SOME PROPERTIES OF THE SET OF FUNCTIONALS WHICH ATTAIN THEIR SUPREMUM ON THE UNIT SPHERE

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A continuous linear functional $f(x) \in E'$ is said to attain its supremum on the unit sphere of the space if there exists an element $x \in E$ such that

$$\|x\| = 1, \quad f(x) = \|f\|.$$

We denote by $\mathfrak{M}$ the set of all functionals of E' which attain their supremum on the unit sphere. If the space E is nonreflexive, set $\mathfrak{M}$ is in general a nonlinear entity with a complicated structure, and investigation of the properties of $\mathfrak{M}$ involves obvious difficulties. At present, not much information is available on the structure of $\mathfrak{M}$; we only know that it is everywhere dense in the strong topology of the conjugate space [1]. It should also be mentioned that functionals which attain their supremum on the unit sphere play an important role in theory of approximation and interpolation in Banach space [2, 3], in view of which, some results on the properties of such functionals are described in the articles quoted. In the present article we discuss some facts concerning the structure, both of $\mathfrak{M}$ itself, and of certain special subsets of it.

1. Let $S_1(E')$ be the unit sphere of the conjugate space E'.

<u>THEOREM 1.</u> The set

$$\mathfrak{M}_1 = \mathfrak{M} \cap S_1(E')$$

is connected in the weak topology $\sigma(E', E)$.

<u>Proof.</u> Assume that there exist two sets A and B, open in the weak topology $\sigma(E', E)$, nonintersecting, and such that

$$A \cup B \supset \mathfrak{M}_1, \quad A \cap \mathfrak{M}_1 \neq \varnothing, \quad B \cap \mathfrak{M}_1 \neq \varnothing.$$

Let $f_0 \in A \cap \mathfrak{M}_1$, $f_1 \in B \cap \mathfrak{M}_1$, and let x_i be the element of the unit sphere of space E on which f_i attains the norm ($f_i(x_i) = 1$, $i = 0, 1$). We shall show that the functionals f_i can be so chosen that $x_0 \neq -x_1$. For, let us first take as f_1 an arbitrary functional of $A \cap \mathfrak{M}_1$; assume that, for any $f \in B \cap \mathfrak{M}_1$ $f(-x_0) = \|f\| = 1$; then f will belong to the hyperplane $M_{-x_0} = \{f : f(-x_0) = 1,\ f \in E'\}$, and hence $B \cap \mathfrak{M}_1 \subset M_{-x_0}$. If it turns out that, for any $g \in A \cap \mathfrak{M}_1$,

$$g(x_0) = 1,$$

then $A \cap \mathfrak{M}_1 \subset M_{x_0} = \{f : f(x_0) = 1,\ f \in E'\}$. We obtain the imbedding

$$\mathfrak{M}_1 \subset M_{x_0} \cup M_{-x_0},$$

which contradicts the above-mentioned theorem due to Bishop and Phelps [1] to the effect that the set $\mathfrak{M}_1$ is dense in the unit sphere $S_1(E')$. Hence the functionals f_0 and f_1 can in fact be chosen so that $x_0 \neq -x_1$.

Consider the family of elements

$$x_\lambda = \frac{(1-\lambda)x_0 + \lambda x_1}{\|(1-\lambda)x_0 + \lambda x_1\|} \in E \quad (0 \leqslant \lambda \leqslant 1)$$

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(since $x_0 \neq -x_1$, the x_λ are defined for all $\lambda \in [0, 1]$ and form a continuous curve in space E). Denote by F_λ the set of functionals of $\mathfrak{M}_1$, for which

$$f_\lambda(x_\lambda) = 1, \quad F_\lambda = \{f_\lambda\}.$$

The set F_λ is convex, hence it is contained either in A or in B. With $\lambda = 0$, $F_0 \subset A$, $\lambda = 1$, $F_1 \subset B$; we set

$$\lambda_0 = \sup\{\lambda : F_\lambda \subset A\}.$$

Assume that $F_{\lambda_0} \subset B$. By definition of λ_0, there exists a sequence $\lambda_n \to \lambda_0$ satisfying the condition $F_{\lambda_n} \subset A$. Let $f_n \in F_{\lambda_n}$; since the sphere $\bar{S}_1(E')$ is compact in the topology $\sigma(E', E)$ [4], it can be assumed without loss of generality that the f_n converge weakly to some functional $\tilde{f} \in \bar{S}_1(E')$. We have

$$|f_n(x_{\lambda_0}) - 1| = |f_n(x_{\lambda_0} - x_{\lambda_n})| \leqslant \|x_{\lambda_0} - x_{\lambda_n}\| \to 0$$

as $n \to \infty$, so that $\tilde{f}(x_{\lambda_0}) = \lim\limits_{n\to\infty} f_n(x_{\lambda_0}) = 1$ and $\tilde{f} \in F_{\lambda_0} \subset B$. On the other hand, every neighborhood $\tilde{f}$ in the topology $\sigma(E', E)$ contains elements $f_n \in A$. We obtain a contradiction, since the set B is open and does not intersect A by hypothesis.

It can be shown similarly that the assumption that $F_{\lambda_0} \subset A$ also leads to a contradiction. QED.

THEOREM 2. If the norm of space E is smooth, the set $\mathfrak{M}_1$ will be linearly connected in the weak topology $\sigma(E', E)$.

Proof. Let f_0 and f_1 be arbitrary elements of the set $\mathfrak{M}_1$ and let x_i $(i = 0, 1)$ be the points of $S_1(E)$ at which f_i attains its supremum. It can be assumed without loss of generality that $x_0 \neq -x_1$. Consider the element

$$x_\lambda = \frac{(1-\lambda)x_0 + \lambda x_1}{\|(1-\lambda)x_0 + \lambda x_1\|}.$$

Denote by f_λ the functional which attains its norms on x_λ (since the norm of E is smooth, such a functional is uniquely defined).

Let us show that the curve f_λ $(0 \leq \lambda \leq 1)$ is continuous in the weak topology $\sigma(E', E)$. Let $\lambda_n \to \lambda_0$. Since $\bar{S}_1(E')$ is compact in the weak topology $\sigma(E', E)$, it can be assumed that $f_{\lambda_n} \to \tilde{f}$ in the topology $\sigma(E', E)$. Using similar arguments to those employed in the proof of Theorem 1, we can show that $\tilde{f} = f_{\lambda_0}$. QED.

2. Let us now turn to properties of the set $\mathfrak{M}$ linked with characterization of the conjugate Banach spaces.

THEOREM 3. Let M be a closed subspace having the following properties:

1) M is dense in the weak topology $\sigma(E', E)$;

2) the norm $\|f\|$ $(f \in M)$ is smooth on the subspace M.

Then E is the conjugate space to the subspace M.

Proof. Consider the space M' conjugate to space M, furnished with the norm $\| \ \|_{E'}$. It is easily seen that $E \subset M'$. Denote by $\mathfrak{M}(M)$ the set of all continuous linear functionals which attain their supremum on the unit sphere of space M. Let φ be an arbitrary element of $\mathfrak{M}(M)$; then there exists an element $f_0 \in M$ with the norm $\|f_0\| = 1$, for which $\varphi(f_0) = \|\varphi\|_{M'}$. On the other hand, there exists in the space E an element x_0 such that

$$f_0(x_0) = \|x_0\|_E = \|\varphi\|_{M'}.$$

It follows from condition 2) that $x_0 = \varphi$, so that $\mathfrak{M}(M) \cap S_1(M') \subset S_1(E)$. To complete the proof, it remains to apply the theorem of Bishop and Phelps to the effect that the set $\mathfrak{M}(M)$ is dense with respect to the norm of space M (see [1]). QED.

It may be recalled that the set $M \subset E'$ is said to be normative if $\sup\limits_{f \in M} |f(x)|/\|f\| = \|x\|$ for all $x \in E$ [5].

THEOREM 4. The necessary and sufficient condition for a separable Banach space E to be conjugate to a Banach space F is that there exist in E' a closed subspace $M \subset \mathfrak{M}$, everywhere dense in the weak topology $\sigma(E', E)$.

Proof. The necessity is obvious. To prove the sufficiency, we note that the following can be proved, by arguments similar to those used in the proof of Theorem 1 of [6]: if M is a total closed subspace of $\mathfrak{M}$, every continuous linear functional φ of M may be written as

$$\varphi(f) = x(f), \quad x \in E,$$

so that M' is the same as E. In view of this, and the familiar results on the equivalence of two comparable complete metrizable topologies, defined on a vector space E, the norms $\|x\|_E$ and $\|x\|_{M'}$ must be equivalent. We shall show that the subspace $M \subset E'$ is normative. This latter assertion is equivalent to the relation $S_1(E) = S_1(M')$. The inclusion $S_1(E) \subset S_1(M')$ is obvious; to prove the reverse inclusion, let us assume that there is an element x of $S_1(M')$ which does not belong to $S_1(E)$. Then, since $S_1(E)$ is closed with respect to the norm $\|x\|_{M'}$ in M", there must exist a continuous linear functional f_0 with norm $\|f_0\|_{M''} = 1$, such that

$$\sup_{x \in S_1(E)} |f_0(x)| < \theta^2,$$

where $0 < \theta < 1$. Since the space M' = E is separable, the unit sphere of space M" is metrizable in the weak topology $\sigma(M'', M')$. We choose a sequence $f_n \in M$ $\|f_n\| = 1$ convergent to the element f_0 in the weak topology $\sigma(M'', M')$ (see [4, Chap. 4, Sec. 5, Proposition 5]); here it can be assumed without loss of generality that $\|f\| \geq \theta$ for every functional $f \in \text{conv}\{f_n\}$. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_i \lambda_i = 1$; by Lemma 1 of [6], there exist a number $\theta \leq \alpha \leq 1$ and a sequence $\{g_n\}$, satisfying the following conditions: $g_n \in \text{conv}\{f_n, f_{n+1}, \ldots\}$ for every n, $\|\sum_{i=1}^{\infty} \lambda_i g_i\| = \alpha$, and for any n,

$$\left\|\sum_{i=1}^{n} \lambda_i g_i\right\| \leqslant \alpha\left(1 - \theta \sum_{i=n+1}^{\infty} \lambda_i\right). \tag{1}$$

It immediately follows from the choice of the g_n that $g_n \to f_0$ in the weak topology $\sigma(M'', M')$. We shall have reached a contradiction if it can be shown that the element

$$g = \sum_{i=1}^{\infty} \lambda_i g_i,$$

belonging to M, does not attain its supremum on the unit sphere $S_1(E)$. For, let x be an element of $S_1(E)$; we choose an n such that

$$g_i(x) < \theta^2 \leqslant \alpha\theta,$$

if $i > n$. Then,

$$\sum_{i=1}^{\infty} \lambda_i g_i(x) < \sum_{i=1}^{n} \lambda_i g_i(x) + \alpha\theta \sum_{i=n+1}^{\infty} \lambda_i \leqslant \left\|\sum_{i=1}^{n} \lambda_i g_i\right\| + \alpha\theta \sum_{i=n+1}^{\infty} \lambda_i.$$

From this and the inequality (1) we have

$$\sum_{i=1}^{\infty} \lambda_i g_i(x) < \alpha\left(1 - \theta \sum_{i=n+1}^{\infty} \lambda_i\right) + \alpha\theta \sum_{i=n+1}^{\infty} \lambda_i = \alpha.$$

QED.

We shall conclude with a further condition under which Banach spaces are conjugate.

THEOREM 5. The necessary and sufficient condition for Banach space F to be conjugate to a Banach space E is that there exist in F a separable locally convex topology T, in which the unit sphere $S_1(F)$ of space F is compact.

The necessity follows readily from familiar results (see [4, Chap. 4, Sec. 5, Proposition 1]). To prove the sufficiency, we shall first show that the space F'_T conjugate to space F, furnished with the topology T, is imbedded in F'. In fact, if $f \in F'_T$, then f is bounded on S(F), since S(F) is compact in the topology T; hence $f \in F'$.

Denote by $F'_T(F')$ the normed space which is obtained when the norm of space F' is introduced into the vector space F'_T. We consider the duality between spaces F and $f \in F'_T$ and show that E' = F. To this end, we introduce into E a Mackey topology $\tau(E, F)$ and a weak topology $\sigma(E, F)$, generated by the duality between E and F. It can easily be seen that the topology of space E, generated by the norm $\|f\|_{F'}$, majorizes the weak topology $\sigma(E, F)$ and is majorized by the Mackey topology $\tau(E, F)$. By Mackey's theorem (see [4, Chap. 4, Sec. 2]), the conjugate space E' is the same as F. Let us show that

$$\sup_{f \in S(E)} |x(f)| = \|x\|_F. \quad (2)$$

This equation is equivalent to the fact that the linear manifold E $\subset$ F' is normative (see [5]). Since the sphere S(F) is compact in the topology T, the sphere S(F) is closed in the topology $\sigma(F, F'_T)$; in view of the criterion for contiguousness (see [5, Sec. 3, Note]), it follows from this that the set E is normative. QED.

Note 1. It can be shown that Theorem 3 ceases to hold if condition 2) in it is infringed. For, consider the space $l_1[0, 1]$, described in the book [7], the conjugate space to which consists of all bounded functions defined in [0, 1]. The subspace $M = C[0, 1] \subset (l_1[0, 1])' = l_\infty[0, 1]$ and satisfies all the conditions of Theorem 3 except for condition 2), yet $(C[0, 1])' \neq l_1[0, 1]$. This example also shows that Theorem 4 does not in general hold for a nonseparable Banach space.

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