We prove Milman's theorem on the extension, in a given direction, of M-basic sequence to M-basis in a separable Banach space.

Keywords: quasicomplement, Markushevich basis.

Mathematics Subject Classification: 46B15.

Let X be a real separable Banach space and X^* its dual. A subset $F \subset X^*$ is *total* on a subspace $Y \subset X$ if for every $y \in Y$, $y \neq 0$, there is an f in F such that $f(y) \neq 0$. A biorthogonal system $(x_n, f_n)_1^\infty$, $x_n \in X$, $f_n \in X^*$ is said to be *a Markushevich basis* (*M-basis*) if the closed linear span $[x_n]_1^\infty = X$ and (f_n) is total on X. In this case, we also call the sequence (x_n) an M-basis, because f_n are determined uniquely. Closed subspaces Y and Z of X are *quasicomplemented* if $Y \cap Z = 0$ and the closure $\overline{Y+Z} = X$. The subspaces Y and Z are quasicomplemented if and only if for their annihilators there is $Y^\perp \cap Z^\perp = 0$ and $\overline{Y^\perp + Z^\perp}^* = X^*$, where $\overline{\circ}^*$ stands for weak* closure. In [1], Theorem 1.8, the following theorem is stated.

Theorem 1. *Let Y and Z be closed quasicomplemented subspaces of a separable Banach space X. Let $(y_n, \hat{g}_n)$ be an M-basis in Y. Then there exists a sequence $(z_n) \subset Z$ such that $(y_n) \cup (z_n)$ is M-basis in X.*

I. Singer ([2], p.234) noted that Theorem 1 is not valid under the additional condition $[z_n]_1^\infty = Z$. We present a complete proof of Theorem 1. A sketch of the proof was published in ([2], p. 860). For another proof of Theorem 1, see [3]. We use the same symbol to state for an element $\hat{g} \in Y^*$ and its preimage under the quotient map $X^* \to X^*/Y^\perp = Y^*$, hoping that this does not lead to misunderstanding. We also consider elements of X as functionals on X^* and denote by $G^\top$ the annihilator of subset $G \subset X^*$ in X.

Lemma 1. *Under the conditions of Theorem 1, there are representatives $g_n \in \hat{g}_n$ for which*

$$\overline{[g_n]_1^\infty + Z^\perp}^* \cap Y^\perp = 0 \,. \tag{1}$$

Proof. Since X is separable, one can present $Y^\perp \backslash \{0\}$ as a union of convex weakly* compact sets K_n: $Y^\perp \backslash \{0\} = \cup_n K_n$.

Let us construct elements $x_n \in X$ and representatives $g_n \in \hat{g}_n$ so that for every n:

a) x_n separates $G_{n-1} := [g_i]_1^{n-1} + Z^\perp$ and K_n,

b) the restriction $x_n|_{Y^\perp} \notin [x_i|_{Y^\perp}]_1^{n-1}$,

c) $G_n \subset ([x_i]_1^n)^\perp$ and

d) $G_n \cap Y^\perp = 0$.

Start from $n = 1$. Let us separate, by the Hahn-Banach theorem, the weakly* closed subspace $Z^\perp$ and K_1 by a functional $x_1 \in X$, and consider two cases.

1) $\hat{g}_1 \cap Z^\perp \neq \oslash$.
Take, as g_1, any element of this intersection. Then $G_1 \subset x_1^\perp$ and $G_1 \cap Y^\perp = 0$.

2) $\hat{g}_1 \cap Z^\perp = \oslash$.
Then

$$[\hat{g}_1] \cap Z^\perp = 0 \,. \tag{2}$$

The intersection $x_1^\perp \cap [\hat{g}_1]$ cannot contain elements of $Y^\perp$ only, because then $x_1(Y^\perp + Z^\perp) \equiv 0$, hence $x_1 = 0$. Therefore, there exists $g_1 \in x_1^\perp \cap [\hat{g}_1]$, $g_1 \notin Y^\perp$. Then $G_1 \subset x_1^\perp$ and, by (2) , $G_1 \cap Y^\perp = 0$.

Let the collections $(x_i)_1^{n-1}$ and $(g_i)_1^{n-1}$ with conditions a)–d) be constructed. Using condition d), separate the (weakly* closed) subspace G_{n-1} and weakly* compact set K_n by a functional $x \in X$: $\inf\{x(f) : f \in K_n\} = a > 0$ and

$$x(G_{n-1}) \equiv 0 \,. \tag{3}$$

If $x|_{Y^\perp} \notin [x_i|_{Y^\perp}]_1^{n-1}$, put $x_n = x$. In the opposite case, choose $z \in G_{n-1}^\top$ with $\sup\{z(f) : f \in K_n\} < a/2$ and $z|_{Y^\perp} \notin [x_i|_{Y^\perp}]_1^{n-1}$ (of course, the subspaces Y and Z are assumed to be infinite-dimensional). Put $x_n = x + z$. Obviously, for x_n conditions a) and b) are satisfied.

As for $n = 1$, let us consider two cases.

1) $\hat{g}_n \cap G_{n-1} \neq \oslash$.
Take, as g_n, any element of this intersection. The verification of conditions c), d) is trivial.

2) $\hat{g}_n \cap G_{n-1} = \oslash$.
Then

$$[\hat{g}_i]_1^n \cap Z^\perp = 0 \,. \tag{4}$$

The intersection $([x_i]_1^n)^\perp \cap [\hat{g}_i]_1^n$ cannot contain elements of $[\hat{g}_i]_1^{n-1}$ only, because in this case, $([x_i]_1^n)^\perp$, which cuts out from $Y^\perp$ a subspace of codimension n (condition b)), shall cut out from $[\hat{g}_i]_1^n$ a subspace of codimension $n+1$ (since $(y_n, \hat{g}_n)$ is M-basis, $\hat{g}_n \notin [\hat{g}_i]_1^{n-1}$!). It is impossible.

Take an element

$$g_n \in ([x_i]_1^n)^\perp \cap [\hat{g}_i]_1^n \ , \tag{5}$$

$g_n \notin [\hat{g}_i]_1^{n-1}$. Since $(g_i)_1^{n-1} \subset ([x_i)]_1^n)^\perp$, we can assume $g_n \in \hat{g}_n$.

Condition c) follows from (3) and (5); condition d) follows from (4).

Therefore, the elements with conditions a)–d) are constructed. Condition c) implies that $\overline{[g_n]_1^\infty + Z^\perp}^* \subset ([x_n]_1^\infty)^\perp$. This and a) imply (1). $\square$

The Proof of Theorem 1. Let (g_n) be the sequence from Lemma 1 and $Z_0 = ([g_n]_1^\infty + Z^\perp)^\top$. By (1), the subspaces Y and Z_0 are quasicomplemented. In the standard way ([2], p.224), choose an M-basis $(\hat{z}_n, h_n)$, in X/Y, $\hat{z}_n \in X/Y$, $h_n \in (X/Y)^* = Y^\perp$ such that there are representatives $z_n \in \hat{z}_n \cap Z_0$ with $[z_n]_1^\infty = Z_0$. Since $[y_n]_1^\infty = Y$, $[(y_n) \cup (z_n)] = X$. Since $((h_n)_1^\infty)^\top = Y$ and (g_n) is total on Y, $(g_n) \cup (h_n)$ is total on X. For every n, $g_n \in Z_0^\perp$ and $h_n \in Y^\perp$. Hence, our system is biorthogonal. $\square$

REFERENCES

[1] Milman V. D., *Geometric theory of Banach spaces. Part I: The theory of basis and minimal systems*, Russ. Math. Surv. **25** (1970) 3, 111–170.

[2] Singer I., *Bases in Banach spases, II*, Berlin e.a., Springer–Verlag 1981.

[3] Terenzi P., *On bounded and total biorthogonal systems spanning given subspaces.* Rend. Accad. Naz. Lincei. **67** (1979), 1–11.

Anatolij Plichko
aplichko@usk.pk.edu.pl

Cracow University of Technology
Institute of Mathematics
ul. Warszawska 24, 31-155 Cracow, Poland

Received: March 11, 2005.